Differential Equations and Computer Algebra

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Preface

The papers in this volume are an outgrowth of invited talks at a workshop entitled COMPUTER ALGEBRA AND DIFFERENTIAL EQUATIONS (CADE-90) held at the Mathematical Sciences Institute at Cornell University from May 6 to May 9, 1990. Approximately 50 people attended from Europe, North America and the Soviet Union.

This is the second in what we hope will be a biennial series of workshops devoted to this topic. The first such workshop was held in Grenoble from May 24 to May 27 1988 and papers from that conference were collected in a volume entitled "Computer Algebra and Differential Equations", E. Tournier, ed., Academic Press, 1989. The aim of these workshops was twofold: to allow computer algebra users to learn about recent theoretical developments concerning differential equations and to make researchers in theoretical areas aware of questions arising in the design of computer algebra systems.

We would like to thank the Mathematical Sciences Institute for its financial support of this workshop. We would also like to thank the staff of this institute for their untiring assistance.

Michael F. SINGER

En Memoire de Jean Martinet

Lors du premier "Workshop" sur le calcul formel et les équations différentielles "CADE 88" l'un des cours avait été fait par Jean Martinet. Tout le monde se souvient de son enthousiasme communicatif: il savait rendre clairs et vivants les concepts les plus délicats. Jusque tard dans la soirée il écoutait les questions de tous, revenant sur les leçons de la journée avec son inépuisable gentillesse, sachant se mettre à l'écoute de chacun quelle que soit sa spécialité ou sa formation. Depuis CADE 88 les choses avaient beaucoup évolué et de notre côté nous avions pas mal progressé dans notre "Théorie de Cauchy sauvage". Jean aurait beaucoup aimé participer à CADE 90 et discuter de tout cela. Hélas la maladie l'en a empêché. Cette maladie devait l'emporter le 3 juillet 1990. Je suis certain que les participants de CADE 88 se souviendront de Jean, de son regard clair et de ses émerveillements d'enfant devant les beautés des mathématiques. Nous aurions tellement aimé l'avoir avec nous dans les prochaines rencontres...

Jean-Pierre RAMIS

CONTENTS

Internal Symmetries of Differential Equations

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Abstract

A new Bäcklund Theorem for internal symmetries of systems of differential equations is discussed. Every internal symmetry of any "reasonable" system of differential equations comes from a first order generalized symmetry and, conversely, every first order generalized symmetry satisfying certain explicit contact conditions determines an internal symmetry. Applications to a remarkable differential equation due to Hilbert and Cartan whose internal symmetry group is the exceptional simple Lie group G_2 are given.

In this paper I will survey some very recent work done in collaboration with Ian Anderson, of Utah State University, and Niky Kamran, of McGill University. A preprint containing detailed proofs and

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many more examples is available, [1]. The work had its genesis in a series of lectures on the variational bicomplex given by Ian Anderson while visiting the University of North Carolina at Chapel Hill. Robert Bryant, who was in the audience, asked Ian to compute the symmetry group of the innocent looking underdetermined ordinary differential equation $u' = (v'')^2$. Robert knew well the history of this equation, which we have decided to call the Hilbert-Cartan equation; in particular, Elie Cartan had proved that the "symmetry group" of this equation is a realization of the non-compact real form of the exceptional simple Lie group G_2! Robert was suitably impressed when Ian came back with a fourteen dimensional symmetry algebra for the equation. There matters rested until, during a Conference on Symbolic Manipulation hosted by the Institute for Mathematics and Its Applications, Robby Gardner asked Fritz Schwarz to answer the same question using his fancy computer algebra package for computing symmetry groups in SCRATCHPAD. Fritz only found a six dimensional symmetry group. After Ian sent the results of his earlier (hand!) computations, we realized that the discrepancy was due to the fact that Ian had computed the first order generalized symmetries of the equation, whereas Fritz' program was designed to compute classical point symmetries; this is why he failed to detect the eight remaining vector fields. However, upon reflection, it occurred to us that much more was at stake than the difference between point symmetries and generalized symmetries. Cartan was not aware of the concept of a generalized symmetry, and all his symmetries were realized as geometrical transformations of some finite-dimensional space, which the generalized symmetries are not. Contact transformations fit into Cartan's framework, but these were not the objects Cartan had computed for this particular equation since, according to Bäcklund's Theorem, there are no contact transformations (beyond prolonged point transformations) if the number of dependent variables is greater than one. What Cartan had computed were what we will call "internal symmetries", which are transformations which preserve the contact ideal *only* when restricted to the equation submanifold. (These are also known as "dynamical symmetries" in the mathematical physics literature, and have also received mention in the rather abstruse work of Vinogradov and his collaborators, cf. [9].) The restrictions of Bäcklund's Theorem no longer apply, and there are internal symmetries which depend explicitly on higher order derivatives. Thus, a new question arose: for the Hilbert-Cartan equation, why did Ian's computed Lie algebra of generalized symmetries coincide with Cartan's Lie algebra of internal symmetries? Answering this question was the motivation for our work.

