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PREFACE

Le cinquième Colloque "Analyse Harmonique Non Commutative et Groupes de Lie" s'est tenu à Marseille - Luminy du 21 au 26 Juin 1982 dans le cadre du nouveau Centre International de Rencontres Mathématiques.

Les participants noteront que la liste des articles publiés ci-dessous ne correspond pas complètement aux conférences présentées durant le Colloque. C'est le cas, en particulier, pour des travaux dont la publication détaillée était prévue par ailleurs, ou de résultats dûs à plusieurs coauteurs.

Outre les participants à cette rencontre, nous tenons à remercier l'U.E.R. de Marseille - Luminy et le Centre International de Rencontres Mathématiques qui ont rendu possible la tenue de ce Colloque, ainsi que le secrétariat du Laboratoire de Mathématiques qui a assuré la préparation du présent volume.

Jacques CARMONA

Michèle VERGNE

NON COMMUTATIVE HARMONIC ANALYSIS AND LIE GROUPS

L^2 index and unitary representations

M. W. Baldoni Silva

§1. Introduction.

In this paper we investigate the representations that contribute to the index of the Dirac operator. Let G be a connected real semisimple Lie group with finite center, and let K be a maximal compact subgroup of the same rank as G. We assume that $G \subseteq G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a linear complexification of G. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebra of G and K respectively. Passing if needed to a suitable double covering of G, we can assume that the isotropy representation $K \to So(\mathfrak{g}_0/\mathfrak{k}_0)$ lift to $Spin(\mathfrak{g}_0/\mathfrak{k}_0)$. Since $\dim(G/K)$ is even, the spin representation (s,S) of $Spin(\mathfrak{g}_0/\mathfrak{k}_0)$ breaks up into two half spin representations s^+, s^-, and correspondingly $S = S^+ \oplus S^-$.

Let η be an irreducible finite dimensional representation of K and let V_η be the corresponding space. If we let $\tilde{\mathcal{E}}_\eta^{\pm}$ denote respectively the homogeneous vector bundle on G/K defined by the K-modules $E^{\pm} = V_\eta \otimes S^{\pm}$ then we may identify the L^2 cross-section of such a bundle as $(L^2(G)\otimes E^{\pm})^K$ and we have a corresponding Dirac operator $\tilde{\mathcal{D}}^+ : (L^2(G)\otimes E^+)^K \to (L^2(G)\otimes E^-)^K$ defined as in [P]. Since $\tilde{\mathcal{D}}^+$ is G invariant, it drops down to an elliptic differential operator $\mathcal{D}_\Gamma^+$ on $\Gamma\backslash G/K$ with coefficients in the bundle $\mathcal{E}^{\pm} = \tilde{\mathcal{E}}^{\pm}/\Gamma$, Γ being a discrete torsion-free subgroup of G of finite covolume. The L^2 index of $\mathcal{D}_\Gamma^+$ is finite ([M], [A]). More precisely let $\hat{G}$ denote the set of all equivalence classes of irreducible unitary representations π of G and write $m_\Gamma(\pi)$ for the multiplicity of π in $L^2_d(\Gamma\backslash G)$. Then

$$\text{index } \mathcal{D}^+_\Gamma = \sum_{\pi\in\hat{G}} m_\Gamma(\pi)\,(\dim \operatorname{Hom}_K(H(\pi),E^+) - \dim \operatorname{Hom}_K(H(\pi),E^-))$$

$H(\pi)$ being the representation space of π.

The aim of this paper is, fixed a K-module η, to describe all the π's for which $m(\pi,\eta) = \dim \operatorname{Hom}_K(H(\pi),E^+) - \dim \operatorname{Hom}_K(H(\pi),E^-) \neq 0$ and to compute it, when the real rank of G is one.

