# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1265

Walter Van Assche

# Asymptotics for Orthogonal Polynomials

Springer-Verlag

# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1265

## Walter Van Assche

# Asymptotics for Orthogonal Polynomials

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

**Author**

Walter Van Assche
Senior Research Assistant of the Belgian National Fund for Scientific Research)
Departement Wiskunde, Katholieke Universiteit Leuven
Celestijnenlaan 200 B, 3030 Leuven, Belgium

Mathematics Subject Classification (1980): Primary: 42C05
Secondary: 33A65; 39A10; 40A15; 40A30; 41A55; 41A60; 60J80

ISBN 3-540-18023-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18023-0 Springer-Verlag New York Berlin Heidelberg



Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

# Lecture Notes in Mathematics

**For information about Vols. 1–1062 please contact your bookseller or Springer-Verlag.**

Vol. 1063: Orienting Polymers. Proceedings, 1983. Edited by J.L. Ericksen. VII, 166 pages. 1984.

Vol. 1064: Probability Measures on Groups VII. Proceedings, 1983. Edited by H. Heyer. X, 588 pages. 1984.

Vol. 1065: A. Cuyt, Padé Approximants for Operators: Theory and Applications. IX, 138 pages. 1984.

Vol. 1066: Numerical Analysis. Proceedings, 1983. Edited by D.F. Griffiths. XI, 275 pages. 1984.

Vol. 1067: Yasuo Okuyama, Absolute Summability of Fourier Series and Orthogonal Series. VI, 118 pages. 1984.

Vol. 1068: Number Theory, Noordwijkerhout 1983. Proceedings. Edited by H. Jager. V, 296 pages. 1984.

Vol. 1069: M. Kreck, Bordism of Diffeomorphisms and Related Topics. III, 144 pages. 1984.

Vol. 1070: Interpolation Spaces and Allied Topics in Analysis. Proceedings, 1983. Edited by M. Cwikel and J. Peetre. III, 239 pages. 1984.

Vol. 1071: Padé Approximation and its Applications, Bad Honnef 1983. Prodeedings. Edited by H. Werner and H.J. Bünger. VI, 264 pages. 1984.

Vol. 1072: F. Rothe, Global Solutions of Reaction-Diffusion Systems. V, 216 pages. 1984.

Vol. 1073: Graph Theory, Singapore 1983. Proceedings. Edited by K.M. Koh and H.P. Yap. XIII, 335 pages. 1984.

Vol. 1074: E.W. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations. III, 143 pages. 1984.

Vol. 1075: H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points. IX, 159 pages. 1984.

Vol. 1076: Infinite-Dimensional Systems. Proceedings, 1983. Edited by F. Kappel and W. Schappacher. VII, 278 pages. 1984.

Vol. 1077: Lie Group Representations III. Proceedings, 1982–1983. Edited by R. Herb, R. Johnson, R. Lipsman, J. Rosenberg. XI, 454 pages. 1984.

Vol. 1078: A.J.E.M. Janssen, P. van der Steen, Integration Theory. V, 224 pages. 1984.

Vol. 1079: W. Ruppert. Compact Semitopological Semigroups: An Intrinsic Theory. V, 260 pages. 1984

Vol. 1080: Probability Theory on Vector Spaces III. Proceedings, 1983. Edited by D. Szynal and A. Weron. V, 373 pages. 1984.

Vol. 1081: D. Benson, Modular Representation Theory: New Trends and Methods. XI, 231 pages. 1984.

Vol. 1082: C.-G. Schmidt, Arithmetik Abelscher Varietäten mit komplexer Multiplikation. X, 96 Seiten. 1984.

Vol. 1083: D. Bump, Automorphic Forms on GL (3,IR). XI, 184 pages. 1984.

Vol. 1084: D. Kletzing, Structure and Representations of Q-Groups. VI, 290 pages. 1984.

Vol. 1085: G.K. Immink, Asymptotics of Analytic Difference Equations. V, 134 pages. 1984.

Vol. 1086: Sensitivity of Functionals with Applications to Engineering Sciences. Proceedings, 1983. Edited by V. Komkov. V, 130 pages. 1984

Vol. 1087: W. Narkiewicz, Uniform Distribution of Sequences of Integers in Residue Classes. VIII, 125 pages. 1984.