The main results of our investigations can now be summarized by the following, all of which hold for any "reasonable" = "nondegenerate" (see below) system of differential equations. 1. It is easy to see that every external symmetry of a system of differential equations gives rise to an internal symmetry by restricting to the equation manifold. Indeed, in many cases, including normal systems of partial (not ordinary) differential equations of order at least two, all internal symmetries arise this way. 2. Every internal symmetry comes from a first order generalized symmetry. This is essentially a generalization of Bäcklund's Theorem for internal symmetries of differential equations. (In particular, contact transformations are "internal symmetries" of the entire jet bundle.) 3. Every first order generalized symmetry which satisfies additional explicit contact conditions gives an internal symmetry. A detailed analysis of the contact conditions leads to results on the existence of genuine internal symmetries.

In order to keep the exposition as brief as possible, I will assume that the reader is reasonably familiar with the standard theory of symmetry groups of differential equations as presented, for instance, in my book, [10]. We will work with local coordinates throughout, although all of these results have analogous, more general, statements for arbitrary fiber bundles over smooth manifolds. Consider a system of differential equations

$$\mathcal{R}: \qquad \Delta_\kappa(x, u^{(n)}) = 0, \qquad \kappa = 1, \ldots, r, \tag{1}$$

in p independent variables $x = (x^1,\ldots,x^p)$, and q dependent variables $u = (u^1,\ldots,u^q)$. The derivatives of the dependent variables are denoted by $u^\alpha_J = \partial^J u^\alpha/\partial x^J$, where $J = (j_1, \ldots, j_k)$, $1 \le j_\nu \le p$, is a symmetric multi-index, of order $k = \#J$. We let $u^{(n)}$ denote the collection of all such derivatives of orders $k \le n$, which provide coordinates on the associated jet space J^n. We will assume that the system (1) satisfies the nondegeneracy conditions of being both maximal rank and locally solvable, cf. [10; §2.6], and can identify it with the corresponding implicitly defined submanifold $\mathcal{R} \subset J^n$. (These nondegeneracy conditions are quite mild and are satisfied by virtually every system of differential equations arising in applications.)

In general, by a symmetry of the system of differential equations (1) we mean a transformation which maps solutions to solutions. The most basic type of symmetry is a point transformation, meaning a local diffeomeorphism of the space of independent and dependent variables:

$$\Phi: \ (x, u) \longmapsto (\tilde{x}, \tilde{u}) .$$

Such transformations act on solutions $u = f(x)$ by pointwise transforming their graphs. Let G denote a local group of point transformations. We will always assume that our transformation group G is connected, thereby consciously omitting discrete symmetry groups, which, while also of great interest for differential equations, are unfortunately not amenable to Lie's techniques. Connectivity implies that it is sufficient to work with the associated infinitesimal generators, which, in the case of groups of point transformations, form a Lie algebra of vector fields of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha} , \qquad (2)$$

on the space of independent and dependent variables. The group transformations in G are recovered from the infinitesimal generators by the usual process of exponentiation.

