INTEGRAL AND DIFFERENTIAL CALCULUS:

an intuitive approach

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PREFACE

This book is designed as a one-semester course for students of varying backgrounds. Those who wish to take this course with a reasonable chance to succeed should be familiar with elementary algebraic techniques as provided by a good high school preparation or a one-semester course customarily called *College Algebra*. Chapter I introduces as much analytic geometry as is needed in the sequel. Certain parts, e.g., the section on the quadratic equation (Sect. 9) can be omitted if this is warranted by the students' background. Appendix III is devoted to an introduction to trigonometry. This Appendix III should be taken up between Chapter II and Chapter III, if needed.

This book is intended for students who take the second semester of a one-year terminal course in mathematics as well as high school teachers of a specific category who attend refresher courses or summer institutes. This "specific category" embraces high school teachers who never had a formal course in calculus, or had a standard calculus course and either did not derive much profit from it or quickly forgot most of it, and are now called upon to teach the calculus in high school.

Lastly, this text is designed for a calculus course as offered by many high schools as part of an accelerated sequence for students of exceptional ability.

Once upon a time there was a young lady who enrolled in one of the universities in the Northwest. Because of her poor high school record in algebra, she was put into a refresher course in this subject. She was struggling along, just barely keeping from drowning, until suddenly, halfway through the course, her features lit up and her eyes sparkled with astonished enlightenment as she exclaimed: "Why didn't they tell me before that those letters stand for numbers?"

When writing this book, I was thinking of this young lady's plight and made an effort to stay in as close contact with numbers as possible. I frequently resorted to experimental methods at the expense of mathematical rigor to provide for a practical understanding of the limit processes that are basic for a good comprehension of the calculus.

The calculus is developed here in a fashion as it could have happened—and to some extent did happen—historically. Physical and geometric applications are interwoven with the text to provide sufficient motivation for the introduction of new mathematical concepts.

This treatment is not cluttered up with technical details and tricks. The reader will not learn to differentiate

$$y = \frac{\sqrt{\sin^5 (\sqrt{x} + e^{\cos^2 x})^3 - \sqrt{27 + \log |\tan \sqrt{\pi/2 - x^2}|}}}{\log \left| \log \dfrac{x^2}{a^x} \right| + e^{\sqrt{\cot(x + \sqrt{1+x^2})^3 + e^{1000\sqrt{3x^x}}}}}$$

nor to integrate

$$\int (e^{x^2} + \sin \sqrt{x})^{15} \left(e^{2x^2} + \sin^2 \sqrt{x} + 2e^{x^2} \sin \sqrt{x} + \frac{1}{\sqrt{3}} \right)^{36} \cdot \left(2xe^{x^2} + \frac{1}{2\sqrt{x}} \cos \sqrt{x} \right) dx.$$

Instead, it is the aim of this book to provide for a thorough understanding of the *limit of a sum process*, the *limit of the difference quotient*, and the *possible practical applications of the calculus.*

The sections on the *chain rule* (differentiation of a function of a function, Chap. III, Sect. 4) and the *inverted chain rule* (integration by substitution, Chap. III, Sect. 6), which deal with some more formal aspects of the calculus, are entirely independent of the main text and may or may not be included in the course. A bare minimum of formulas is developed, but great emphasis is placed on such items as the trapezoidal rule and Simpson's rule, which promote a very practical understanding of the limit process on which the definition of the definite integral is based.

Chapter II deals with integration. The concept of an area is developed in a semirigorous manner. The main purpose of the introductory sections, 1 and 2, is to make students aware of the fact that *area* is not something that "*is*" but, rather, an artificial concept which has to be defined in a manner that will meet our intuitive demands. A thorough discussion of the definite integral precedes the introduction of the derivative in this book because it was felt that it is easier to convince students of the necessity for measuring areas, rather than slopes. Another reason is a very practical one: if the derivative is discussed first and then the indefinite integral is introduced as the antiderivative and, subsequently, the definite integral is defined in terms of the indefinite integral, students will hardly pay much attention to what might be said later about the definite integral as a limit of a sum, being already in possession of a very simple routine for evaluating it. This would seem very unfortunate indeed, as the concept of the limit of a sum is really the key to most important applications of the definite integral.

