Mathematical Methods in Engineering

EDITED BY
Glyn A. O. Davies

MATHEMATICAL METHODS IN ENGINEERING

Edited by
GLYN A. O. DAVIES
Department of Aeronautics
Imperial College of Science and Technology

A Wiley–Interscience Publication

JOHN WILEY & SONS
Chichester · New York · Brisbane · Toronto · Singapore

Library of Congress Cataloging in Publication Data:

Main entry under title:

Mathematical methods in engineering.
Guidebook 5 to Handbook of applicable mathematics.
'A Wiley–Interscience publication.'
Includes index.
1. Engineering mathematics. I. Davies, Glyn A. O.
II. Handbook of applicable mathematics.
TA330.M32 1984 620'.001'51 83-23250

ISBN 0 471 10331 4 (U.S.)

British Library Cataloguing in Publication Data

Mathematical methods in engineering.—(Handbook of applicable mathematics; guidebook 5)
1. Engineering mathematics
I. Davies, Glyn A. O. II. Series
510'.2462 TA330

ISBN 0 471 10331 4

Filmset by Eta Services (Typesetters) Ltd., Beccles, Suffolk.
Printed and bound in Great Britain.

Contributing Authors

S. Barnett, Professor of Applied Mathematics, School of Mathematical Sciences, University of Bradford, West Yorkshire, U.K.

M. J. M. Bernal, Reader in Numerical Analysis, Department of Mathematics, Imperial College of Science and Technology, London, U.K.

P. Bettess, Senior Lecturer in Department of Civil Engineering, University College of Swansea, U.K.

R. Bettess, Senior Scientific Officer, Hydraulics Research Station, Wallingford, Oxfordshire, U.K.

G. A. O. Davies, Head of Department of Aeronautics, Imperial College of Science and Technology, London, U.K.

J. M. R. Graham, Reader in Department of Aeronautics, Imperial College of Science and Technology, London, U.K.

R. Hillier, Lecturer in Department of Aeronautics, Imperial College of Science and Technology, London, U.K.

D. P. Isherwood, Lecturer in Department of Mechanical Engineering, Imperial College of Science and Technology, London, U.K.

V. G. Jenson, Senior Lecturer in Department of Chemical Engineering, University of Birmingham, U.K.

N. B. Jones, Reader in Control Engineering, School of Engineering and Applied Sciences, University of Sussex, Brighton, U.K.

J. D. Lewins, University Lecturer in Nuclear Engineering, Engineering, Laboratories, University of Cambridge, U.K.

J. O. Medwell, Lecturer in Department of Mechanical Engineering, University College of Swansea, U.K.

J. Munro, Professor of Civil Engineering Systems and Head of Department of Civil Engineering, Imperial College of Science and Technology, London, U.K.

E. G. Prater, Senior Research Engineer, Institute of Foundation Engineering and Soil Mechanics, Federal Institute of Technology, Zurich, Switzerland.

C. Taylor, Reader in Department of Civil Engineering, University College of Swansea, U.K.

W. L. Wood, Honorary Fellow, formerly Senior Lecturer in Department of Mathematics, University of Reading, U.K.

Editorial Note

Mathematical skills and concepts are increasingly being used in a great variety of activities. Applications of mathematics range from intricate research projects to practical problems in commerce and industry.

Yet many people who are engaged in this type of work have no academic training in mathematics and, perhaps at a late stage of their careers, have neither the inclination nor the time to embark upon a systematic study of mathematics.

To meet the needs of these users of mathematics we have produced a series of texts or *guidebooks* with uniform title. One of these is *Mathematical Methods in Engineering* (with similar books on Medicine, Economics and so on). The purpose of this volume is to describe how mathematics is used as a tool in Engineering and to illustrate it with examples from this field. The guidebooks do not, as a rule, contain expositions of mathematics *per se*. A reader who wishes to learn about a particular mathematical topic, or consolidate his knowledge, is invited to consult the *core volumes* of the *Handbook of Applicable Mathematics*; they bear the titles

I Algebra
II Probability
III Numerical Methods
IV Analysis
V Combinatorics and Geometry (Parts A and B)
VI Statistics (Parts A and B)