Since the transformations in G act on functions $u = f(x)$, they also act on their derivatives, and so induce so-called prolonged transformations

$$\mathrm{pr}^{(n)}\Phi : \quad (x, u^{(n)}) \longmapsto (\tilde{x}, \tilde{u}^{(n)}) ,$$

which is defined on an appropriate open subset of J^n. The explicit formula for the prolonged group transformations is very complicated; however the corresponding prolonged infinitesimal generators have a rather simple "prolongation formula". Explicitly, the n^{th} prolongation of the vector field (2), which is the infinitesimal generator of its prolonged action of the associated one-parameter group, is the vector field

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \varphi_J^\alpha \frac{\partial}{\partial u_J^\alpha} , \qquad (3)$$

on J^n. The coefficients φ_J^α are determined recursively via the well-known formula

$$\varphi_{J,i}^\alpha = D_i \varphi_J^\alpha - \sum_{k=1}^{p} D_i \xi^k \, u_{J,k}^\alpha . \tag{4}$$

Here D_i denotes the total derivative with respect to x^i.

Theorem 1. Assume that the system of partial differential equations (1) is nondegenerate. Then the vector field $\mathbf{v}$ in (2) will generate a one-parameter symmetry group of the system (1) if and only if the classical infinitesimal symmetry criterion

$$\mathrm{pr}^{(n)}\, \mathbf{v}\, (\Delta_\nu) = 0, \qquad \nu = 1, \ldots, r, \qquad \text{whenever} \quad \Delta = 0 . \tag{5}$$

holds.

The "determining equations" (5) form a large over-determined linear system of partial differential equations for the coefficients ξ^i, φ_α of $\mathbf{v}$, and can, in practice, be explicitly solved to determine the complete (connected) symmetry group of the system (1). There are now a wide variety of computer algebra packages available which will automate most of the routine steps in the calculation of the symmetry group of a given system of partial differential equations. See [5], [8], [12] for examples in MACSYMA, REDUCE and SCRATCHPAD. Reference [5] gives a good survey of the different packages available at present, and a discussion of their strengths and weaknesses. (Conspicuously lacking are packages in MAPLE or MATHEMATICA.) More recent approaches based on the application of Gröbner basis techniques to the theory of overdetermined system of partial differential equations are also implemented, cf. [11].

The theory of point symmetries of differential equations is classical, and, in more or less the same form, dates back to the original work of Sophus Lie. After this theory is well understood, a number of possible generalizations come to mind. In one direction, we define a *generalized vector field* by allowing the coefficients of the original vector field (2) to also depend on derivatives of u:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u^{(k)}) \frac{\partial}{\partial u^\alpha} . \tag{6}$$

The condition that $\mathbf{v}$ be a generalized symmetry of the system of differential equations (1) is the same as before, (5), although now one must also take into account the derivatives (prolongations) of the system:

$$\mathrm{pr}^{(k)} \mathcal{R}: \quad D_J \Delta_\kappa(x, u^{(n)}) = 0, \qquad \kappa = 1, \ldots, r, \qquad \# J \le k, \tag{7}$$

with $D_J = D_{j_1} \ldots D_{j_\ell}$ denoting a total derivative of order $\ell = \# J$. Every generalized symmetry is equivalent to one in evolutionary form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x,u^{(k)}) \frac{\partial}{\partial u^\alpha} , \tag{8}$$

where the q-tuple $Q = (Q_1,\ldots,Q_q)$, known as the *characteristic* of $\mathbf{v}$, is given by

$$Q_\alpha(x, u^{(k)}) = \varphi_\alpha(x, u^{(k)}) - \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial u^\alpha}{\partial x^i} , \quad \alpha = 1, \ldots, q . \tag{9}$$

Replacing the generalized vector field $\mathbf{v}$ by its evolutionary form $\mathbf{v}_Q$ leads to a simpler set of determining equations in that they only involve the q unknown functions Q_α rather than the p + q unknown coefficients ξ^i, φ_α of **v**. (This technique even works for point symmetries, where the associated characteristic depends linearly on first order derivatives.) An evolutionary vector field $\mathbf{v}_Q$ is a *trivial symmetry* of the system (1) if the characteristic $Q(x,u^{(n)})$ vanishes on all solutions. Two generalized symmetries **v** and **w** are *equivalent* if their respective evolutionary forms differ by a trivial evolutionary symmetry.