If π is an irreducible unitary representation with regular integral infinitesimal character, then [V1] gives a necessary condition for $m(\pi,\eta)$ not to be zero. This condition says that π is obtained in terms of parabolic induction via the Vogan-Zuckerman functor and allows one to proceed and compute $m(\pi,\eta)$ as shownin theorem 2.1 below. If the real rank of G is one, this necessary condition is also sufficient in view of the classification of [B-B] of the irreducible unitary representations for real rank one groups. Thus one obtains a complete classification. If π has singular integral infinitesimal character we have to use in some cases a more direct approach.

The new results in this paper are joint work with D. Barbasch.

§2. Notation and main results.

If H is a real Lie group we will write $\mathfrak{h}_0$ for the corresponding Lie algebra and $\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$ for the complexification.

Let now G be a Lie group satisfying the same assumption as in the introduction, and let θ be a Cartan involution of $\mathfrak{g}_0$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus s_0$. Fix a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$ and let K and T be the analytic subgroups corresponding to $\mathfrak{k}_0$ and $\mathfrak{t}_0$. Let $\Delta = \Delta(\mathfrak{g},\mathfrak{t})$ be the roots of $\mathfrak{g}$ relative to $\mathfrak{t}$, and let $\Delta(\mathfrak{k}) = \Delta(\mathfrak{k},\mathfrak{t})$ be the corresponding compact roots. If $V \subseteq \mathfrak{g}$ is a $\mathfrak{t}$-invariant subspace, write $\Delta(V)$ for the roots of $\mathfrak{t}$ in V, and $\rho(V) = 1/2\ \Sigma_{\alpha\in\Delta(V)}\ \alpha$.

Fix a system of positive compact roots $\Delta^+(\mathfrak{k})$. Given a θ stable (cf. [V 2]) parabolic subalgebra $\mathfrak{q} = \ell + u$ of $\mathfrak{g}$ with $\ell \supseteq \mathfrak{t}$, we fix a positive system $\Delta^+(\ell \cap \mathfrak{k}) \subseteq \Delta^+(\mathfrak{k})$ for $\Delta(\ell \cap \mathfrak{k},\mathfrak{t})$.

Let L be the subgroup corresponding to $\ell_0 = \ell \cap \mathfrak{g}_0$. Irreducible K types or $L \cap K$ types will be parametrized by the highest weight with respect to $\Delta^+(\mathfrak{k})$ or $\Delta^+(\ell\cap\mathfrak{k})$.

Fix a positive system of roots Δ^+ for Δ such that $2\rho_c = \Sigma_{\alpha\in\Delta^+(\mathfrak{k})}\ \alpha$ is dominant for Δ^+ and normalize s^+ and s^- such that on T', the regular elements of T, the characters of $s^{\pm}$ satisfy:

$$\mathrm{ch}(s^+-s^-)_{|T'} = \prod_{\alpha\in\Delta^+-\Delta^+(\mathfrak{k})} (e^{\alpha/2}-e^{-\alpha/2}).$$

We want to compute

$$m(\pi,\eta) = \dim \mathrm{Hom}_K(H(\pi),V_\eta \otimes S^+) - \dim \mathrm{Hom}_K(H(\pi),V_\eta \otimes S^-)$$

for π a irreducible unitary representation of G and for η a fixed K-module, $H(\pi)$ and $V(\eta)$ being the corresponding representation spaces.

Without loss of generality we may and do exclude the case that π is a discrete series or a nondegenerate limit of discrete series.

Let $\mathrm{ch}\,\pi$ be the distribution character of π, and let $\mathrm{ch}\,\eta$ be the character of η, η being a K-module. Let $D = \prod_{\alpha\in\Delta^+} (e^{\alpha/2}-e^{-\alpha/2})$. We recall the following facts from [A-S]. If X_π denote the infinitesimal character of π, then

1) $m(\pi,\eta) \neq 0$ only if $X_\pi = X_{\eta+\rho_c}$

2) $m(\pi,\eta) = (-1)^q a_\eta$ where a_η is defined by

$\mathrm{ch}(s^+-s^-)\mathrm{ch}_\pi|_{T'} = \sum_{\gamma\in\hat{K}} a_\gamma \mathrm{ch}\,\gamma|_{T'}$ $\quad q = 1/2(\dim \mathfrak{g} - \dim \mathfrak{k})$

Let Φ be a system of positive roots that makes $\eta + \rho_c$ dominant, and set $\lambda(\eta) = \eta + \rho_c - \rho(\Phi)$, where $\rho(\Phi) = 1/2 \sum_{\alpha\in\Phi} \alpha$. We denote by $\mathcal{R}^*$ the Vogan-Zuckerman functor as defined in [V2].