The core volumes are specifically designed to elaborate on the mathematical concepts presented in the guidebooks. The aim is to provide information readily, to help towards an understanding of mathematical ideas and to enable the reader to master techniques needed in applications. Thus the guidebooks are furnished with references to the core volumes. It is essential to have an efficient reference system at our disposal. This system is explained fully in the Introduction to each core volume; we repeat here the following points. The core volumes are denoted by the Roman numerals mentioned above. Each mathematical item belongs to one of the six categories, namely:

(i) Definitions
(ii) Theorems, propositions, lemmas and corollaries
(iii) Equations and other displayed formulae
(iv) Examples
(v) Figures
(vi) Tables

A typical item is designated by a Roman numeral followed by three Arabic numerals *a*, *b*, *c*, where *a* refers to the chapter, *b* to the section and *c* to the individual item enumerated consecutively in each category. For example, 'IV Theorem 6.2.3' is the third theorem of section 2 in Chapter 6 of the core volume IV. We refer to equation 6.2.3 of Volume IV as IV (6.2.3), and section 6.2 of Volume IV as IV §6.2.

We trust that these guidebooks will contribute to a deeper appreciation of the mathematical methods available for elucidating and solving problems in all the various disciplines covered by the series.

Preface and Acknowledgements

As explained in the introductory chapter of this book, the following chapters are a selection of those branches of engineering which particularly rely on the application of mathematical techniques. There is no coverage of the more discursive qualitative aspects of engineering, nor is there a chapter on electrical engineering or information technology which are separately identifiable and comprehensive disciplines which deserve guidebooks to themselves.

The chapters in this book are but brief summaries of various fields in engineering, just sufficiently long to describe the nature of the subject and to put into context the modelling which is necessary before a mathematical problem emerges. The mathematical solutions are not dwelt upon, but reference is simply made to the Handbook as described in the Editorial Notes. In this fashion it is hoped that engineers will find illumination where they look for it and, further, may discover helpful techniques which they had not looked for. There is often a singular lack of cross-fertilization between the various branches of engineering, let alone between engineers and other scientists. An equally serious communications gap also exists between practising engineers and applied mathematicians, and we hope that the Guidebook and Handbooks will contribute to the closure of this gap.

One consequence of the barriers that inevitably arise when we drift into separate areas of specialization is that of notation. The common symbols in use have been kept uniform wherever possible, but there is a limit—if only that imposed by the twenty-six characters in the Roman alphabet. In cases where traditions have become firmly entrenched, I have not striven to alter them simply for uniformity. For example, $\sqrt{-1}$ is denoted by 'i' in the core volumes and in all of these chapters with the exception of control engineering which adopts the time-honoured electrical practice of reserving that symbol for *current*.

It is not possible for a single engineer to present a precis of fifteen separate branches of engineering, and my first thanks must go to the authors of each chapter for their contributions and their willingness to shorten the content to a bare minimum. In many cases the engineering treatment is cursory and the reader is assumed to already have the knowledge or is directed to engineering

references at the conclusion of each chapter. For the brevity, and for often wielding an editorial knife, I take complete responsibility.

I would especially like to thank my friend Roland Lewis, of the University College of Swansea, who undertook the task of compiling this guidebook in the first place, but who had eventually to bow to other pressures. He persuaded the authors of Chapters 3, 5, 6, 7, 8 and 9 to undertake their tasks, and for this gentle persuasion I am extremely grateful.

To Carol van der Ploeg I am indebted for inserting all the core volume references and for bravely reading all chapters as a lay person unbiased by specialized engineering prejudices. I am also grateful to Walter Ledermann for a similar role, but more important for conveying much enthusiasm for the total concept of guidebooks and handbooks.

Finally, of course, I am grateful to John Wiley for their forbearance over a long gestation period and their trust in believing that sixteen separate authors could be united in a common aim.

G. A. O. Davies
August 1983

Contents

Mathematical Methods in Engineering
Edited by G. A. O. Davies

1

Introduction

1.1 Objectives

The six volumes of the *Handbook of Applicable Mathematics* are a comprehensive account of the many branches of applicable mathematics. The word *applicable*—rather than 'applied'—has been chosen to emphasize that these core volumes are directed at *practitioners* who use mathematics to explain and solve problems, rather than as an abstract discipline. The terms 'pure' and 'applied' mathematics have been used in the past but such labels can rapidly become 'old hat'. The computer, for example, has dated many forms of analysis thought to be applied, whilst some pure abstract algebra has now been found to be applicable.