At the end of Chapter II, the integrand is characterized as the rate of change of the definite integral with a variable upper limit. Thus a continuous transition from Chapter II to Chapter III is provided at the expense of the fundamental theorem of the calculus which loses its character as a theorem under such treatment.

Chapter III deals with the derivative and its geometric and physical interpretations. The basic theme concerning the discrepancy between physical reality and its mathematical description, which was already introduced in Chapter I, is now carried to a crescendo in the sections on *motion* and *freely falling bodies*. The ideas put forth here are those of logical positivism, presented in a simplified and personalized form.

Chapter IV finally deals with volumes as far as this is practical without having to introduce any more essentially new ideas beyond those that have been already developed in Chapters II and III.

There are certain sections and portions of sections that can be omitted without seriously jeopardizing the continuity of the development. These sections are clearly set apart from the main text by solid triangles which are set at the beginning and the end of each such portion. This does not mean, of course, that these sections should be omitted. On the contrary, they ought to be studied if this is feasible under the given circumstances, because most of these specially designated sections serve to round out the treatment or open up new avenues of thought that should stimulate the better students to deeper thinking and inspire them to further studies in mathematics.

Many problems are listed at the end of every section. The answers to most of the even-numbered problems are supplied in the back of the book. Some of these problems complement the text and serve to help the reader familiarize himself with the new notions and techniques that are introduced. Some problems supplement the text in exploring certain aspects of the material in greater depth than the main text. Still other problems lead the reader away from the text in a pursuit of sidelines which are only loosely connected with the material that is studied. There are many more supplementary problems supplied at the end of every chapter.

It is my belief that students have to be in possession of facts before they can make any attempt to fit them into a beautifully constructed deductive system. It was my aim to present in this book the facts, or some facts anyway, but I tried to give the reader occasionally a fleeting glimpse of the deductive system by leading him through some simple deductive arguments.

I hope that my book will promote interest in mathematics among noncommitted students as well as assist teachers in giving stimulating presentations of the calculus on the elementary level.

Moscow, Idaho
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H. S.

CONTENTS

CHAPTER I

FUNCTIONS

1. REPRESENTATION OF FUNCTIONS BY TABLES

The so-called exact sciences engage in a quantitative analysis of nature. There are basically two types of quantities that play a significant role in scientific systems: those that change their value, the so-called *variables*, and those that do not change their value, the so-called *constants*. Our attention in this treatment will be devoted primarily to a study of variables. In order to reach some understanding of this concept, let us discuss a few examples.

It is an experimentally established fact that the boiling point of water, i.e., the temperature at which water starts boiling, depends on the atmospheric pressure under which the water is brought to a boil. Every traveler knows that it takes 6 minutes to prepare a soft boiled egg in Bozeman, Montana, while it takes only $2\frac{1}{2}$ minutes to accomplish the same result in Redding, California. The realization of this phenomenon made men envision the pressure cooker which is in our days a gruesome reality that reduces the great variety of potato dishes to something which is hardly distinguishable from mashed potatoes. We are not about to introduce the American menu as our first example of a variable. So, let us return to the point at which we embarked on this culinary discussion: the boiling point of water. It can be experimentally established, as we mentioned above, that the boiling point of water changes with the atmospheric pressure. Specifically, the entries in the left column of Table I.1 indicate the different values of the atmospheric pressure under which the experiment was carried out and the entries in the right column give the corresponding temperatures at which boiling occurs.