Theorem 2.1. Suppose G has real rank one and $\eta + \rho_c$ is regular. Then the irreducible unitary representations for which $m(\pi,\eta) \neq 0$ are given by the following procedure.

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra such that $\lambda = \lambda(\eta)$ satisfies $(\lambda,\alpha) = 0$ for $\alpha \in \Delta(\mathfrak{l})$ and $(\lambda,\alpha) \geq 0$ for $\alpha \in \Delta(\mathfrak{u})$. Then $\pi = \mathcal{R}^s \pi^L_\lambda$ and $m(\pi,\eta) = (-1)^{q_L}(-1)^{|-\Delta^+ \cap \Phi|}$, where $q_L = 1/2(\dim \mathfrak{l} - \dim \mathfrak{l} \cap \mathfrak{k})$ and $s = \dim(\mathfrak{u} \cap \mathfrak{k})$.

Remark. In [V1] it is proved in full generality, i.e., without the rank one assumption, the fact that if $m(\pi,\eta) \neq 0$ then there exists a θ-stable parabolic subalgebra and a λ' as in the theorem so that $\pi = \mathcal{R}^s \pi^L_{\lambda'}$. This is the necessary condition to which we alluded in the introduction.

Proof: By the classification of the irreducible unitary representations of real rank one groups of [B-B] all the representations in the conclusion of the theorem are unitary and conversely all the unitary irreducible representations with regular integral infinitesimal character are obtained in this way.

Therefore it is enough to prove that any representation of the form $\mathcal{R}^s \pi^L_{\lambda'}$ with λ' as in the theorem has $m(\pi,\eta) = 0$ unless $\lambda' = \lambda$ and in this case $m(\pi,\eta) = (-1)^{q_L}(-1)^{|-\Delta^+ \cap \Phi|}$.

Thus suppose that $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra and $\lambda' \in \hat{T}$ is such that $(\lambda',\alpha) = 0$ for $\alpha \in \Delta(\mathfrak{l})$ and $(\lambda',\alpha) \geq 0$ for $\alpha \in \Delta(\mathfrak{u})$. Let $\Psi(\mathfrak{l})$ be a positive system for $\Delta(\ell)$ that makes $2\rho(\ell \cap k) = \Sigma_{\alpha \in \Delta^+(\ell \cap k)} \alpha$ dominant, and set $\Psi = \psi(\ell) \cup \Delta(u)$. Then $\pi = \mathcal{R}^s \pi^L_{\lambda'}$ is an irreducible unitary representation by [B-B] and $\mathcal{R}^i \pi^L_{\lambda'} = 0$ for $i < s$ by [V2]. We want to compute $m(\pi, \eta)$ for such a representation.

For any irreducible representation π^L of L we denote by $e(\pi^L)$ the virtual character

$$e(\pi^L) = \sum_i (-1)^i ch(\mathcal{R}^i \pi^L)$$

and refer to it as the <u>Euler characteristic</u>. The Euler characteristic is a well defined map from the ring of virtual characters of L to the ring of virtual characters of G. Furthermore it commutes with coherent continuation. Suppose

$$ch\ \pi^L_{\lambda'} = \Sigma\ c_j\ ch\ \pi^L(\gamma_j)$$

where c_j are integers and $\pi^L(\gamma_j)$ is a discrete series or a principal series representation. Then

$$e(\pi^L_{\lambda'}) = \Sigma\ c_j e(\pi^L(\gamma_j)).$$

We note that $\gamma_j + \rho(u) = w(\lambda' + \rho(\psi))$ with $w \in W(\ell)$, $W(\ell)$ being the Weyl group of ℓ. Since $(\lambda' + \rho(\psi), \alpha) > 0$ for $\alpha \in \Delta(u)$, we obtain $(\gamma_j + \rho(u), \alpha) > 0$ for $\alpha \in \Delta(u)$. It follows from [V 2, theorem 8.2.15] that

$$e(\pi^L(\gamma_j)) = (-1)^s \mathrm{ch}\, \mathcal{R}^s \pi^L(\gamma_j) = (-1)^s \mathrm{ch}\ \pi(\gamma_j+\rho(u)).$$

