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DISPERSION RELATIONS AND THE ABSTRACT APPROACH TO FIELD THEORY

edited by
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PREFACE

It often happens in physics that a field grows with such rapidity that it becomes impossible to collect and organize the developments into a book before they are superseded by newer results. The present volume is an attempt to fill the need for an introduction to such a field: that of the axiomatic approach to field theory and its application to proving dispersion relations. The basic papers as well as the more recent advances have been included here in an attempt to present a coherent picture of the development of a field which at first sight seems chaotic. It is hoped that the reader will be able to find here the fundamental material to enable him to acquire a sufficiently broad knowledge of this new approach to elementary particle physics to read the new material which is now appearing in almost every issue of the journals.

It is, moreover, often easier to understand a new theory if one reads the original papers written when there was considerable confusion as to which path to follow in arriving at the correct formulation. After the theory reaches a more advanced stage of development, the literature tends to become less detailed in the treatment of the foundations and hence difficult to follow in the development of the physical content even though the logic becomes more powerful. It is hoped, therefore, that this volume—apart from any historical value it may have—will serve to illustrate the difficulties which exist during the formation of a new physical theory, and that it will point up more sharply the problems and complexities involved in the growth of physics.

Field Theory in recent years has tended to follow two independent lines in the approach to Elementary Particle Physics.

One approach has been to treat the strong interactions as, though pathological, fundamentally akin to the weak interactions. Quantum Electrodynamics, with its perturbation expansions and renormalization procedure, has been the model for further developments. The hope of this approach is that new physical ideas, developing from the mass of data being delivered by the huge new accelerators with their bubble chambers and related apparatus, will enable one to modify the existing field theories, and thus ultimately lead to the successful description of elementary particles. However, although Quantum Electrodynamics is able to attain a fantastic degree of accuracy in describing experiments involving weak interactions, the strong interactions have thus far failed to yield to this type of analysis.

The other, newer approach to the interpretation of the elementary particles utilizes a few general principles drawn from the experience with previous field theories and attempts a systematic study of the mathematical implications of these assumptions. The view here is that there is perhaps already enough experimental data to construct a field theoretic description if one were clever enough. This is the Abstract Approach to Field Theory, or Asymptotic Mechanics as it is sometimes called. Here, one attempts to avoid all the divergencies which arise from splitting a Hamiltonian into an unperturbed part plus a perturbation. In fact, no Hamiltonian is introduced at all. In addition, all fields introduced are already renormalized, and all particles are "dressed." No infinities should arise in this theory. This is a pure S-matrix type of theory, where one does not consider the details of the interaction at all, but merely the consequences of certain restrictions on the transformation from initial to final states.

This volume, contains the original papers which developed this latter approach. In particular, the axiomatic approaches of Lehmann, Symanzik, and Zimmermann, and A. S. Wightman have been included here since they are considered to be the formulations which are most fruitful.

It is important to keep in mind, however, that as yet no one has been able to calculate any physical processes with any of these formalisms. Only general theorems about field theory have been proved and it is not yet known whether or not these sets of axiomatics are empty. The principal use of these approaches has been in the proofs of the dispersion relations, i.e., in providing spectral representations and establishing the analyticity properties of matrix elements of the S-matrix. In this way, statements can be made about the scattering amplitudes for certain processes. The articles included in this volume besides developing the axiomatic formalism provide an excellent introduction to its principle application, the proofs of the dispersion relations.

The first paper in this volume, by H. Lehmann, introduces the general methods of proof used in Abstract Field Theory. With the technique of spectral representations developed here, Lehmann is able to prove that the propagators of the interacting fields are at least as singular as those of the free fields. The only assumptions needed to prove this surprising theorem were, aside from relativistic invariance, the completeness of the set of positive energy states and the absence of negative energy states. This result meant that there is no hope of removing the singularities in the calculation of observable quantities by writing down the correct interacting fields, and hence, no hope of eliminating the difficulties in perturbation expansions.