Of all practitioners who use mathematics as a working tool, engineers must surely form the largest group, and it is appropriate that a guidebook be devoted to engineering. Unfortunately the field of engineering today is so vast that two problems of scale present themselves immediately.

First, the scope of applicable mathematics ranges from the elementary to the sophisticated, and it is hoped that the six core volumes of the Handbook encompass something like the range currently used by modern engineers. This hope is probably a pious one. Many engineers are having to pioneer new mathematical techniques to meet the demand of high technology—particularly in the fields of numerical analysis and systems analysis, for example. These new techniques may be justified, developed and proven using physical arguments, and it may be some time before they are given mathematical rigor and catalogued in a cohesive way with familiar mathematics. Such is the nature of progress under pressure. The reader of this guidebook will therefore find techniques discussed which are not as yet incorporated into the core volumes, but references to other textbooks will be given where necessary.

Second, the field of engineering is itself so large that it might be a daunting task to include it all in a single guidebook. However, this text contains only those branches and aspects of engineering which specifically use mathematics as a routine tool. This leaves a smaller—and rather curious looking—field to cover and also explains why some non-mathematical areas in engineering are completely absent. There is also the problem of classification, of course. Should a guidebook have *vocational* chapters like mechanical engineering, civil engineering, aeronautical, marine, and so on? It was decided against this in view of the various topics that should be covered. For example, the subject of Fluid Mechanics alone occurs in all of the divisions just mentioned, and many more. It would be absurd to suggest that mechanical and civil engineers use fluid mechanics in quite different ways—far better to present the topic and its associated mathematics in a general way if possible. A glance at the chapter headings will reveal that most of this book has an interdisciplinary flavour. Fluid mechanics, boundary layers, compressible flow, free-surface flow, heat transfer, structural mechanics and systems engineering are all subjects which stride across the various provinces of engineering. Hydrology and porous media are of interest mostly—but not exclusively—to civil engineers, and control engineering is not exclusively the domain of electrical engineers. The chapters on polymer engineering and nuclear power are, however, fairly specific. Notable omissions are the fields of electrical engineering and electronics, computers and information science, as it was felt that these readily identifiable subject areas deserved their own guidebooks.

It has not been the intention to produce an engineering textbook, but instead to limit discussion to an explanation of the problems which give rise to a mathematical need. Therefore chapters in general contain the physics of the problem or whatever is necessary to explain the modelling and put it into its engineering context. Whenever modelling is based on lengthy physical or logical arguments, the reader is referred to engineering references at the conclusion of that particular chapter. Cross-referencing to other chapters is also used, of course. Having described the engineering formulation of the problem and put the modelling into some form of quantitative perspective, the mathematical techniques are usually briefly described and often listed in some order of merit, like expense, accuracy or convenience. However, no mathematical details are entered into, but references are made to the core volumes, as explained in the introductory editorial note.

1.2 Modelling

The aim of mathematical modelling is to enable the engineer to *predict* without having actually to construct or reproduce. Although there is no substitute for actually building an aircraft, nuclear reactor, dam, chemical plant, transport system, and so on, the modern engineer has to get it right first time. If the economics or safety are wrong then such is the complexity of projects like those mentioned that only minor modifications can usually be tolerated before the financial viability of a project becomes at risk. Gone are the days when one could build a quick prototype and change it rapidly and often until it performed adequately. The ability to predict does not of course necessarily mean the exclusive use of mathematical models. Simulation may take the form of physical models or analogues, but even here the use of mathematics is likely to be important in validating the use of models which must differ from the real thing—otherwise they would have no advantage. If a reliable mathematical model or simulation can be found and tested then there is a valuable bonus in addition to the immediate cost saving. Other alternatives can be examined. The simulation of complex systems as an aid to *optimum design* is a rapidly growing industry not only in engineering but also in the less deterministic fields of socioeconomic and fiscal models, in project planning, and so on.

The formulation of a mathematical model involves three interrelated phases:

(i) Define the important variables in the problem, their expected influence and the relations between them.
(ii) Solve the relationship between the known inputs and the desired consequences.
(iii) Test the solutions for reliability and range of application.