Thus

$$\mathrm{ch}\ \pi = (-1)^s e(\pi^L_{\lambda'}) = \Sigma\ c_j\ \mathrm{ch}\ \pi(\gamma_j+\rho(u)).$$

Since we need only $\mathrm{ch}\ \pi|_{T'}$ we are interested only in the c_j's for which $\pi^L(\gamma_j)$ is a discrete series representation.

By comparing the expressions for such characters we can write

$$\mathrm{ch}\ \pi^L_{\lambda'}|_{T'} = (-1)^{q_L} \sum_{\substack{w \in W(\ell) \\ w\psi(\ell) \supset \Delta^+(\ell \cap \mathfrak{k})}} \mathrm{ch}\ \pi(w\psi(\ell),\ \lambda' + \rho(\ell))|_{T'},$$

where $\rho(\ell) = 1/2\ \Sigma_{\alpha \in \psi(\ell)}\ \alpha$ and where the discrete series on the right side is the one whose chamber is $w\psi(\ell)$ and whose infinitesimal character is $\lambda' + \rho(\ell)$. Then

$$\mathrm{ch}\ \pi|_{T'} = (-1)^{q_L} \sum_{\substack{w \in W(\ell) \\ w\psi(\ell) \supset \Delta^+(\ell \cap \mathfrak{k})}} \mathrm{ch}\ \pi(w\psi(\ell) \cup \Delta(u),\ \lambda' + \rho(\psi))|_{T'}$$

Writing out explicitly the expression on the right hand side, one obtains immediately that $m(\pi,\eta) = 0$ unless $w(\lambda'+\rho(\psi)) = \lambda' +$

$w\rho(\psi) = \eta + \rho_c$ for some $w \in W(\ell)$. In this case

$$m(\pi,\eta) = (-1)^{q_L}(-1)^{|-\Delta^+ \cap w\psi(\ell) \cup \Delta(u)|} .$$

Because of the regularity of $\eta + \rho_c$ we see that $m(\pi,\eta) = 0$ unless $\lambda = \lambda'$ and $w\psi(\ell) \cup \Delta(u) = \Phi$. Then $m(\pi,\eta) = (-1)^{q_L}(-1)^{|-\Delta^+ \cap \Phi|}$, and the theorem follows.

We now give an idea of how to compute $m(\pi,\eta)$ when $\eta + \rho_c$ is singular. We omit the details. Since $\eta + \rho_c$ is regular with respect to $\Delta(\mathfrak{k})$, $\eta + \rho_c$ cannot be singular with respect to two adjacent noncompact roots in the Dynkin diagram for the simple roots of Φ. Thus in real rank one, $\eta + \rho_c$ is singular with respect to exactly one simple noncompact root in Φ, say β. Then χ_π coincides with the infinitesimal character of a limit of discrete series. We note also that if $w(\eta+\rho_c)$ is dominant with respect to $\Delta^+(\mathfrak{k})$, then $w(\eta+\rho_c)$ must be singular with respect to a compact root unless $w(\eta+\rho_c) = \eta + \rho_c$; hence $w = \text{id}$ or the simple reflection about β. Thus

$$\operatorname{ch} \pi|_{T'} = (-1)^q m(\eta+\rho_c,\pi) \cdot \frac{\sum_{w\in W(\mathfrak{k})} \varepsilon(w) e^{w(\eta+\rho_c)}}{D}$$

where $W(\mathfrak{k})$ is the Weyl group of $\mathfrak{k}$.