Vol. 1088: A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, Characterization of Distributions by the Method of Intensively Monotone Operators. X, 175 pages. 1984.

Vol. 1089: Measure Theory, Oberwolfach 1983. Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XIII, 327 pages. 1984.

Vol. 1090: Differential Geometry of Submanifolds. Proceedings, 1984. Edited by K. Kenmotsu. VI, 132 pages. 1984.

Vol. 1091: Multifunctions and Integrands. Proceedings, 1983. Edited by G. Salinetti. V, 234 pages. 1984.

Vol. 1092: Complete Intersections. Seminar, 1983. Edited by S. Greco and R. Strano. VII, 299 pages. 1984.

Vol. 1093: A. Prestel, Lectures on Formally Real Fields. XI, 125 pages. 1984.

Vol. 1094: Analyse Complexe. Proceedings, 1983. Edité par E. Amar, R. Gay et Nguyen Thanh Van. IX, 184 pages. 1984.

Vol. 1095: Stochastic Analysis and Applications. Proceedings, 1983. Edited by A. Truman and D. Williams. V, 199 pages. 1984.

Vol. 1096: Théorie du Potentiel. Proceedings, 1983. Edité par G. Mokobodzki et D. Pinchon. IX, 601 pages. 1984.

Vol. 1097: R.M. Dudley, H. Kunita, F. Ledrappier, École d'Éte de Probabilités de Saint-Flour XII – 1982. Edité par P.L. Hennequin. X, 396 pages. 1984.

Vol. 1098: Groups – Korea 1983. Proceedings. Edited by A.C. Kim and B.H. Neumann. VII, 183 pages. 1984.

Vol. 1099: C.M. Ringel, Tame Algebras and Integral Quadratic Forms. XIII, 376 pages. 1984.

Vol. 1100: V. Ivrii, Precise Spectral Asymptotics for Elliptic Operators Acting in Fiberings over Manifolds with Boundary. V, 237 pages. 1984.

Vol. 1101: V. Cossart, J. Giraud, U. Orbanz, Resolution of Surface Singularities. Seminar. VII, 132 pages. 1984.

Vol. 1102: A. Verona, Stratified Mappings – Structure and Triangulability. IX, 160 pages. 1984.

Vol. 1103: Models and Sets. Proceedings, Logic Colloquium, 1983, Part I. Edited by G.H. Müller and M.M. Richter. VIII, 484 pages. 1984.

Vol. 1104: Computation and Proof Theory. Proceedings, Logic Colloquium, 1983, Part II. Edited by M.M. Richter, E. Börger, W. Oberschelp, B. Schinzel and W. Thomas. VIII, 475 pages. 1984.

Vol. 1105: Rational Approximation and Interpolation. Proceedings, 1983. Edited by P.R. Graves-Morris, E.B. Saff and R.S. Varga. XII, 528 pages. 1984.

Vol. 1106: C.T. Chong, Techniques of Admissible Recursion Theory. IX, 214 pages. 1984.

Vol. 1107: Nonlinear Analysis and Optimization. Proceedings, 1982. Edited by C. Vinti. V, 224 pages. 1984.

Vol. 1108: Global Analysis – Studies and Applications I. Edited by Yu.G. Borisovich and Yu.E. Gliklikh. V, 301 pages. 1984.

Vol. 1109: Stochastic Aspects of Classical and Quantum Systems. Proceedings, 1983. Edited by S. Albeverio, P. Combe and M. Sirugue-Collin. IX, 227 pages. 1985.

Vol. 1110: R. Jajte, Strong Limit Theorems in Non-Commutative Probability. VI, 152 pages. 1985.

Vol. 1111: Arbeitstagung Bonn 1984. Proceedings. Edited by F. Hirzebruch, J. Schwermer and S. Suter. V, 481 pages. 1985.

Vol. 1112: Products of Conjugacy Classes in Groups. Edited by Z. Arad and M. Herzog. V, 244 pages. 1985.

Vol. 1113: P. Antosik, C. Swartz, Matrix Methods in Analysis. IV, 114 pages. 1985.

Vol. 1114: Zahlentheoretische Analysis. Seminar. Herausgegeben von E. Hlawka. V, 157 Seiten. 1985.

Vol. 1115: J. Moulin Ollagnier, Ergodic Theory and Statistical Mechanics. VI, 147 pages. 1985.