A k^{th} order generalized vector field is will not usually prolong to a well-defined vector field on any jet bundle J^n since its n^{th} prolongation will involve derivatives of orders up to $k + n$. Beyond point transformations, the only exceptions to this are the infinitesimal contact transformations, which correspond to first order generalized symmetries in the case of just one dependent variable. In general, recall that a contact transformation is a (locally defined) map on J^n which preserves the *contact ideal* $\mathcal{I}^{(n)}$. In local coordinates, $\mathcal{I}^{(n)}$ is generated by the *contact forms*

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i, \qquad \alpha = 1, \ldots, q, \qquad 0 \le \#J < n . \qquad (10)$$

Therefore a (locally defined) transformation

$$\Psi : \ J^n \longrightarrow J^n ,$$

on the jet space will determine a contact transformation provided its pull-back maps every contact form to a linear combination of contact forms:

$$\Psi^* (\mathcal{I}^{(n)}) \subset \mathcal{I}^{(n)} . \qquad (11)$$

A contact transformation acts on a function $u = f(x)$ by pointwise transforming the graph of its n-jet $u^{(n)} = j^n f(x)$; the contact condition (11) ensures that the transformed graph is (locally) the n-jet of some function. The infinitesimal version of this criterion is that a vector field

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \varphi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \qquad (12)$$

on J^n generates a one-parameter group of contact transformations provided the Lie derivative of any contact form is contained in the contact ideal, i.e. for each α, K,

$$\mathbf{X}[\theta_K^\alpha] = \sum_{\beta, J} \mu_{K,\beta}^{\alpha,J} \theta_J^\beta, \qquad \alpha = 1, \ldots, q, \quad \#K < n, \qquad (13)$$

for some functions $\mu_{K,\beta}^{\alpha,J}$ on J^n. These conditions are quite restrictive, and Bäcklund's Theorem, [2], [7], implies that any contact transformation on J^n is the prolongation of either a point transformation or, if there is only one dependent variable, of a first order contact transformation on J^1.

Note that the projection

$$\pi(\mathbf{X}) = \sum_{i=1}^{p} \xi^i(x, u^{(1)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha} \qquad (14)$$

of any contact vector field gives a first order generalized vector field, or, if $q > 1$, of a point vector field, as in (2). Conversely, the contact conditions (13) show that $\mathbf{X}$ will coincide with the n^{th} prolongation of its projection $\pi(\mathbf{X})$. The next lemma is utilized to provide a characterization of which generalized vector fields produce contact transformations. As such, it plays a key role in the standard infinitesimal proof of Bäcklund's Theorem, cf. [7].

Lemma 2. An evolutionary vector field $\mathbf{v}_Q$ is equivalent to an infinitesimal contact transformation if and only if its characteristic $Q(x, u^{(1)})$ depends on at most first order derivatives, and there exist functions $\xi^i(x, u^{(1)})$, $i = 1, \ldots, p$, such that the *contact conditions*

$$\frac{\partial Q_\alpha}{\partial u_i^\beta} + \delta_\beta^\alpha \, \xi^i = 0 \,, \tag{15}$$

hold.

Indeed, in this case, the ξ^i's will be the coefficients of the $\partial / \partial x^i$ and the coefficients of the $\partial / \partial u^\alpha$ will be defined by

$$\varphi_\alpha = Q_\alpha + \sum_{i=1}^{p} \xi^i u_i^\alpha \,.$$

The contact vector field $\mathbf{X}$ is then just the n^{th} prolongation of

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(1)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha} \,. \tag{16}$$

Note that left hand sides of the contact conditions (15) appear in the prolongation formula as the coefficients of the terms in $\mathrm{pr}^{(n)}\,\mathbf{v}$ which depend on derivatives of order $n + 1$, hence their vanishing is a necessary and sufficient condition that the prolongation $\mathrm{pr}^{(n)}\,\mathbf{v}$ of the first order generalized vector field (16) define a genuine vector field on the jet space J^n.