The two papers which follow by Lehmann, Symanzik and Zimmermann comprise one of the first completely abstract formulations of field theory. Here one attempts to circumvent all the difficulties involved in the writing down of interacting fields by assuming a completely renormalized, dressed particle with a complete set of states which form the elements of a Hilbert space and is described by an ordinary field operator, $A(x)$. This operator must be Lorentz invariant and must satisfy two important conditions. First is the so-called micro-causality condition which requires that the commutator, $[A(x), A(y)]$ must vanish on spacelike surfaces, that is, when $(x-y)^2>0$. This means that two points cannot affect each other with an interaction which propagates faster than the speed of light no matter how close the points may be. This requirement insures that the resulting field theory will be causal since the field operators which determine the observables of the theory must themselves be causal.

The second condition introduces the physics into the mathematical framework. This is the asymptotic condition which, qualitatively, says that earlier in the infinite past and later in the infinite future, the particles involved in a scattering process do not interact. That is, at $t=-\infty$ and $t=+\infty$ the fields, $A(x)$, are free fields. The precise mathematical formulation of this condition is far from simple, and this problem has been the basis for extensive discussion in the literature.

These postulates form the foundation for a theory which represents the first real extension of the S-matrix theory proposed by Heisenberg in 1943. With these axioms it is possible to derive a reduction formula which enables one to write the S-matrix entirely in terms of vacuum expectation values of time-ordered products of the field operators (the τ functions). Furthermore, a relation is found with which one can write down one of these τ functions in terms of all the other τ functions of the theory. Thus, an infinite set of equations which determine completely the τ functions is found. If these equations could be solved, a test of the formalism could be found. Unfortunately, this set of equations remains unsolved at the present time and it is entirely an open question whether they are merely an identity or whether they contain physical restrictions on the theory.

The paper following, by A. S. Wightman, outlines the "Wightman

Program" in Abstract Field Theory, the most ambitious attempt in axiomatics to date. In an extremely precise mathematical manner the postulates which were outlined above are formulated. The vacuum expectation values of products of field operators (the Wightman functions) are expressed by the powerful apparatus of distribution theory so as to avoid the difficulties associated with singular functions. The field operators, themselves, are expressed by the classifications of representations of the Lorentz Group. The result proved in this paper is that a local relativistic field theory is completely characterized by the set of Wightman functions. The equation for the τ functions and the expression for the S-matrix found in the previous papers by Lehmann, Symanzik, and Zimmermann are clearly a special case of this more general result.

One of the most remarkable theorems in axiomatics is proved in the next paper, by R. Haag. He shows here that any field related at one time to a free field by a unitary transformation must be equivalent to a free field throughout the space-time continuum. This does not preclude the existence of an S-matrix theory, but forces one to be extremely subtle in the definitions in this type of theory. The paper by Hall and Wightman puts much of the work in the previous reference into a rigorous mathematical framework.

In all the preceding work the bound states of the particles have been neglected. The paper by Zimmermann discusses the modifications necessary for these considerations.

The next group of papers is concerned with the application of this abstract formulation to the problem of proving dispersion relations. The axiomatic method permits one to prove certain spectral representations of matrix elements of the S-matrix. These representations, called dispersion relations, describe, of course, the amplitudes of arbitrary transitions in scattering processes. The reduction formula given in the second Lehmann, Symanzik, and Zimmermann paper can be used to write essentially these matrix elements in terms of the commutator of two field operators. This method transcribes automatically all of the axioms from the abstract formulations directly into properties of the scattering amplitudes. The fact that these commutators vanish outside the light cone enables one to continue analytically the amplitudes into the complex plane. This extension is not simple and must be done in a very subtle manner. Bremmermann, Oehme, and Taylor have been able to develop proofs founded on the theory of many complex variables for dispersion relations obtained in this way.

R. Jost and H. Lehmann earlier found a spectral representation for the special case of equal mass scattering, which represented an important advance in the method of proof of dispersion relations. A similar representation was found for the more general case by F. J. Dyson, thus permitting the proofs of dispersion relations to be done by more familiar mathematical methods. The paper following by Lehmann makes use of the Dyson representation for proving analytic continuation and also gives a proof of the convergence of the Legendre polynomial expansion of the imaginary part of the amplitudes when the range of the argument is unphysical. This enables one to show that the imaginary parts of the amplitudes retain their meaning in the extension into the non-physical region. The three papers mentioned before along with this paper by Lehmann constitute an excellent introduction to all essential steps for the proofs of dispersion relations.