Vol. 1116: S. Stolz, Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder. XXIII, 134 Seiten. 1985.

continued on page 205

# PREFACE

In recent years there has been an increased interest in the theory of orthogonal polynomials but the number of textbooks treating orthogonal polynomials is rather limited. Even at present the best reference is Szegö's book [175] which was first published in 1939. Freud's book [61] is also highly recommended. The more recent introduction by Chihara [39] does not include asymptotic results and the monographs by Geronimus [75] and Nevai [138] are rather technical and treat a very specific part of the theory of orthogonal polynomials on [-1,1]. This monograph concentrates on the asymptotic theory of general orthogonal polynomials on the real line. Most of the theorems have been proved, for some of them only a sketch of the proof is given and tedious proofs out of the scope of this monograph have been omitted.

I would like to express my very cordial thanks to all those who, in some way or another, have contributed to this monograph. Many thanks in particular to Jef Teugels who made me appreciate the mathematical analysis of orthogonal polynomials. Some theorems in this monograph are the result of working with other mathematicians. I am very grateful to Makoto Maejima (Chapter 3), Jeffrey Geronimo (Chapter 2, Section 4.4) and Guido Fano (Section 5.1) for a fruitful collaboration. Finally I would like to thank Daniel Bessis, Pierre Moussa, Giorgio Turchetti, Paul Nevai, Doron Lubinsky, Ed Saff, Alphonse Magnus and many others for various interesting discussions and Bea Peeters for an excellent job of typing the manuscript.

Walter Van Assche, March 1987.

TABLE OF CONTENTS

# INTRODUCTION

## 0.1. Definitions and examples

Let $\mu$ be a positive probability measure on the real line with distribution function $\mu(t) = \mu((-\infty,t])$. Suppose that all the *moments*

$$(0.1.1) \qquad m_n = \int_{-\infty}^{\infty} x^n d\mu(x)$$

are finite and that the *support* of the measure $\mu$

$$\mathrm{supp}(\mu) = \{x \in \mathbb{R} \ : \ \forall\varepsilon > 0 \quad \mu((x-\varepsilon,x+\varepsilon)) > 0\}$$

is an infinite set. Then there exists a sequence of polynomials $\{p_n(x) : n = 0,1,2,\ldots\}$ such that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)d\mu(x) = \delta_{m,n} \qquad m,n \geqslant 0$$

(0.1.2)

$$p_n(x) = p_n(x;\mu) = k_n x^n + \ldots \quad , \quad k_n > 0 \ .$$

This sequence consists of *orthogonal polynomials* with *spectral measure* $\mu$ (or *orthogonality measure*) (Szegö [175], p. 23). The definition (0.1.2) actually implies that the polynomials are normed so that one should speak of "orthonormal polynomials". We denote the *monic polynomials* by

$$(0.1.3) \qquad \hat{p}_n(x) = k_n^{-1}\ p_n(x) \ .$$

Since $\mu$ is a probability measure, it follows that

$$p_0(x) = \hat{p}_0(x) = 1.$$

It is possible to extend the notion of orthogonal polynomials by using a measure $\mu$ on some curve in the complex plane, but in this monograph we will always use orthogonality on the real line (the only exception is § 1.4).

The measure $\mu$ can always be decomposed as a linear combination of three different types of measures, $\mu = \mu_{ac} + \mu_d + \mu_s$, $\mu_{ac}$ is an absolutely continuous measure (with respect to Lebesgue measure), $\mu_d$ is an atomic measure with mass on a discrete set which is at most denumerable and $\mu_s$ is a singular measure (with respect to Lebesgue measure) with a continuous distribution function. If the spectral measure is absolutely continuous, then there exists a (Radon-Nikodym) derivative w such that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w(x)dx = \delta_{m,n} \qquad m,n \geqslant 0 .$$

The function w is then the *weight function* for the orthogonal polynomials.

Example 1 (Szegö [175], Chapter 4).