To compute $m(\eta+\rho_c,\pi)$ if real rank of G is one we argue as follows. By the classification in [B-B] we have either that

1) $\pi = \mathcal{R}^S \pi_{\lambda'}^L$, with $\mathfrak{q} = \ell + u$ a θ-stable parabolic and

$$(\lambda',\alpha) = 0, \quad \alpha \in \Delta(\ell), \quad (\lambda'+\rho(\ell)+\rho(u),\alpha) \geq 0 \quad \text{for} \quad \alpha \in \Delta(u)$$

(with equality for at least one $\alpha \in \Delta(u)$) or else

2) $\mathfrak{g} = sp(n,1)$ and π has a lowest K-type of a very special form which can be given explicitly.

In the first situation the argument given in the regular case still works, with obvious modifications, and one gets that $m(\pi,\eta) = 0$ unless there exists $w \in W(\ell)$ such that $w(\lambda'+\rho(\ell)+\rho(u)) = \eta + \rho_c$. (Such w is unique). When w exists then $m(\pi,\eta) = (-1)^{P_L}(-1)^{|-\Delta^+ \cap w\psi(\ell) \cup \Delta(u)|}$.

In the second situation one has to compute directly $D\ ch\ \pi|_{T'}$. This is not very hard because the composition factors of π are known [B-K] and one can proceed to compute $m(\pi,\eta+\rho_c)$.

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SUR LA CLASSIFICATION DES MODULES ADMISSIBLES IRREDUCTIBLES

par

Jacques CARMONA

0. Introduction.

0.1. Etant donné un groupe de Lie semi-simple connexe réel G, de centre fini, d'algèbre $\mathfrak{g}$, on fixe un sous-groupe compact maximal K de G, d'algèbre κ, on désigne par θ la conjugaison et par $\mathfrak{g} = \kappa + \mathfrak{p}$ la décomposition de Cartan correspondante. Tout sous-groupe parabolique P de G admet une décomposition de Langlands $P = M A N$ pour laquelle la composante déployée A est un sous-groupe vectoriel dont l'algèbre $\mathcal{A}$ est contenue dans $\mathfrak{p}$, MA étant le centralisateur de A dans G. Si $\mathfrak{m}$ et η sont les algèbres de Lie de M et N respectivement, on note $\Phi(P,A)$ l'ensemble des poids de $\mathcal{A}$ dans η et on définit

(0.1.1) $$\rho_P = \frac{1}{2} \sum_{\alpha\in\Phi(P,A)} \alpha ,$$

chaque poids α figurant avec sa multiplicité. On fixe une fois pour toutes un sous-groupe parabolique minimal $P_\circ = M_\circ A_\circ N_\circ$ de G; on dira que la paire (P,A) est standard si $P \supseteq P_\circ$ et $A \subseteq A_\circ$. Enfin, on note $\mathfrak{g}_{\mathbb{C}}$ la complexifiée de $\mathfrak{g}$ et $\mathcal{S}_{\mathbb{C}}$ le complexifié de tout sous-espace $\mathcal{S}$ de $\mathfrak{g}$.

0.2. Pour tout $(\mathfrak{g},K)$-module admissible de type fini $\mathcal{S}$, l'espace quotient $\mathcal{S}_\eta = \mathcal{S}/_{\eta.\mathcal{S}}$ est un $(\mathfrak{m},M\cap K)$-module admissible de type fini (voir [1] Ch. IV) somme directe des sous-modules

(0.2.1) $$\mathcal{S}_{\eta\,\xi} = \{ v\in\mathcal{S}_\eta \ / \ \forall H\in\mathcal{A} \ \exists k\in N \quad (H-\xi(H)Id)^k.v = 0 \} ,$$

où ξ parcourt le dual complexe $\mathcal{A}'_{\mathbb{C}}$ de $\mathcal{A}_{\mathbb{C}}$. On sait qu'à tout élément ξ de