Beyond the problem of proving the dispersion relations lies the question of how to predict experimental phenomena from them. The method of extracting this information is based on the presence of singularities in the scattering amplitudes, since these poles govern the behavior of the amplitudes in their neighborhood. The search for the location and classification of these poles is known as polology.

Perturbation theory has been extremely useful in polology because poles that have not been derived on a more rigorous basis are suggested by Feynman diagrams of the scattering process. A simple pole, for example, will occur in a scattering amplitude when the conservation laws permit a single particle intermediate state. A branch point occurs at the point where production amplitudes compete with scattering amplitudes and real particles can occur in the intermedate states. The case of scattering with fixed momentum transfer has been exhaustively studied, but only recently has the more general case become amenable to any kind of analysis and this, due to the spectacular success of a conjecture by S. Mandelstam. The representation for arbitrary energy-momentum transfer which Mandelstam wrote down enables one to extend both complex variables simultaneously into the complex plane. The calculations based on the Mandelstam representation, done with the perturbation series approach, enable one to write a complete dynamical description of strong interaction scattering.

The paper by R. J. Eden discusses the problem of finding a proof of the Mandelstam representation. Eden has recently been investigating the proof of the conjecture within the framework of perturbation theory and encountered certain difficulties in the sixth-order diagram. In any case, the general proof of this representation is extremely difficult and remains to be given.

As an illustration of the scattering theorems that may be proved using dispersion relations, the paper by I. Pomeranchuk contains the proof of the remarkable theorem that in the high energy limit if the π^+ and π^- cross sections approach a constant, they will become equal.

This volume concludes with a rather long review article by S. Gasiorowicz in which the methods of proof and application of the dispersion relations is treated at length. This final article, rounding out the entire group, provides a summary and presents the outlook for the future of this new branch of field theory.

It is not to be expected, of course, that all of the important work in Abstract Field Theory should be included here. This is only a selection of those papers which seem to contain the most fruitful approaches or results, a selection inevitably reflecting the personal bias of the editor. The method of selection employed, however, and the emphasis throughout the volume, has been on those papers which would present a survey of the fundamental physical ideas and mathematical techniques adequate to permit the reader to become conversant with this new and rapidly changing field of physics.

I wish to express my appreciation to James Siagas and Paula Siagas for their translation of the papers from the German.

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PROPERTIES OF PROPAGATION FUNCTIONS AND RENORMALIZATION CONSTANTS OF QUANTIZED FIELDS

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Summary. It is attempted to derive some general properties of the propagation functions for coupled fields (Δ'_F, S'_F) without the use of power series expansions, and to show their connection with the renormalization constants for field operators and masses. Assuming that the coupled functions exist, it appears possible to discuss their behavior near the light cone (or for large momenta) and to obtain some information about the singularities of these functions when continued analytically. Attempts at the treatment of renormalizable theories are criticized on the basis of these results. Formulas are given for the mentioned renormalization constants which contain inequalities for the constants Z_2 and Z_3. Finally, it is pointed out that the methods introduced are advantageous for computations by means of power series expansions. As an example, the lowest order correction to the S_F-function in pseudoscalar meson theory is calculated without the appearance of infinite terms during the calculation.

INTRODUCTION

In recent work which treats the interaction problem in the structure of quantum field theory, the values Δ'_F and S'_F designated as propagation functions (or as Greens functions) play a significant role. These values should in principle be calculated from the fundamental equations of the theory. As yet however, only pertubation theoretical approximations are known, which probably with the exception of Quantum Electrodynamics are completely insufficient. It appears therefore suitable to obtain statements about these functions without assuming the possibility of their application with a coupling parameter or so using them.

The main results are therefore the derivation of formulas, which make possible the representation of these functions as the superposition of propagation functions of free fields with different masses and conclusions resulting from above. These concern especially the behavior of $\Delta'_F(x)$ and $S'_F(x)$ functions for small values of x^2, for example, the behavior of their Fourier transform for large momentum.

In the second part it will be shown that the constants appearing by the renormalization of field operators and masses can be expressed in a simple way through the values introduced for representing the propagation functions. One obtains in this way equations that are independent of pertubation computations and to a few expressions about these constants.