Consider the weight function $(\alpha,\beta > -1)$

$$(0.1.4) \qquad w(x) = \begin{cases} 2^{-\alpha-\beta-1} \dfrac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} (1-x)^{\alpha}(1+x)^{\beta} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(this is the density of a beta-distribution on [-1,1]), then

$$(0.1.5) \qquad p_n(x) = \left\{ \frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1} \frac{n!(\alpha+\beta+2)_n}{(\alpha+1)_n (\beta+1)_n} \right\}^{1/2} P_n^{(\alpha,\beta)}(x)$$

where $\{P_n^{(\alpha,\beta)}(x) : n = 0,1,2,\ldots\}$ are *Jacobi polynomials* with parameters $\alpha$ and $\beta$. We have used the Pochhammer notation

$$(a)_n = a(a+1) \ldots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} .$$

The Jacobi polynomials are explicitely given by

$$(0.1.6) \qquad P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j}(x+1)^j$$

$$= \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(\alpha+\beta+n+j+1)}{\Gamma(\alpha+j+1)} \left(\frac{x-1}{2}\right)^j .$$

They satisfy the differential equation

$$(1-x^2)y'' + [\beta-\alpha-(\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0$$

and can be found from *Rodrigues' formula*

$$(1-x)^{\alpha}(1+x)^{\beta}\, P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} .$$

Special cases include the *Legendre polynomials*

$$P_n(x) = P_n^{(0,0)}(x) ,$$

the *Chebyshev polynomials of the first kind*

$$T_n(x) = \frac{2.4\ldots 2n}{1.3\ldots(2n-1)}\, P_n^{(-1/2,-1/2)}(x) ,$$

and the *Chebyshev polynomials of the second kind*

$$U_n(x) = \frac{1}{2}\,\frac{2.4\ldots(2n+2)}{1.3\ldots(2n+1)}\, P_n^{(1/2,1/2)}(x) .$$

If we set $x = \cos t$ then

$$T_n(x) = \cos(nt) \quad ; \quad U_n(x) = \frac{\sin((n+1)t)}{\sin t} .$$

Jacobi polynomials with $\alpha = \beta$ are called *ultraspherical* or *Gegenbauer polynomials.*

Example 2 (Szegö [175], Chapter 5).

The orthogonal polynomials for the weight function ($\alpha > -1$)

$$(0.1.7)\qquad w(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)}\, x^{\alpha} e^{-x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

are given by

$$(0.1.8)\qquad p_n(x) = (-1)^n \binom{n+\alpha}{n}^{-1/2} L_n^{(\alpha)}(x)$$

where

$$(0.1.9)\qquad L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!} .$$

The polynomials in (0.1.9) are the *Laguerre polynomials* when $\alpha = 0$; for arbitrary $\alpha > -1$ these polynomials are called *generalized Laguerre polynomials* or *Sonin-Laguerre polynomials*. They satisfy the differential equation

$$xy'' + (\alpha+1-x)y' + ny = 0$$

and can be obtained through Rodrigues' formula

$$e^{-x}x^{\alpha}L_n^{(\alpha)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) .$$

Notice that (0.1.7) is the density of a gamma distribution on $[0,\infty)$.

Example 3 (Szegö [175], Chapter 5).

Consider the density of a normal (or Gaussian) distribution

(0.1.10) $$w(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \qquad x \in \mathbb{R}$$

then the orthogonal polynomials are

(0.1.11) $$p_n(x) = (2^n n!)^{-1/2} H_n(x)$$

where $\{H_n(x) : n = 0,1,2,...\}$ are the *Hermite polynomials*. Somewhat more general is the weight function $(\alpha > -1/2)$

(0.1.12) $$w(x) = \frac{1}{\Gamma(\alpha+\frac{1}{2})} |x|^{2\alpha} e^{-x^2}$$

with

(0.1.13) $$p_n(x) = 2^{-n}\{[\tfrac{n}{2}]!\,(\alpha + 1/2)_{[\frac{n+1}{2}]}\}^{-1/2} H_n^{(\alpha)}(x) .$$

The polynomials $\{H_n^{(\alpha)}(x) : n = 0,1,2,...\}$ are *generalized Hermite polynomials* or *Markov-Sonin polynomials*. They satisfy the differential equation

$$xy'' + 2(\alpha - x^2)y' + (2xn - \theta_n x^{-1})y = 0$$

with $\theta_{2m} = 0$ and $\theta_{2m+1} = 2\alpha$. There exists a Rodrigues' formula which is (for $\alpha = 0$)

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}) .$$

There is a simple relation with the (generalized) Laguerre polynomials :

(0.1.14) $$H_{2n}^{(\alpha)}(x) = (-1)^n 2^{2n} n!\, L_n^{(\alpha-1/2)}(x^2)$$

$$(0.1.14') \quad H^{(\alpha)}_{2n+1}(x) = (-1)^n \, 2^{2n+1} \, n! \; x \; L^{(\alpha+1/2)}_n(x^2) \; .$$

The polynomials of Jacobi, Laguerre and Hermite together are the *classical orthogonal polynomials*. They can be characterized as being the only ones that satisfy a homogeneous linear differential equation of the second order. This class also consists of the only orthogonal polynomials for which the derivatives are again orthogonal polynomials. A formula of Rodrigues type is also possible only within this class.