We recognize in this example two physical quantities as variables, i.e., quantities that change their value: the atmospheric pressure and the boiling temperature of water. We observe at the same time that these two variables play a clearly distinct role because if we choose freely any pressure we please, the boiling point of the water is completely determined by the choice we make. In other words: even though the boiling temperature of water is

Table I.1

Under an atmospheric pressure in mm mercury	Water starts boiling at a temperature of degrees Celsius (centigrades)
9.209	10
17.53	20
31.824	30
55.32	40
92.51	50
149.38	60
233.7	70
355.1	80
525.8	90
760	100

a variable, its value is determined by the value of the variable that represents the pressure. We express this situation mathematically by stating that the boiling point of water is a *function* of the atmospheric pressure. The variable to which we can assign values freely (to some extent) we call the *independent variable* (here the atmospheric pressure). The other variable, the value of which is determined by the value of the independent variable, we call for obvious reasons the *dependent variable* (here the boiling temperature).

Table I.2

For an elevation above sealevel in m	The following atmospheric pressure in mm mercury at 0°C is found
0	760
2947	525.8
6087	355.1
9433	233.7
13012	149.38

Of course, the concept of dependent and independent variable is a rather relative one. Thus, the atmospheric pressure appears to be a dependent variable if we venture to measure it at different elevations. Specifically, we obtain the results in Table I.2. Here the elevation plays the role of an independent variable while the atmospheric pressure emerges as the dependent variable: the atmospheric pressure is a *function* of the elevation.

Combining Tables I.1 and I.2, we see that we may consider the boiling point of water as a function of the elevation, as given in Table I.3, and eliminate the pressure entirely.

Table I.3

Elevation in m	Boiling point of water in °C
0	100
2947	90
6087	80
9433	70
13012	60

Of course, we could go on now to consider the elevation as a function of the time, supposing we are sitting in a rocket that is shot straight upward. The application of such an analysis is quite obvious if we get the idea that we must have a boiled egg 6 minutes after launching time.

Table I.4

Pressure in mm mercury	Volume in cm^3
50	152
100	76
200	38
300	25.3
500	15.2
600	12.7
700	10.8
800	9.5
900	8.4
1000	7.6

Other examples of dependent and independent variables are easily found. Let us consider a cylinder that contains some gas and is closed tightly by a piston (see Fig. I.1). Clearly, if we increase the pressure on the piston, then the volume of the enclosed gas will become smaller, and vice versa.

Using crude experimental methods, we will find a relationship for some gas as revealed in Table I.4.

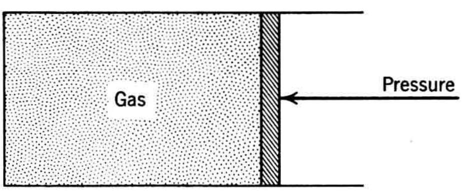

Fig. I.1

Here the volume of the enclosed gas appears as a function of the pressure which is applied to the piston. If we are interested in knowing how much pressure must be applied in order to compress the gas to a given volume, we have only to interchange the two columns in Table I.4 and consider the pressure as a function of the volume. Again, we can see that whether a variable is dependent or independent depends largely on the point of view.

2. REPRESENTATION OF FUNCTIONS BY GRAPHS

A common means of representing the relation between two variable quantities, if such a relation exists, is the graph. Let us return to Table I.1 for the purpose of an introductory discussion. This table contains ten pairs of values, one of which represents a certain atmospheric pressure, the other one the corresponding boiling temperature of water: (9.209, 10), (17.53, 20), (31.824, 30), (55.32, 40), (92.51, 50), (149.38, 60), (233.7, 70), (355.1, 80), (525.8, 90) and (760, 100). Our aim is to give a geometric representation of this relationship.