In contrast to other representations, in this fashion the use of incoming fields is avoided. In conclusion for illustration of methods used we will take up an analysis of pertubation theory. Two things should be pointed out: One, so far, it is not known if the basic equations of some quantized field theories (with the exception of free fields) possess solutions. We do not pursue this question but we strive for expressions of the propagation functions under the assumption that these exist. The other point is that to derive general results it is indispensable to operate with functions that are not explicitly known. Some of the considered mathematical operations (especially interchange of order of integration) have therefore formal character; their correctness could be shown only through performance of detailed calculations.

I. PROPERTIES OF PROPAGATION FUNCTIONS

a) Scalar fields:

In order to investigate the properties of propagation functions it appears useful to introduce other functions besides Δ_F' all of which can be defined as Vacuum Expectation values of the Heisenberg operators. The situation here is very similar as in a free field, where in a customary way one can derive Δ as well as $\Delta^{(1)}$ functions and from them easily change into Δ_F function.

Next, a Hermitian scalar field A(x) will be treated, that can be coupled with itself in a non-linear way or may be in a position of interacting with other (Bose or Fermion) fields. No special assumptions shall be made about the manner of coupling, it can be local or non-local. It is assumed that the treated theory is Lorentz invariant. As a result all energy-momentum four vectors P_μ should exist with the property:

$$\frac{\partial A(x)}{\partial x_\mu} = i\left[A(x), P_\mu\right]; \qquad \left[P_\mu, P_\nu\right] = 0. \tag{1}$$

Furthermore it is assumed that it is possible to define a vacuum state: that is the energy operator should possess a smallest eigenvalue which we normalize to zero. The knowledge of commutation relations for field operators is almost not necessary. Equation (1) is sufficient.

Now we want to examine the vacuum expectation values of quadratic field magnitudes. As the orthogonal system in Hilbert space we shall use in this case, the eigenvectors of the P_μ operators, which should form a complete set.

It follows then

$$P_\mu \Phi_k = k_\mu \Phi_k \qquad (k_0 > 0). \tag{2}$$

The eigenvalues k_μ can naturally be degenerate.

In complete analogy to free fields we have the following:

$$\left| \begin{array}{ll} (\Phi_0, A(x)A(x')\Phi_0) = \langle A(x)A(x')\rangle_0 & = i\Delta^{(+)\prime}(x-x') \\ \langle A(x')A(x)\rangle_0 & = -i\Delta^{(-)\prime}(x-x') \\ \langle [A(x), A(x')]\rangle_0 & = i\Delta'(x-x') = -2i\varepsilon(x_0-x_0')\bar{\Delta}'(x-x') \\ \langle \{A(x), A(x')\}\rangle_0 & = \Delta^{(1)\prime}(x-x') \\ \langle TA(x)A(x')\rangle_0 & = \tfrac{1}{2}\Delta_F'(x-x'). \end{array} \right. \tag{3}$$

From these defining equations of vacuum functions it follows that the same relations exist among them as in a free field. For example one can express all functions through $\Delta^{(+)\prime}$. For the propagation functions one can state: $[\theta(x_0) = \tfrac{1}{2}(1 + x_0/|x_0|)]$

$$\Delta_F'(x) = 2i\left[\theta(x_0)\Delta^{(+)\prime}(x) - \theta(-x_0)\Delta^{(-)\prime}(x)\right] = \Delta^{(1)\prime}(x) - 2i\bar{\Delta}'(x). \tag{4}$$

The functions $\Delta^{(+)\prime}$ or $\Delta^{(-)\prime}$ contain only positive or negative frequencies (see below) so that Δ_F' can be interpreted as a causal function in the same manner as in a free field. To obtain an explanation of the structure of these functions we consider the $\Delta^{(+)\prime}$ function.

$$\begin{aligned} \langle A(x)A(x')\rangle_0 &= \sum_k (\Phi_0, A(x)\Phi_k)(\Phi_k, A(x')\Phi_0) \\ &= \sum_k A_{0k}(x)A_{k0}(x') = \sum_k a_{0k}a_{0k}^* \exp[ik(x-x')]. \end{aligned} \tag{5}$$

Here we set $(\Phi_0, A(x)\Phi_k) = A_{0k}(x) = a_{0k}\, e^{ikx}$

The possibility of this conversion follows in the known way from

$$\frac{\partial A_{0k}(x)}{\partial x_\mu} = i(\Phi_0, [A(x), P_\mu]\Phi_k) = ik_\mu A_{0k}(x).$$

The summation in (5) is to extend over all states.