There are other important sequences of orthogonal polynomials. We might call those semi-classical orthogonal polynomials. They are obtained by allowing a discrete version of the notion of derivative (ordinary difference or q-difference). Some examples are

Example 4 (Chihara [39], p. 170).

Suppose the spectral measure is discrete and supported on the positive integers with jumps ($a > 0$)

$$(0.1.15) \qquad \mu(\{n\}) = e^{-a} \frac{a^n}{n!} \qquad n = 0,1,2,\ldots$$

then

$$(0.1.16) \qquad p_n(x) = (a^n n!)^{-1/2} \, C^{(a)}_n(x)$$

where $\{C^{(a)}_n(x) : n = 0,1,2,\ldots\}$ are the *Charlier polynomials*. Sometimes these polynomials are referred to as *Poisson-Charlier polynomials* because the spectral measure corresponds to the Poisson distribution. An explicit expression for the Charlier polynomials is

$$(0.1.17) \qquad C^{(a)}_n(x) = \sum_{j=0}^{n} \binom{n}{j}\binom{x}{j} j!(-a)^{n-j} \; .$$

Example 5 (Chihara [39], p. 175).

For the discrete measure ($\beta > 0$, $0 < c < 1$)

$$(0.1.18) \qquad \mu(\{n\}) = (1-c)^{\beta} \, (\beta)_n \frac{c^n}{n!} \qquad n = 0,1,2,\ldots$$

(which is the Pascal distribution or the negative-binomial distribution) the orthogonal polynomials are

$$(0.1.19) \qquad p_n(x) = \left(\frac{c^n}{n!}\right)^{1/2} m_n(x;\beta,c)$$

with $\{m_n(x;\beta,c) : n = 0,1,2,\ldots\}$ the *Meixner polynomials*, given by

$$(0.1.20) \qquad m_n(x;\beta,c) = n! \sum_{j=0}^{n} \binom{x}{j}\binom{n+\beta-1}{j+\beta-1}\left(1 - \frac{1}{c}\right)^j .$$

Chihara [39] refers to these polynomials as the *Meixner polynomials of the first kind*. The *Meixner polynomials of the second kind* (in Chihara's terminology) have the weight function

$$(0.1.21) \qquad w(x) = C \left|\Gamma\left(\frac{\eta + ix}{2}\right)\right|^2 \exp(-x \text{ Arctan } \delta)$$

with C a normalizing constant. Askey and Wilson [9] refer to these polynomials as the *Meixner-Pollaczek* polynomials.

Example 6 (Chihara [39], p. 184)

Consider the weight function $(a \geq |b|,\ \lambda > 0)$

$$(0.1.22) \qquad w(x) = \begin{cases} \dfrac{2^{2\lambda}(2\lambda+a)}{4\pi\ \Gamma(2\lambda)}(1 - x^2)^{\lambda-1/2} \exp\left(-\text{Arcsin } x \dfrac{ax + b}{\sqrt{1 - x^2}}\right) & \\ \qquad \times \left|\Gamma\left(\lambda + i \dfrac{ax + b}{2\sqrt{1 - x^2}}\right)\right|^2 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

then the orthogonal polynomials are given by

$$(0.1.23) \qquad p_n(x) = \left\{\frac{n!(2n+2\lambda+a)}{(2\lambda)_n\ (2\lambda+a)}\right\}^{1/2} P_n^{\lambda}(x;a,b)$$

where $\{P_n^{\lambda}(x;a,b) : n = 0,1,2,\ldots\}$ are the *Pollaczek polynomials* on [-1,1]. These can be written explicitely as

$$(0.1.24) \qquad n!P_n^{\lambda}(x;a,b) = (2\lambda)_n\ e^{in\theta} \sum_{j=0}^{n} \binom{n}{j} \frac{(\lambda + it)_j}{(2\lambda)_j} (e^{-2i\theta} - 1)^j$$

with $x = \cos\theta$ and $t = \dfrac{ax + b}{2\sqrt{1 - x^2}}$ .

There are many more sequences of orthogonal polynomials. More examples are given in Chihara's book [39]. In Chapter 1 (§ 1.4) there will be a short discussion about orthogonal polynomials with a singular spectral measure supported on a Cantor set.

## 0.2. General properties

We will now mention some general properties of orthogonal polynomials. We will not give proofs here but refer to the literature (Szegö [175], Freud [61], Chihara [39]).

Lemma 0.1. (Szegö [175], p. 44). The zeros of orthogonal polynomials are real and simple and belong to the interval $(a,b)$, where $a$ and $b$ are respectively the infimum and supremum of $\text{supp}(\mu)$ (we take $a = -\infty$ and/or $b = \infty$ when these do not exist). If we order the zeros of $p_n$ in such a way that

$$(0.2.1) \qquad a < x_{1,n} < x_{2,n} < \ldots < x_{n,n} < b$$

then $x_{j,n+1} < x_{j,n} < x_{j+1,n+1}$ $(j = 1,\ldots,n)$, which means that the zeros of $p_n$ and $p_{n+1}$ interlace.

The zeros of orthogonal polynomials serve very well as nodes for a numerical quadrature formule. They give rise to the *Gauss-Jacobi quadrature* :

Lemma 0.2. (Szegö [175], p. 47). Let

$$(0.2.2) \qquad \lambda_{j,n} = \frac{k_{n+1}}{k_n} \frac{-1}{p_{n+1}(x_{j,n})p_n'(x_{j,n})} = \frac{k_n}{k_{n-1}} \frac{1}{p_{n-1}(x_{j,n})p_n'(x_{j,n})}$$

then for every polynomial $\pi$ of degree less than or equal to $2n-1$

$$(0.2.3) \qquad \int_{-\infty}^{\infty} \pi(x)\,d\alpha(x) = \sum_{j=0}^{n} \lambda_{j,n}\pi(x_{j,n}) .$$

The numbers $\{\lambda_{j,n} : j = 1,\ldots,n\}$ are called *Christoffel numbers* and they are all positive.

One of the most important properties of orthogonal polynomials is the existence of a recurrence relation for three consecutive polynomials :

Lemma 0.3. (Szegö [175], p. 42; Freud [61], p. 60). Orthogonal polynomials always satisfy a three term recurrence relation

$$(0.2.4)\qquad xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x) \qquad n = 0,1,2,\ldots$$

with starting values $p_{-1}(x) = 0$ and $p_0(x) = 1$. The recurrence coefficients are given by

$$(0.2.5)\qquad \begin{cases} a_n = \dfrac{k_{n-1}}{k_n} > 0 & n = 1,2,\ldots \\[2ex] b_n = \displaystyle\int_{-\infty}^{\infty} xp_n^2(x)d\mu(x) \in \mathbb{R} & n = 0,1,2,\ldots\ . \end{cases}$$

If, on the other hand, $\{p_n(x) : n = 0,1,2,\ldots\}$ is a sequence of polynomials that satisfies a recurrence relation of the form (0.2.4) with $a_n > 0$ and $b_{n-1} \in \mathbb{R}$ $(n = 1,2,\ldots)$, then there exists a probability measure $\mu$ such that these polynomials are orthogonal with spectral measure $\mu$.

The second part of the previous lemma is often referred to as *Favard's theorem*. The recurrence relation for the monic polynomials becomes

$$(0.2.6)\qquad \hat{p}_{n+1}(x) = (x - b_n)\hat{p}_n(x) - a_n^2\hat{p}_{n-1}(x) \qquad n = 0,1,2,\ldots\ .$$

The recurrence relation always implies the existence of a spectral measure with respect to which the polynomials are orthogonal. This measure need not be unique. The measure will be unique if and only if the *Hamburger moment problem* associated with this measure has a unique solution. A sufficient condition has been given by Carleman, namely

$$(0.2.7)\qquad \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$$

(Shohat-Tamarkin [170], p. 59).

The recurrence relation and the Gauss-Jacobi quadrature have some consequences. Define the *associated polynomials* $\{p_n^{(1)}(x) : n = 0,1,2,\ldots\}$ by means of the recurrence formula

$$(0.2.8)\qquad xp_n^{(1)}(x) = a_{n+2}p_{n+1}^{(1)}(x) + b_{n+1}p_n^{(1)}(x) + a_{n+1}p_{n-1}^{(1)}(x)$$

with $p_{-1}^{(1)}(x) = 0$ and $p_0^{(1)}(x) = 1$, then the decomposition into partial fractions of the ratio $p_{n-1}^{(1)}(x)/p_n(x)$ is given by

$$(0.2.9) \qquad \frac{p_{n-1}^{(1)}(x)}{p_n(x)} = a_1 \frac{\hat{p}_{n-1}^{(1)}(x)}{\hat{p}_n(x)} = a_1 \sum_{j=1}^{n} \frac{\lambda_{j,n}}{x - x_{j,n}}$$

which means that the Christoffel numbers are the residues of the ratio $\hat{p}_{n-1}^{(1)}(x)/\hat{p}_n(x)$. Another important rational function is

$$(0.2.10) \qquad \frac{\hat{p}_{n-1}(x)}{\hat{p}_n(x)} = \sum_{j=1}^{n} \frac{a_{j,n}}{x - x_{j,n}}$$

with

$$(0.2.11) \qquad a_{j,n} = \frac{\hat{p}_{n-1}(x_{j,n})}{\hat{p}_n'(x_{j,n})} = \lambda_{j,n} p_{n-1}^2(x_{j,n}) > 0 .$$

If the recurrence relation (0.2.4) is known, then one can introduce the *Jacobi matrix* (of order n) for the orthogonal polynomials :

$$(0.2.12) \qquad J_n = \begin{bmatrix} b_0 & a_1 & 0 & 0 & \cdots & & \\ a_1 & b_1 & a_2 & 0 & \cdots & & \\ 0 & a_2 & b_2 & a_3 & \cdots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & & a_{n-2} & b_{n-2} & a_{n-1} \\ \vdots & \vdots & & & 0 & a_{n-1} & b_{n-1} \end{bmatrix}$$

The eigenvalues of this matrix are equal to the zeros of $p_n$; this follows immediately by expanding the determinant of $J_n - xI$ along the last row. A normalized eigenvector for the eigenvalue $x_{j,n}$ is given by

$$\sqrt{\lambda_{j,n}}\, (p_0(x_{j,n}), p_1(x_{j,n}), \ldots, p_{n-1}(x_{j,n}))$$

where $\{\lambda_{j,n} : j = 1,\ldots,n\}$ are the Christoffel numbers (0.2.2). The monic orthogonal polynomials therefore can be written as

$$(0.2.13) \qquad \hat{p}_n(x) = \det(xI - J_n) .$$

Another matrix representation for the orthogonal polynomials is given in terms of the moments (0.1.1) :

$$(0.2.14)\qquad p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}}\begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & & & & \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}$$

where $D_n$ is the determinant of the *Hankel matrix* $H_n$ with elements $\{(H_n)_{i,j} = m_{i+j} : i,j = 0,1,2,\ldots,n\}$.

We can obtain some more interesting formulas from the recurrence formula :

Lemma 0.4. (Szegö [175], p. 43). The following formulas always hold for orthogonal polynomials :

$$(0.2.15)\qquad \sum_{j=0}^{n} p_j(x)p_j(y) = \frac{k_n}{k_{n+1}}\ \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}$$

$$(0.2.16)\qquad \sum_{j=0}^{n} p_j^2(x) = \frac{k_n}{k_{n+1}}\ \{p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\}\ .$$

The first equality is called the *Christoffel-Darboux formula*, the second formula is a confluent form of the Christoffel-Darboux formula.

We finally mention the following minimal property :

Lemma 0.5. (Szegö [175], p. 39). Let $\Pi_n^+$ be the set of all polynomials with leading term $x^n$, then

$$(0.2.17)\qquad \frac{1}{k_n^2} = \inf_{q_n \in \Pi_n^+} \int_{-\infty}^{\infty} |q_n(x)|^2\, d\mu(x)$$

and the infimum is attained by the monic orthogonal polynomial $\hat{p}_n$ with spectral measure $\mu$.