However, before we can endeavor to present pairs of numbers geometrically, we first have to settle a much simpler problem, namely: how do we represent a single numerical value geometrically? The answer is simply given by the ruler with an engraved scale or the thermometer. We consider a line (see Fig. I.2) and choose one point on this line quite arbitrarily. We call this point 0 and let it represent the number 0. Next we choose one more point which shall lie to the right of 0, but can otherwise be chosen quite arbitrarily, and call it 1. This point shall represent the number 1. The distance between the point 0 and the point 1 we call *unit distance*. Clearly, the number 2 will now be represented by a point one unit distance to the right of 1, etc. It is really quite clear how we have to proceed to locate the points which are supposed to represent all the positive integers.

Negative integers, as suggested by the scale of the thermometer, will be represented by points to the left of 0. (Historically, the concept of the *line of numbers* preceded the scale of the thermometer; however, although few students are acquainted with the line of numbers, it can be assumed that everybody has seen a thermometer at least once.) Specifically, the representative of the number -1 will be a point one unit to the left of 0, the representative of -2 one unit to the left of -1, etc.

Thus we have attained a geometric interpretation of all positive and negative integers. How do we now represent fractions? Clearly, the representative of $\frac{1}{2}$ will be a point halfway between 0 and 1, the representative

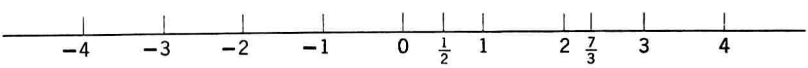

Fig. I.2

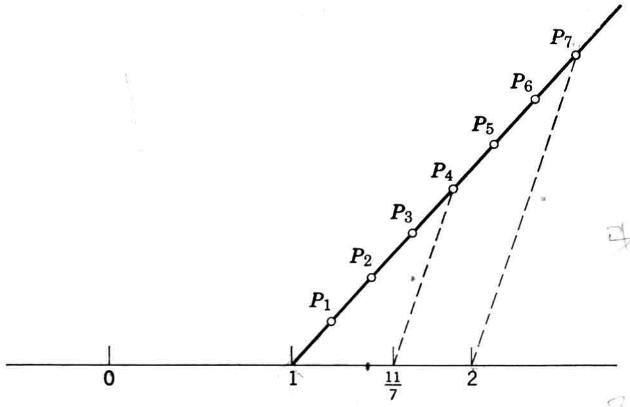

Fig. I.3

of $\frac{7}{3}$ will be a point between 2 and 3 such that its distance from 2 is one half of its distance from 3, and, finally, the point representing $\frac{11}{7}$ will be located between 1 and 2 so that its distance from 1 is $\frac{4}{7}$ of the unit distance. These few examples clearly indicate how we have to proceed in finding the representatives of *rational numbers** (fractions).

It is quite easy to construct those points which are supposed to represent fractions by the following device. Let us again consider the number $\frac{11}{7}$. In the following argument we refer to Fig. I.3. We draw a line through the point representing 1 at an acute angle with the line of numbers. Then we proceed to mark 7 points on this line at equal distances, starting with the point 1 (*equidistant* points). We call these points $P_1, P_2, P_3, \cdots, P_7$. We join the last point P_7 and the point representing the number 2 with a straight line and then draw a line parallel to the line through 2 and P_7 through the point P_4. This line will intersect the line of numbers in the point which represents the number $\frac{11}{7}$, i.e., the point $\frac{4}{7}$ of a unit to the right of 1. This can be seen quite easily by considering the two similar triangles $(1, 2, P_7)$ and $(1, \frac{11}{7}, P_4)$.

The problem of locating points that represent *irrational numbers*† (numbers which are not fractions) is not so simple. While it is quite easy to construct the representative of $\sqrt{2}$ (see Fig. I.4), it is not so clear how and if one can construct the representative of $\sqrt[3]{7}$ or, to make matters

* a is a rational number if, and only if, it can be represented in the form $a = \frac{m}{n}$, where m and n are positive or negative integers. $(n \neq 0)$.

† A number b is irrational if, and only if, it is *not* possible to write it in the form $b = \frac{m}{n}$ where m, n are positive or negative integers. $(n \neq 0)$.