We introduce now a function

$$\varrho(-k^2) = (2\pi)^3 \sum a_{0k}a_{0k}^*. \tag{6}$$

Here we sum over all states that belong to eigenvalue k_μ.

It follows now from equations (3), (5) and (6)[1]

$$\Delta^{(+)\prime}(x-x') = -\frac{i}{(2\pi)^3}\int \theta(k_0)\varrho(-k^2)\exp[ik(x-x')]\,d^4k. \tag{7}$$

By this the summation over the eigenvalues is replaced by an integration with the condition that for all eigenvalues, $k_0 \geq 0$ is true.

We place in (7)

$$\varrho(-k^2) = \int_0^\infty \varrho(\varkappa^2)\,\delta(k^2+\varkappa^2)\,d(\varkappa^2)\,,$$

and obtain

$$\Delta^{(+)\prime}(x) = \int_0^\infty \Delta^{(+)}(x;\varkappa^2)\varrho(\varkappa^2)\,d(\varkappa^2)\,. \tag{8}$$

Analogous formula apply for all vacuum functions (in the following denoted as $\Delta^{(\ )\prime}$) since they can be formed in a linear way from $\Delta^{(+)\prime}$ according to (3).

$$\Delta^{(\ )\prime}(x) = \int_0^\infty \Delta^{(\ )}(x;\varkappa^2)\varrho(\varkappa^2)\,d(\varkappa^2)\,. \tag{9}$$

The primed functions allow themselves to be represented through a mass density by means of a linear transformation of the corresponding free functions.[2]

We have in the derivation of formula (9) already used that ρ is only dependent on argument k_μ^2 and for $k_\mu^2 > 0$ is identically zero. Both properties follow from the Lorentz invariance. Otherwise the vacuum functions (especially the commutation function Δ') will not be invariant quantities. Furthermore it follows from the defining equation (6) that ρ is a positive function. It is true

$$\varrho(\varkappa^2) > 0\,. \tag{10}$$

For the special case of a free field of mass m, it is naturally $\rho(\kappa^2) = \delta(\kappa^2 - m^2)$.

In general the discrete eigenvalues of the operators P_μ^2 will give rise to δ functions in $\rho(\kappa^2)$ inasmuch as the matrix element of the operators A(x) between the vacuum and the corresponding state (i.e. on the basis of selection rules) do not disappear. The discrete eigenvalues of P_μ^2 comply with the stable particles that were developed from the theory. From a physically meaningful theory one would expect that it at least describes a stable particle, namely ρ will contain at least one δ function. If no other stable particles appear, then it is true

$$\varrho(\varkappa^2) = \delta(\varkappa^2 - m^2) + \sigma(\varkappa^2); \qquad \sigma(\varkappa^2) = 0 \quad \text{for} \quad 0 < \varkappa^2 < (2m)^2;$$

where $\sigma(\kappa^2)$ is free of δ functions.

This structure describes the circumstance that in one such case the operator P_μ^2 possesses a discrete eigenvalue, to which a continuum fixes itself, when at least two particles are present.[3] From equation (9) and from the just given property of function ρ we can draw several conclusions. First of all it is clear that (9) is true also in Fourier space. ρ is therefore up to a certain factor equal to the transform of $\Delta^{(1)\prime}$ functions.

$$\varrho(-k^2) = \frac{1}{2\pi}\Delta^{(1)'}(k^2)\,. \tag{11}$$

Furthermore it is true for example:

$$\Delta'_F(k^2) = -2i\int_0^\infty \frac{\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)}{k^2+\varkappa^2-i\varepsilon} \tag{12}$$

With the given relations we can now disregard the behavior of Δ' functions in the neighborhood of the light cone, or for their Fourier transforms having large values of k^2 because the behavior of "free" functions is known and does not depend on mass. So one obtains for example:

$$\bar{\Delta}'(x) = \frac{1}{4\pi}\int_0^\infty \{\delta(x^2)+\ldots\}\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2) = \frac{1}{4\pi}\,\delta(x^2)\int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2) + \ldots \tag{13}$$

and in momentum space

$$\bar{\Delta}'(k^2) = P\int_0^\infty \frac{\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)}{k^2+\varkappa^2} = \frac{1}{k^2}\int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2) + \ldots \tag{14}$$

if the separated integrals converge.