The first stage is the true modelling stage and mostly consists in recognizing the nature of the problem so that existing theory, programs, etc., can be found and applied. Sometimes the problem may be entirely novel, of course, but the great majority of engineering problems have been met before, and any innovative analysis is usually an extension or improvement on established methods. It is this first stage which occupies most of the chapters in this book. Another glance at the chapter headings reveals that much of the material comes under the heading of 'applied mechanics' in the general fields of fluids, gases or solids. In principle many of the laws of mechanics governing the behaviour of fluids and solids are precise and exact, such as the equations of motion, conservation of mass, laws of thermodynamics, etc. These laws themselves are not usually sufficient and equations of state are also necessary. These latter equations usually involve some empirical relationship based on experience which relates *changes of state*, stress to strain rate, pressure to volume change and almost anything to temperature. Such laws are usually approximate and often linearized since all mathematical problems are simpler if they are linear. Outside the field of applied mechanics, some engineering problems may have to be formulated in an entirely empirical way—although analogies with applied mechanics are sometimes possible and helpful. Such empirical views of real life enter into several of the chapters in this book.

The second phase involves solving the equations after formulating them. Here further approximations have frequently to be introduced, even to the 'exact' governing equations. In practice the exact equations are usually non-linear and may have to be solved as such to preserve the expected behaviour of the physics. Often, however, it is possible to obtain approximate linearized solutions, and the engineer should know whether the exact equations are theoretically tractable, albeit expensively, say by using a tailor-made computer program (or by calling in a consultant). The approximations may be justified in mathematical terms, by using physical reasoning, or by both.

The final phase, having formulated a possible solution procedure, is to test it. The practising engineer will as often as not forego this stage if the problem and solution are familiar territory. If the modelling—or perhaps the range of validity of the solution—is doubtful, then assessment may be difficult. Comparison between a novel solution procedure, or modelling technique, and an alternative is one way of testing, but frequently the only way is to compare with the real thing if the experimental data are available. Perfect agreement can never be secured, of course, and mathematical techniques are often used in their own right to correlate observed results with the theoretical predictions.

The dangers and pitfalls in constructing reliable robust mathematical models, some useful rules and some cautionary tales can be found in Volume IV, Chapter 14.

1.3 Analytical resources

The mathematical techniques and descriptions referred to throughout this guidebook usually fall into one of two categories—*analytical* and *numerical*. Many engineering problems lead to simple governing equations which possess a closed-form analytical solution. The ubiquitous boundary-value problem is an example in which a 'field' behaviour is described by a differential equation and there are accompanying boundary (surface) conditions to be satisfied. To take a simplistic view, this class of problems is often solvable in a closed form in one dimension, and possibly also in two dimensions, but rarely in three dimensions. If the difficulties arise in the governing differential equations themselves, then the number of dimensions is often not the crucial factor. If the difficulties arise in the boundary conditions then they are often entirely geometric in nature, and an impasse in satisfying boundary conditions is no less important than in satisfying the differential equation. In many important engineering problems in the past fifty years or so, it has been necessary to accept crude analytical solutions solely because the geometry of the problem presented formidable difficulties.

Fortunately the advent of the computer has released us from the bondage of geometrical stumbling blocks. A numerical description and solution is nowadays entirely adequate and a closed-form approximate solution is no longer necessarily a desirable aim. The use of numerical methods aimed at awkwardly shaped regions is likewise true of awkward differential equations; again an analytical solution may actually be inferior, particularly if it has demanded questionable approximations. Unfortunately in the profession of engineering there is sometimes still a residual resistance to computational analysis, as if the computer were a crude blunt instrument, powerful but lacking finesse. This attitude dies hard even though the pre-computer solutions involved special functions which are just as expensive to compute as a numerical solution.

However, analytical solutions of even limited scope are extremely valuable in forming guidelines for a numerical solution in complex problems. They may tell us the order of approximations we may be able to tolerate and the likely errors. If the analysis reveals the presence of necessary singularities or discontinuities we would be foolish to ignore these features in a numerical description. The presence also of boundary layers and their extent can be predicted, which is of enormous help. The same is true of shock waves in fluids, gases, solids and traffic flow.

The engineer should therefore see classical analysis and numerical analysis as different but complementary. Superiority of one over the other is not a mark of esteem; both have their own logic, language and beauty. Also, nothing is static—not even attitudes. Our view of numerical analysis and modelling must undergo as many mutations as did the change from pre- to post-computer analysis. The advent of cheap reliable computer hardware—once the province of the large organization—has revolutionized analysis in non-linear problems, and the accessibility to vector and array processors is also colouring the way engineers are tackling complex modelling problems. The use of expert knowledge-based systems has started to make inroads into some computational packages.