(0.2.2) $$e(P,\mathcal{S}) = \{ \xi\in\mathcal{A}'_{\mathbb{C}} \ / \ \mathcal{S}_{\eta\,\xi} \neq 0 \} ,$$

et à tout M-module admissible de type fini $(\sigma,\mathcal{H}_\sigma)$ dont l'espace des vecteurs $M\cap K$-finis $(\mathcal{H}_\sigma)_\circ$ est isomorphe à un quotient non nul de $\mathcal{S}_{\eta\xi}$, est associé un $(\mathfrak{m},K)$-morphisme non nul $\mathcal{S} \to \mathfrak{I}_{P,\sigma,\nu}$ où $\nu = \xi - \rho_P$ et $\mathfrak{I}_{P,\sigma,\nu}$ désigne l'espace de la série principale induite de P à G par le couple (σ,ν), (voir [1] Ch.IV).

0.3.Pour établir sa classification, Langlands utilise une construction géométrique permettant d'associer à tout $(\mathfrak{g},K)$-module irréductible $\mathcal{S}$, par l'intermédiaire d'un élément extrémal de $e(P_\circ,\mathcal{S})$, une paire parabolique standard (P,A), une représentation tempére $(\sigma,\mathcal{H}_\sigma)$ de M et un caractère complexe strictement $\Phi(P,A)$-dominant ν de $\mathfrak{a}$, de telle sorte que $\mathfrak{X}$ soit isomorphe à un quotient de $\mathfrak{I}_{P,\sigma,\nu}$. Dans la première partie de ce travail, nous proposons une interprétation de cette construction qui nous permet de simplifier notablement l'exposé de Borel-Wallach ([1] Ch.IV).

0.4. Soit B un sous-groupe de Cartan fondamental θ-stable de G, $\mathfrak{b}$ son algèbre de Lie, $\mathfrak{t} = \mathfrak{b} \cap \kappa$ et Δ (respmt. Δ_κ) le système des racines de $(\mathfrak{g},\mathfrak{b})$, (respmt. $(\kappa,\mathfrak{t})$). On fixe un système Δ_κ^+ de racines positives pour Δ_κ et on définit:

$$(0.4.1) \qquad \rho_c = \frac{1}{2} \sum_{\nu\in\Delta_\kappa^+} \nu \quad .$$

Toute classe de représentation irréductible de dimension finie (appelée κ-type) de κ est alors caractérisée par son poids Δ_κ^+-dominant μ .

0.5.Etant donné un κ-type μ, on choisit un système Δ^+ de racines positives pour Δ, θ-stable et rendant $\mu+2\rho_c$ Δ^+- dominant. Si

$$(0.5.1) \qquad \lambda_\mu = \mu + 2\rho_c - \rho$$

où

$$(0.5.2) \qquad \rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha \quad ,$$

Vogan ([10] Prop. 1.4) démontre qu'il existe une décomposition

$$(0.5.3) \qquad \lambda_\mu = (\lambda_\mu)_\circ - \sum_{j=1}^{p} c_j \beta_j \quad ,$$

où $(\lambda_\mu)_\circ$ est Δ^+-dominant, $c_j \geq 0$,$(1\leq j\leq p)$, et $\{\beta_1,\ldots,\beta_p\}$ un système orthogonal de racines imaginaires de Δ^+, orthogonales à $(\lambda_\mu)_\circ$. En outre, $(\lambda_\mu)_\circ$ ne dépend que de μ et non du choix de Δ^+ .Dans la deuxième partie de ce travail, nous proposons une approche différente de ces résultats. Notre méthode nous permet de simplifier notablement les démonstrations de Vogan et, notamment, de trivialiser la démonstration de l'unicité de $(\lambda_\mu)_\circ$, (Prop. 2.2).

0.6. Il est remarquable de constater que les deux classifications dont nous disposons sont basées sur la même construction géométrique: la projection d'une forme "extrémale" sur le cône convexe fermé des poids dominants. On peut espérer que cette présentation plus simple facilitera la comparaison des deux classifications, en permettant par exemple de préciser comment les informations obtenues par l'une ou l'autre de ces méthodes se complètent. Signalons que, dans l'un et l'autre cas, le caractère infinitésimal peut être défini par une