The primed functions have either the same behavior as the corresponding free functions or in case $\int_0^\infty \rho(\kappa^2)\mathrm{d}(\kappa^2)$ is not convergent—they are more strongly singular at the light cone (if they fall off more weakly for large momenta) than the free functions.

But it is not possible that the primed functions are less singular than the free functions because, from (10), $\int_0^\infty \rho\, \mathrm{d}(\kappa^2) > 0$.

It will be shown that the question of convergence of $\int_0^\infty \rho\, \mathrm{d}(\kappa^2)$ for theories capable of being renormalized is the same as the question whether the renormalization constants for the field operators are finite.

That the function Δ'_F should exist at all, it is, according to (12), clearly necessary, that the integral

$$\int^\infty \frac{\varrho(\varkappa^2)}{\varkappa^2}\,\mathrm{d}(\varkappa^2)\,,$$

should converge at the upper limit. A further conclusion concerns the properties of the function $\Delta'_F(k^2)$ by analytical continuation in k^2_μ plane.

It is for real k^2

$$\Delta'_F(k^2) = \Delta^{(1)'}(k^2) - i2\bar{\Delta}'(k^2).$$

Furthermore it is true

$$2\bar{\Delta}'(k^2) = \frac{1}{\pi} P \int_0^\infty \frac{\Delta^{(1)\prime}(l^2)\, d(-l^2)}{k^2 - l^2}\,; \quad \Delta^{(1)\prime}(k^2) = -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{2\bar{\Delta}'(l^2)\, d(-l^2)}{k^2 - l^2}\,(^4). \tag{15}$$

The functions $\Delta^{(1)\prime}(k^2)$ and $2\bar{\Delta}'(k^2)$ are conjugates in the sense of Hilbert transformation.[5] From this (under certain assumptions) the function $\Delta'_F(k^2)$ can be analytically continued and is regular in the lower half plane. This result is gratifying because of the behavior of the pole in this region could lead to new (nonrenormalizable) divergence (i.e. of S matrix elements).[6]

Our analysis shows that by using the exact Δ'_F function, such difficulties should not arise. This is true also for the similarly discussed S'_F functions.

We want now to draw a conclusion for the commutation relations at equal times and use for this purpose the presentation (9) for the Δ' function.

$$\Delta'(x) = \int_0^\infty \Delta(x; \kappa^2)\,\varrho(\kappa^2)\, d(\kappa^2)\,.$$

It follows directly:

$$\begin{cases} \langle [A(\mathbf{x}, t), A(\mathbf{x}', t)] \rangle_0 = \langle [\dot{A}(\mathbf{x}, t), \dot{A}(\mathbf{x}', t)] \rangle_0 = 0\,, \\ \langle [\dot{A}(\mathbf{x}, t), A(\mathbf{x}', t)] \rangle_0 = -i\delta(\mathbf{x} - \mathbf{x}') \int_0^\infty \varrho(\kappa^2)\, d(\kappa^2)\,. \end{cases} \tag{16}$$

The commutation relations at equal times between operators A and $\dot{A}$ for the vacuum state can be therefore derived (up to a factor) from equation (1).

The operators commute for spacelike points, and the Dirac δ function necessarily occurs. This is true, according to their derivation also for theories with non-local interactions.[7] The same state of affairs will occur for spinor fields.

We remark for equation (16) that according to the usual assumption for the commutation relation $[\dot{A}(\mathbf{x}), A(\mathbf{x}')] = -i\delta(\mathbf{x} - \mathbf{x}')$, the following will be true:

$$\int_0^\infty \rho\, d(\kappa^2) = 1.$$

The vacuum functions that were here considered should however relate to renormalized operators, to which, as is known, one must transform.

b) Spinor fields

We now consider a spinor field under the same assumptions as were made for the already treated scalar field. Besides this we require the invariance of the theory toward the particle-antiparticle conjugation. The procedure and the results are accordingly similar as in a scalar field.

Again we begin with the definition of vacuum functions:

$$\left\{\begin{array}{l} \langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = -iS^{(+)'}_{\alpha\beta}(x-x'); \qquad \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 = -iS^{(-)'}_{\alpha\beta}(x-x') \\ \langle\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}\rangle_0 = -iS'_{\alpha\beta}(x-x'); \qquad \langle[\psi_\alpha(x), \bar{\psi}_\beta(x')]\rangle_0 = -S^{(1)'}_{\alpha\beta}(x-x') \\ \qquad \langle T\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = -\tfrac{1}{2}\, S'_{F\alpha\beta}(x-x')\,. \end{array}\right. \tag{17}$$

From the invariance about particle-antiparticle conjugation it follows:

$$\langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 = \langle\bar{\psi}'_\beta(x')\psi'_\alpha(x)\rangle_0 = -C^{-1}_{\beta\gamma}\langle\psi_\gamma(x')\bar{\psi}_\delta(x)\rangle_0 C_{\delta\alpha}\,.$$

Therefore:

$$S^{(-)'}_{\alpha\beta}(x-x') = -C^{-1}_{\beta\gamma}S^{(+)'}_{\gamma\delta}(x'-x)C_{\delta\alpha}\,. \tag{17a}$$

Thus one can express all functions with $S^{(+)'}$. Now there is:

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = \sum_k (\Phi_0, \psi_\alpha(x)\Phi_k)(\Phi_k, \bar{\psi}_\beta(x')\Phi_0) = \sum_k c^\alpha_{0k}\bar{c}^\beta_{0k}\exp[ik(x-x')]. \tag{18}$$

Next we should again perform the summation over those states that belong to eigenvalue k_μ. Accordingly we introduce two functions ρ_1 and ρ_2.

$$(i\gamma_{\alpha\beta}k - \sqrt{-k^2}\,\delta_{\alpha\beta})\varrho_1(-k^2) + \delta_{\alpha\beta}\varrho_2(-k^2) = -(2\pi)^3\sum c^\alpha_{0k}\bar{c}^\beta_{0k}\,. \tag{19}$$

Because of the relativistic invariance of the theory the expression (19) can depend on the γ-matrices only in the specified way. The division of the γ-free portion into ρ_1 and ρ_2 is of course arbitrary.

Exactly as in the scalar case it follows now

$$\begin{aligned} S^{(+)'}(x) &= -\frac{i}{(2\pi)^3}\int\theta(k_0)\{(i\gamma k - \sqrt{-k^2})\varrho_1(-k^2) + \varrho_2(-k^2)\}\exp[ikx]\,d^4k = \\ &= \int_0^\infty\{S^{(+)}(x;\varkappa)\varrho_1(\varkappa^2) + \Delta^{(+)}(x;\varkappa)\varrho_2(\varkappa^2)\}\,d(\varkappa^2)\,. \end{aligned} \tag{20}$$

A corresponding representation is true according to (17) and (17a) for all primed functions:

$$S^{(\ )'}(x) = \int_0^\infty\{S^{(\ )}(x;\varkappa)\varrho_1(\varkappa^2) + \Delta^{(\ )}(x;\varkappa^2)\varrho_2(\varkappa^2)\}\,d(\varkappa^2)\,. \tag{21}$$

For ρ_1 and ρ_2 it follows directly from (19) that both functions are real. Furthermore we want to prove the following inequalities.

$$\varrho_1(\varkappa^2) > 0; \qquad 0 \leq \varrho_2(\varkappa^2) < 2\varkappa\varrho_1(\varkappa^2)\,. \tag{22}$$

We multiply in (19) on the left with $(i\gamma k + \alpha)$, on the right with $(i\gamma k + \alpha)\gamma_4$ and we obtain ($f_{0k} = (i\gamma k + \alpha)c_{0k}$)

$$\sum_\beta f^\beta_{0k}f^{*\beta}_{0k} = k_0[(\varkappa-\alpha)^2\varrho_1 + 2\alpha\varrho_2] > 0; \qquad (\varkappa^2 = -k^2)\,.$$

This inequality is true for arbitrary real α. Because $k_0 > 0$ we have: