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Singularity Theory and its Applications

Warwick 1989, Part I: Geometric Aspects of Singularities

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The figure on the front cover shows two non-isomorphic disentanglements of a projection to $\mathbb{C}^3$ of the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$. For details see the paper, *Disentanglements* by T. de Jong and D. van Straten.

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Preface

A year-long symposium on Singularity Theory and its Applications was held at the University of Warwick in the academic year 1988–89. Two workshops were held during the Symposium, the first primarily geometrical and the second concentrating on the applications of Singularity Theory to the study of bifurcations and dynamics. Accordingly, we have produced two volumes of proceedings. One of the notable features of Singularity Theory is the close development of the theory and its applications, and we tried to keep this as part of the philosophy of the Symposium. We believe that we had some success.

It should perhaps be pointed out that not all the papers included in these two volumes were presented at the workshops; these are not Proceedings of the workshops, but of the Symposium as a whole. In fact a considerable amount of the material contained in these pages was developed during the Symposium.

For the record, the Symposium was organized by the four editors of the two volumes: David Mond, James Montaldi, Mark Roberts and Ian Stewart. There were over 100 visitors and 120 seminars. The Symposium was funded by the S.E.R.C., and could not have been such a success without the hard work of Elaine Shiels, to whom we are all very grateful.

Every paper published here is in final form and has been refereed.

David Mond
James Montaldi

University of Warwick,
August 1990

Contents

Symmetric Lagrangian singularities and Gauss maps of theta divisors

Malcolm R. Adams, Clint McCrory, Theodore Shifrin and Robert Varley

Recently three of the authors studied the Thom–Boardman singularities and the local $\mathbb{Z}/2$-stability of the Gauss map of the theta divisor of a smooth algebraic curve of genus three [12]. In this paper we develop a theory of $\mathbb{Z}/2$-symmetric Lagrangian maps appropriate to the study of theta divisors of curves of arbitrary genus, and we apply this theory to the genus three case, obtaining Lagrangian analogues of the results of [12]. We find that the local classification of $\mathbb{Z}/2$-Lagrangian-stable Gauss maps coincides with our previous local classification of $\mathbb{Z}/2$-stable Gauss maps in genus three. The corresponding classifications in higher genus are expected to diverge, as in the nonsymmetric case (cf. [3]).

Let C be a curve of genus g, $J(C)$ its Jacobian variety, and $\Theta \subset J(C)$ the theta divisor. Torelli's theorem states that the curve C is determined by the pair $(J(C), \Theta)$. More precisely, C is determined by the Gauss map γ which assigns to a point of Θ its tangent hyperplane, translated to the origin of $J(C)$. Andreotti proved that the branch locus of γ is the dual hypersurface of the canonical embedding of C, provided C is nonhyperelliptic. Thus the singularities of the Gauss map are directly related to the extrinsic geometry of the canonical embedding, and hence to the intrinsic geometry of the curve C.

Locally Θ can be given as the graph of a function $f\colon \mathbb{C}^{g-1} \to \mathbb{C}$, and the Gauss map $\gamma\colon \Theta \to \mathbb{P}^{g-1*}$ is given locally as the gradient of f. Since the gradient df has a canonical Lagrangian structure, namely the factorization through the Lagrangian submanifold graph$(df) \subset T^*\mathbb{C}^{g-1}$, γ is locally Lagrangian. However, this local Lagrangian structure depends on the choice of local coordinates used to define f; moreover, the $\mathbb{Z}/2$-symmetry of $T^*\mathbb{C}^{g-1}$ is antisymplectic. To obtain a global symmetric Lagrangian structure, we consider the conormal bundle $\mathcal{C}_\Theta \subset T^*J(C)$ of the theta divisor. The Gauss map γ lifts to the *homogeneous Gauss map* Γ, which is the restriction of the projection to the fiber:

$$\begin{array}{ccc} \mathcal{C}_\Theta & \xrightarrow{\Gamma} & T_0^*J(C) \\ \downarrow & & \downarrow \\ \Theta & \xrightarrow{\gamma} & \mathbb{P}(T_0^*J(C)) \end{array}.$$

If we remove the zero-section of $T^*J(C)$, Γ is a *conic Lagrangian map*, with $\mathbb{Z}/2$-symmetry induced by the (-1)-map of $J(C)$. (If Θ is singular, then Γ is defined over the Nash blowup of the Gauss map.) The homogeneous Gauss map is defined in the same way for a complex affine hypersurface $M \subset \mathbb{C}^n$. In contrast to the real case, the Gauss map of a complex affine hypersurface does not seem to have a natural global Lagrangian structure.

For conic Lagrangian submanifolds $\Lambda \subset T^*X$, projection to the *base* X has been studied by several authors (cf. [8], [15]), but projection to the *fiber* has not been previously considered. For example, in the work of Janeczko and Kowalczyk [9] [10], the symmetry of T^*X is also induced from a symmetry of the base X, but Λ is projected to the base.

The homogeneous Gauss map $\Gamma: C_\Theta \to T^*J(C) \to T^*_0J(C)$ is equivariant with respect to commuting actions of $\mathbb{C}^*$ and $\mathbb{Z}/2$, and $\mathbb{Z}/2$ acts trivially on the $\mathbb{C}^*$-orbit space of the target. If $\ell: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ is any such conic $\mathbb{Z}/2$-Lagrangian map-germ, there are two cases to consider: $\mathbb{Z}/2$ acts either trivially or non-trivially on the $\mathbb{C}^*$-orbit space of the total space $\mathcal{E}$. In the latter case we say that ℓ is *odd*. In both cases we prove that stable conic $\mathbb{Z}/2$-Lagrangian germs are classified by versal *generating families* of functions; if ℓ is odd, its generating family is a family of odd functions. Our method of proof is to pass from a conic Lagrangian map to a Legendrian map by *dehomogenization*, and then to use the work of Zakalyukin on the classification of Legendrian maps (cf. [16], [3]). (The Gauss map of a complex projective hypersurface $M \subset \mathbb{P}^n$ has a natural Legendrian structure; the case $n = 4$ is studied in [11].) Our main result for the homogeneous Gauss map Γ of a theta divisor is the following.

Theorem. For a nonhyperelliptic curve of genus three, Γ is a locally infinitesimally stable conic $\mathbb{Z}/2$-Lagrangian map if and only if the canonical curve $C \subset \mathbb{P}^2$ has no higher flexes. For a hyperelliptic curve of genus three, Γ is a locally infinitesimally stable conic $\mathbb{Z}/2$-Lagrangian map.

In section 1 we show that dehomogenization gives a bijection from isomorphism classes of conic G-Lagrangian germs to isomorphism classes of G-Legendrian germs (1.5), and we construct symmetric Darboux coordinates for an odd conic Lagrangian fibration germ (1.12). In the second section, we show that isomorphism classes of odd conic Lagrangian germs are in one-to-one correspondence with stable isomorphism classes of generating families (2.2), and that infinitesimally stable germs correspond to versal families (2.11). We then derive normal forms for odd versal families with two parameters (2.16). Section three contains the application of our results to genus 3 theta divisors. As in [12], we use the extrinsic geometry of the canonical curve $C \subset \mathbb{P}^2$ to describe the singularities of the homogeneous Gauss map Γ.

All manifolds, maps and group actions are assumed to be complex analytic. (The results in sections 1 and 2 are also valid in the real C^∞ category, with $\mathbb{C}^*$ replaced by $\mathbb{R}^+$, the multiplicative group of positive real numbers.)

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McCrory and Varley wish to thank both the Mathematics Institute of the University of Warwick for its hospitality and support, and also the paticipants in the singularity theory workshops at Warwick for many stimulating conversations. We are especially grateful to J. Montaldi, C. T. C. Wall, and V. M. Zakalyukin for their interest in our work.

1. Conic G-Lagrangian maps

Let G be a finite group. A *conic* G-*manifold* is a manifold together with a proper, free $\mathbb{C}^*$ action and a G action such that the two actions commute. It follows that the orbit map of the $\mathbb{C}^*$ action is a G-equivariant principal $\mathbb{C}^*$-bundle. A *conic* G-*map* is a map of conic G-manifolds which is equivariant with respect to the actions of $\mathbb{C}^*$ and G. A *conic symplectic* G-*manifold* is a conic G-manifold with a (holomorphic) symplectic structure such that the symplectic form Ω is homogeneous and G-invariant. In other words, if κ_t is the action of $t \in \mathbb{C}^*$ then $(\kappa_t)^*\Omega = t\Omega$, and if ν_g is the action of $g \in G$ then $(\nu_g)^*\Omega = \Omega$.

A *conic* G-*Lagrangian map* is a conic G-map $\mathcal{L}: \mathcal{M} \to \mathcal{B}$ of manifolds, together with a factorization $\mathcal{L} = \pi \circ i$,

$$\begin{array}{ccc} & & \mathcal{E} \\ & {}^{i}\nearrow & \downarrow\pi \\ \mathcal{M} & \xrightarrow[\mathcal{L}]{} & \mathcal{B} \end{array}$$

where $\mathcal{E}$ is a conic symplectic G-manifold, $i: \mathcal{M} \to \mathcal{E}$ is a Lagrangian conic G-immersion, and $\pi: \mathcal{E} \to \mathcal{B}$ is a Lagrangian conic G-fibration (cf. [3, ch. 18]). If $\mathcal{E}$ has dimension 2n, then $\mathcal{M}$ and the fibers of π have dimension n.

We will be primarily concerned with two special types of conic G-Lagrangian maps $\mathcal{L}$: either G is the trivial group, or $G = \mathbb{Z}/2$ and the symmetry of $\mathcal{L}$ is <u>odd</u>. An *odd conic Lagrangian fibration* is a conic Lagrangian $\mathbb{Z}/2$-fibration $\mathcal{E} \to \mathcal{B}$ such that

(1) $\mathbb{Z}/2$ acts nontrivially on the $\mathbb{C}^*$-orbit space of $\mathcal{E}$, and

(2) $\mathbb{Z}/2$ acts trivially on the $\mathbb{C}^*$-orbit space of $\mathcal{B}$.

An *odd conic Lagrangian map* is a conic $\mathbb{Z}/2$-Lagrangian map $\mathcal{L} = \pi \circ i: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ such that $\pi: \mathcal{E} \to \mathcal{B}$ is an odd conic Lagrangian fibration.

(1.1) <u>Example</u>. Let $\iota: \mathbb{C}^n \to \mathbb{C}^n$ be the involution $\iota(x) = -x$, and let M be a smooth hypersurface in $\mathbb{C}^n$ such that $\iota(M) = M$. Let $\mathcal{E} = (T^*\mathbb{C}^n) - \mathcal{Z}$, the cotangent bundle of $\mathbb{C}^n$ minus the zero-section, with standard Darboux coordinates $(x,\xi) = (x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ and involution $\tilde{\iota}(x,\xi) = (-x,-\xi)$, and let $i: \mathcal{M} \subset \mathcal{E}$ be the conormal variety of M,

$$\mathcal{M} = \{(x,a) \mid x \in M,\ a \in T_x^*\mathbb{C}^n - \{0\},\ a(T_xM) = 0\}.$$

Let $\pi\colon \mathcal{E} \to \mathcal{B}$ be the projection to the *fiber* of the cotangent bundle, $\pi\colon (T^*\mathbb{C}^n) - \mathcal{Z} \to (T_0^*\mathbb{C}^n) - \{0\}$, $\pi(x,\xi) = \xi$. Then the *homogeneous Gauss map* $\Gamma = \pi \circ i\colon \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ is an odd conic Lagrangian map. The map on $\mathbb{C}^*$-orbits induced by Γ is the *Gauss map*

$$\gamma\colon M \to \mathbb{P}T_0^*\mathbb{C}^n = G(n-1,\mathbb{C}^n),$$

which is $\mathbb{Z}/2$-invariant (cf. [13, p. 720]).

An *isomorphism* of conic G-Lagrangian maps $\mathcal{L}_1 = \pi_1 \circ i_1$ and $\mathcal{L}_2 = \pi_2 \circ i_2$ is a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_1 & \to & \mathcal{E}_1 & \to & \mathcal{B}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_2 & \to & \mathcal{E}_2 & \to & \mathcal{B}_2 \end{array}$$

where the vertical maps are conic G-isomorphisms, and the isomorphism $\mathcal{E}_1 \to \mathcal{E}_2$ is symplectic.

Given a conic G-manifold $\mathcal{M}$ of dimension n and a conic Lagrangian fibration $\pi\colon \mathcal{E} \to \mathcal{B}$ with $\mathcal{E}$ of dimension 2n, a topology on the set of conic G-Lagrangian maps $\mathcal{L} = \pi \circ i\colon \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ is induced from the topology of uniform convergence on compact subsets (or the Whitney topology in the real C^∞ category) on the set of conic G-Lagrangian immersions $i\colon \mathcal{M} \to \mathcal{E}$.

Let $\mathcal{M}$ be a conic G-manifold, and let $\mathcal{O}$ be a $\mathbb{C}^*$-orbit of $\mathcal{M}$ such that $G\mathcal{O} = \mathcal{O}$. A *conic G-Lagrangian map-germ* $\ell\colon \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ at $\mathcal{O}$ is an equivalence class of conic G-Lagrangian maps $\ell = \pi \circ i\colon U \to V \to \mathcal{B}$, such that $U \subset \mathcal{M}$ and $V \subset \mathcal{E}$ are open sets invariant under the actions of $\mathbb{C}^*$ and G, with $\mathcal{O} \subset U$ and $i(\mathcal{O}) \subset V$. Two such maps $\ell_1 = \pi_1 \circ i_1$ and $\ell_2 = \pi_2 \circ i_2$ are equivalent if i_1 and i_2 agree on a neighborhood of $\mathcal{O}$ and π_1 and π_2 agree on a neighborhood of $i_1(\mathcal{O}) = i_2(\mathcal{O})$. Isomorphism of map-germs is defined in the same way as for maps. Given $\mathcal{M}$ and $\mathcal{O}$ as above, and $\pi\colon \mathcal{E} \to \mathcal{B}$ a conic G-Lagrangian fibration, let $\mathcal{L}(\mathcal{M},\mathcal{O},\pi)$ be the space of conic G-Lagrangian germs $\ell = \pi \circ i\colon \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ at $\mathcal{O}$.

A conic G-Lagrangian germ $\ell\colon \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ at $\mathcal{O} \subset \mathcal{M}$ is *stable* if it has a representative $\pi \circ i\colon U \to V \to \mathcal{B}$ with the following property. For every open set $U' \subset U$ such that $\mathcal{O} \subset U'$ and U' is $\mathbb{C}^*$- and G-invariant, there is a neighborhood N of $i|_{U'}$ in the space of conic G-Lagrangian immersions $U' \to V$ such that if $j \in N$ there exists an orbit $\mathcal{O}' \subset U'$ with $G\mathcal{O}' = \mathcal{O}'$ and ℓ isomorphic to the germ of $\pi \circ j$ at $\mathcal{O}'$.

We will be primarily interested in infinitesimal stability of Lagrangian germs (cf. [5, p. 271]).

Let $i: \mathcal{M} \to \mathcal{E}$ be a conic G-Lagrangian immersion. An *infinitesimal* (first order) *Lagrangian deformation* of i is a $\mathbb{C}^*$- and G-equivariant section u of $i^*T\mathcal{E}$ such that $i^*\mathcal{L}_u\Omega = 0$, i.e., the pullback to $\mathcal{M}$ of the Lie derivative of the symplectic form Ω with respect to (an extension of) u is zero. An *infinitesimal isomorphism* of $\mathcal{M}$ is a $\mathbb{C}^*$- and G-equivariant vector field v on $\mathcal{M}$. An *infinitesimal symplectomorphism* of $\mathcal{E}$ is a $\mathbb{C}^*$- and G-equivariant vector field w on $\mathcal{E}$ such that $\mathcal{L}_w\Omega = 0$. If $\pi: \mathcal{E} \to \mathcal{B}$ is a conic G-Lagrangian fibration, the vector field w on $\mathcal{E}$ is (infinitesimally) *fiber-permuting* if $\pi_* v = 0$ implies $\pi_*[w,v] = 0$. A conic G-Lagrangian germ $\ell: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ at $\mathcal{O} \subset \mathcal{M}$ is *infinitesimally stable* if, for every representative $\pi \circ i: U \to V \to \mathcal{B}$ and every infinitesimal Lagrangian deformation u of i, there exist a $\mathbb{C}^*$- and G-invariant open set $U' \subset U$, an infinitesimal isomorphism v of U' and an infinitesimal fiber-permuting symplectomorphism w of a neighborhood of $i(U')$ in V such that $u = i_* v + w$.

Now we reformulate infinitesimal Lagrangian stability of the germ $\ell: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ in terms of Hamiltonian functions, using the method of Arnold [1, §10]. Given an infinitesimal Lagrangian deformation u of i as above, we define a 1-form τ locally on $\mathcal{M}$ as follows. Let $\tilde{u}$ be a local extension of u to a $\mathbb{C}^*$- and G-equivariant vector field on $\mathcal{E}$. Define $\tilde{\tau}$ on $\mathcal{E}$ by $\tilde{\tau} = \Omega(\tilde{u}, \cdot)$, and let $\tau = i^*\tilde{\tau}$. The form τ is independent of the choice of extension $\tilde{u}$, and τ is G-invariant and homogeneous, i.e., $(\kappa_t)^*\tau = t\tau$. The deformation u is Lagrangian if and only if $d\tau = 0$. Two infinitesimal deformations u and u' determine the same 1-form τ if and only if $u - u'$ is tangent to $\mathcal{M}$. We conclude that, up to infinitesimal isomorphisms of $\mathcal{M}$, an infinitesimal Lagrangian deformation of i is the same as a G-equivariant homogeneous closed 1-form τ on $\mathcal{M}$ (cf. [5]).

Let w be an infinitesimal symplectomorphism of $\mathcal{E}$, and let H be a local Hamiltonian function for w, i.e., $H: \mathcal{E} \to \mathbb{C}$ and $dH = \Omega(w, \cdot)$. Then dH is G-invariant and H can be chosen so that it is homogeneous, i.e., $H(\kappa_t e) = tH(e)$. (In fact $H = \Omega(w, t)$ is a homogeneous Hamiltonian for w, where t is the infinitesimal generator of the $\mathbb{C}^*$-action.) The vector field w is fiber-permuting if and only if H is linear with respect to the canonical affine linear structure on each fiber of the Lagrangian fibration $\mathcal{E} \to \mathcal{B}$.

(1.2) <u>Proposition</u>. The conic G-Lagrangian germ $\ell = \pi \circ i: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ is infinitesimally stable if and only if every homogeneous germ $\Phi: \mathcal{M} \to \mathbb{C}$ such that $d\Phi$ is G-invariant can be written as $\Phi = H \circ i$ for some homogeneous germ $H: \mathcal{E} \to \mathbb{C}$ such that dH is G-invariant and H is affine linear on the fibers of $\pi: \mathcal{E} \to \mathcal{B}$.

<u>Proof</u>. Such a germ Φ corresponds to a closed 1-form $\tau = d\Phi$. If u is an infinitesimal deformation of i corresponding to τ, and w is the Hamiltonian vector field of H, then $u = w$ (modulo infinitesimal isomorphisms of $\mathcal{M}$) if and only if $\Phi = H \circ i$. □

Let $\mathcal{L}: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ be a conic G-Lagrangian map, let $\mathcal{O}$ be a $\mathbb{C}^*$-orbit of $\mathcal{M}$, and let

$G_{\mathcal{O}} = \{g \in G : g\mathcal{O} = \mathcal{O}\}$. Then the germ of $\mathcal{L}$ at $\mathcal{O}$ is a conic $G_{\mathcal{O}}$-Lagrangian germ. We will use the following abbreviated terminology. We say that the conic G-Lagrangian map $\mathcal{L}$ is *locally infinitesimally stable* if the germ of $\mathcal{L}$ at $\mathcal{O}$ is an infinitesimally stable conic $G_{\mathcal{O}}$-Lagrangian germ for all $\mathbb{C}^*$-orbits $\mathcal{O}$ of $\mathcal{M}$.

A G-*Legendrian map* is a G-map $L: M \to B$ of manifolds, together with a factorization $L = \pi \circ i: M \to E \to B$, where E is a contact G-manifold (the action of G on E preserves the contact structure), $i: M \to E$ is a Legendrian G-immersion, and $\pi: E \to B$ is a Legendrian G-fibration (cf. [3, ch. 20]). If E has dimension 2n−1, then M and the fibers of π have dimension n−1. G-Legendrian germs and isomorphism of G-Legendrian maps and germs are defined in the same way as for conic G-Lagrangian maps. Given a G-manifold M, a fixed point x of G, and a G-Legendrian fibration $\pi: E \to B$, let $L(M,x,\pi)$ be the space of G-Legendrian germs $\ell: M \to E \to B$ at x.

Stability and infinitesimal stability of a G-Legendrian germ are defined in the same way as for conic G-Legendrian germs, replacing the symplectic form Ω with a contact form α. A G-Legendrian germ $\ell: M \to E \to B$ at $x \in M$ is *infinitesimally stable* if, for every representative $\pi \circ i: U \to V \to B$ and every infinitesimal Legendrian deformation u of i, there exist a G-invariant open set $U' \subset U$, an infinitesimal isomorphism v of U' and an infinitesimal fiber-permuting contact transformation w of a neighborhood of $i(U')$ in V such that $u = i_* v + w$.

(1.3) <u>Example.</u> Let $M \subset \mathbb{C}^n$ be as in example (1.1), let $E = \mathbb{P}T^*\mathbb{C}^n = G(n-1, T\mathbb{C}^n)$, with involution induced by the map $x \mapsto -x$ on $\mathbb{C}^n$, and let $i': M \to E$ be defined by $i'(x) = T_x M$. Let B be the total space of the tautological quotient line bundle on $T_0^*\mathbb{C}^n = G(n-1, \mathbb{C}^n)$, and define $\pi': E \to B$ as follows. If $x \in \mathbb{C}^n$ and H is a hyperplane of $T_x\mathbb{C}^n$, then $\pi'(H,x)$ is the coset of x in $\mathbb{C}^n/H_0$, where H_0 is the translate of H to 0. Then $\Gamma' = \pi' \circ i': M \to E \to B$ is a $\mathbb{Z}/2$-Legendrian map. The composition of Γ' with the projection $B \to G(n-1, \mathbb{C}^n)$ is the Gauss map γ of M.

Let $\pi: \mathcal{E} \to \mathcal{B}$ be a conic G-Lagrangian fibration, with $\mathcal{E}$ of dimension 2n, and let $\mathcal{O} \subset \mathcal{E}$ be a $\mathbb{C}^*$ orbit with $G\mathcal{O} = \mathcal{O}$. The *dehomogenization* of π at $\mathcal{O}$ is a G-Legendrian fibration germ $\pi': E \to B$ which we now proceed to define. Let E be the orbit space of the $\mathbb{C}^*$ action, and let $e \in E$ correspond to $\mathcal{O}$. There is a unique 1-form β on $\mathcal{E}$ such that,

(1) $d\beta = \Omega$,

(2) β is homogeneous ($(\kappa_t)^*\beta = t\beta$), and

(3) $\beta(v) = 0$ for all vectors v tangent to an orbit of $\mathbb{C}^*$.

Furthermore β is G-invariant. (In fact, $\beta(w) = \Omega(t,w)$, where t is the infinitesimal generator of the $\mathbb{C}^*$ action.) Since the field of hyperplanes $B = \{\beta = 0\}$ contains the tangent spaces to the orbits of $\mathbb{C}^*$, B projects to a field of hyperplanes A on the orbit space E. This field of hyperplanes A defines a G-invariant contact structure on E.

The Legendre fibration germ $\pi': E \to B$ is defined as follows. Let B^* be the $\mathbb{C}^*$-orbit space of $\mathcal{B}$. Note that the quotient fibration $\sigma: E \to B^*$ is not Legendrian, since the dimension of the fibers of σ is n, not $n-1$. But if F is a fiber of σ, the field of hyperplanes A is transverse to F, and the intersection of A with the tangent bundle of F defines an integrable field A_F of hyperplanes on F. (The quotient map $\mathcal{E} \to E$ takes each fibre $\mathcal{F}$ of the Lagrangian fibration π isomorphically onto a fibre F of σ, and the field of hyperplanes A_F corresponds to $B_{\mathcal{F}}$, the intersection of B with the tangent bundle of $\mathcal{F}$. The distribution $B_{\mathcal{F}}$, satisfies the integrability condition $d\beta \wedge \beta = 0$, since $d\beta = \Omega$ is zero on $\mathcal{F}$.) We define a Legendrian foliation of E with leaves contained in the fibers of σ: the leaves contained in F are the integral manifolds of A_F. On a sufficiently small G-invariant neighborhood U of e in E, these leaves are the fibers of a map $\pi': U \to B$ which represents the desired G-Legendrian fibration germ at e.

Let $\mathcal{M}$ be a conic G-manifold, let $\mathcal{O}$ be a $\mathbb{C}^*$-orbit of $\mathcal{M}$ such that $G\mathcal{O} = \mathcal{O}$, and let $\pi: \mathcal{E} \to \mathcal{B}$ be a conic G-Lagrangian fibration with dehomogenization $\pi': E \to B$. Let M be the $\mathbb{C}^*$-orbit space of $\mathcal{M}$, and let $x \in M$ correspond to the orbit $\mathcal{O}$. We define a function

$$T: \mathcal{L}(\mathcal{M},\mathcal{O},\pi) \to L(M,x,\pi')$$

as follows. Given a conic G-Lagrangian germ $\ell = \pi \circ i: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$, let $T(\ell) = \pi' \circ i': M \to E \to B$, where i' is the map on $\mathbb{C}^*$-orbits induced by i. We will call $T(\ell)$ the *dehomogenization* of ℓ.

(1.4) <u>Example.</u> Let M be a smooth hypersurface through the origin in $\mathbb{C}^n$, and let $\mathcal{O} = \{a \in T_0^*\mathbb{C}^n - \{0\} \mid a(T_0M) = 0\}$. Let $\Gamma: \mathcal{M} \to \mathcal{E} \to \mathcal{B}$ be the conic $\mathbb{Z}/2$-Lagrangian map of example (1.1). The dehomogenization of the germ at $\mathcal{O}$ of Γ is the germ at 0 of the $\mathbb{Z}/2$-Legendrian map Γ' of example (1.3).

(1.5) <u>Proposition.</u> Dehomogenization $T: \mathcal{L}(\mathcal{M},\mathcal{O},\pi) \to L(M,x,\pi')$ induces a bijection of isomorphism classes. A conic G-Lagrangian germ is stable (resp. infinitesimally stable) if and only if its dehomogenization is stable (resp. infinitesimally stable).

(1.6) <u>Remark.</u> Presumably stability is equivalent to infinitesimal stability for conic G-Lagrangian maps and for G-Legendrian maps. We have not checked these assertions.

The proof of the proposition relies on a homogenization function. Let $\pi: E \to B$ be a G-Legendrian fibration, let e be a point of E, and let α be a 1-form on a neighborhood of e which is a contact form on each tangent space. The α-*homogenization* of π is the conic G-Lagrangian fibration germ $\tilde{\pi}: \mathcal{E} \to \mathcal{B}$ defined as follows. Let $\mathcal{E}$ be the symplectization of E, i.e., the set of all contact forms on the contact manifold germ E [2, p.356], with $\mathbb{C}^*$ action given by $\kappa_t(a) = ta$ and G action given by $\nu_g(a) = (\nu_{g^{-1}})^* a$. Let $\mathcal{O} \subset \mathcal{E}$ be the set of contact forms on T_eE. The symplectic form Ω on $\mathcal{E}$ is homogeneous and G-invariant. (Recall that $\Omega = d\beta$, where β is the tautological 1-form on $\mathcal{E}$.) The composition of the given G-Legendrian fibration $\pi: E \to B$ with the projection $\rho: \mathcal{E} \to E$ is a Lagrangian fibration $\pi^{\#}: \mathcal{E} \to B$; if F is a fiber of π then $\rho^{-1}(F)$ is a fiber of $\pi^{\#}$. But $\mathbb{C}^*$ acts trivially on B, and we want $\mathbb{C}^*$ to act freely on $\mathcal{B}$, so we must define the Lagrangian fibration $\tilde{\pi}: \mathcal{E} \to \mathcal{B}$ differently.

The choice of 1-form α on E defining the given contact structure determines an isomorphism $E \times \mathbb{C}^* \to \mathcal{E}$ which sends the pair (x,t) to the contact form $t\alpha_x$ on the tangent space to E at x. Thus we obtain a local coordinate function $t: \mathcal{E} \to \mathbb{C}^*$. Let X_t be the Hamiltonian vector field associated to the function t; that is, $dt(\xi) = -\Omega(X_t, \xi)$ [3, §18.2]. Using X_t we define the fibers of the Lagrangian fibration $\tilde{\pi}: \mathcal{E} \to \mathcal{B}$ as follows. Let $C_x \subset T_x\mathcal{E}$ be the span of the vector field X_t and the (n-1)-plane $(\ker(\pi^{\#})_*) \cap (\ker dt)$. Then C is a Lagrangian, and hence integrable, field of n-planes on $\mathcal{E}$, invariant under $\mathbb{C}^*$ and G. Thus there is a $\mathbb{C}^*$- and G-invariant neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that on $\mathcal{V}$ the integral manifolds of C are the fibers of a map $\tilde{\pi}: \mathcal{V} \to \mathcal{B}$ which represents the desired conic G-Lagrangian fibration germ.

Let M be a G-manifold, let x be a fixed point of G, and let $\pi: E \to B$ be a G-Legendrian fibration with α-homogenization $\tilde{\pi}: \mathcal{E} \to \mathcal{B}$. We define a function

$$S_\alpha: L(M,x,\pi) \to \mathcal{L}(M\times\mathbb{C}^*,\{x\}\times\mathbb{C}^*,\tilde{\pi})$$

as follows. Given a G-Legendrian germ $\ell = \pi \circ i: M \to E \to B$, let $\alpha_{i(x)}$ be the restriction of α to the tangent space of E at i(x). Define $\tilde{i}: M\times\mathbb{C}^* \to \mathcal{E}$ by $\tilde{i}(x,t) = t\alpha_{i(x)}$, and let $S_\alpha(\ell) = \tilde{\pi}\circ\tilde{i}: M\times\mathbb{C}^* \to \mathcal{E} \to \mathcal{B}$. We call $S_\alpha(\ell)$ the *α-homogenization* of ℓ.

Proof of (1.5). The proposition is a consequence of the following properties of the functions T and S_α, the proofs of which are straightforward:

$$T: \mathcal{L}(\mathcal{M},\mathcal{O},\pi) \to L(M,x,\pi')$$
$$S_\alpha: L(M,x,\pi') \to \mathcal{L}(M\times\mathbb{C}^*,\{x\}\times\mathbb{C}^*, \widetilde{\pi'}),$$

(1.7)
(a) $\mathcal{l}_1 \cong \mathcal{l}_2 \Rightarrow T(\mathcal{l}_1) \cong T(\mathcal{l}_2)$.
(b) $\ell_1 \cong \ell_2 \Rightarrow S_\alpha(\ell_1) \cong S_\alpha(\ell_2)$.
(c) $TS_\alpha(\ell) = \ell$.
(d) $S_\alpha T(\mathcal{l}) \cong \mathcal{l}$
(e) T and S_α are continuous.

To prove the stability part of Proposition (1.5), one actually uses (1.7)(c) and (d) for germ representatives. If V is a G-invariant open subset of E such that α is defined on E and the corresponding Lagrangian foliation on the preimage $\mathcal{V}$ of V is a fibration, then (c) holds for any representative $\pi \circ i: U \to E \to B$ such that $i(U) \subset V$. Similarly, if $\mathcal{V}$ is a $\mathbb{C}^*$- and G-invariant open subset of $\mathcal{E}$ such that the Legendrian foliation defined on its image V in E is a fibration, then (d) holds for any representative $\pi \circ i: U \to \mathcal{E} \to \mathcal{B}$ such that $i(U) \subset \mathcal{V}$. The proof of the infinitesimal stability part of (1.5) is easy. □

Proposition (1.5) reduces the classification of stable conic G-Lagrangian germs to the classification of stable G-Legendrian germs. To carry out this classification for odd conic Lagrangian germs in section 2, it will be convenient to have a Darboux coordinate description of a conic Lagrangian fibration germ.

(1.8) Proposition. Let $\pi: \mathcal{E} \to \mathcal{B}$ be a conic Lagrangian fibration germ at $\mathcal{O} \subset \mathcal{E}$. Let Ω be the symplectic form of $\mathcal{E}$. There are local coordinates (p,y,q,t) on $\mathcal{E}$ near $\mathcal{O}$ such that $\Omega = dp\wedge d(-tq) + dy\wedge dt$, the action of $\mathbb{C}^*$ is $s\cdot(p,y,q,t) = (p,y,q,st)$, and $\pi(p,y,q,t) = (q,t)$.

Proof. Let $\pi': E^{2n-1} \to B^n$ be the dehomogenization of π at $\mathcal{O}$. Given a 1-form α defining the contact structure near $\pi'(\mathcal{O})$ on E, there are local coordinates $(p,q,z) = (p_1,...,p_{n-1},q_1,...,q_{n-1},z)$ on E such that $\pi(p,q,z) = (q,z)$, and $\alpha = dz - pdq$ [3, §20]. In these coordinates, the α-homogenization of π' has the following form. The symplectization $\mathcal{E}$ of the contact manifold E has coordinates (p,q,z,t), with $\Omega = d(t\alpha) = dt\wedge dz + d(-tp)\wedge(dq)$, so that $(-tp,t,q,z)$ are Darboux coordinates for $\mathcal{E}$. The action of $\mathbb{C}^*$ is $s\cdot(-tp,t,q,z) = (-stp,st,q,z)$. The composition of π' with the projection $\rho: \mathcal{E} \to E$ is the Lagrangian fibration $\pi^\#: \mathcal{E} \to B$ given by $\pi^\#(-tp,t,q,z) = (q,z)$. Thus $(\ker(\pi^\#)_*) \cap (\ker dt)$ is parallel to the p-coordinate plane. To give coordinates for the Lagrangian fibration $\pi: \mathcal{E} \to \mathcal{B}$, we let $y = pq - z$, the Legendre transform of z. Then $\alpha = -(dy - qdp)$, and $\Omega = d(t\alpha) = dy\wedge dt + dp\wedge d(-tq)$, so that $(p,y,-tq,t)$ are Darboux coordinates on $\mathcal{E}$, and $X_t = -\partial/\partial y$. Therefore the distribution C is parallel to the (p,y)-coordinate plane, and the homogenization $\pi: \mathcal{E} \to \mathcal{B}$ of $\pi: E \to B$ is given by $\pi(p,y,-tq,t) = (-tq,t)$. □

Before adding $\mathbb{Z}/2$ symmetry to the picture, some discussion of the Darboux coordinates $(p,y,-tq,t)$ is in order. To see why the *Legendre transform* enters into homogenization, consider the reverse process of dehomogenization. If the conic Lagrangian fibration $\mathcal{E} \to \mathcal{B}$ is given in coordinates by $(p,y,-tq,t) \mapsto (-tq,t)$, with $s\cdot(p,y,-tq,t) = (p,y,-stq,st)$, then the quotient fibration is $(p,q,y) \mapsto q$. The restriction of the contact form $\alpha = -(dy - qdp)$ to a fiber $F = \{q = q_0\}$ is exact: $\alpha_F = dz$, $z = pq_0 - y$. Thus the dehomogenization of $\mathcal{E} \to \mathcal{B}$ is given by $(p,q,z) \mapsto (q,z)$, with $\alpha = dz - pdq$.

It is instructive to work out examples (1.1) and (1.3) using the coordinates $(p,y,-tq,t) \to (-tq,t)$ for the conic Lagrangian bundle $(T^*\mathbb{C}^n) - \mathcal{Z} \to (T_0^*\mathbb{C}^n) - \{0\}$. To avoid confusing p and q, let $(p,q) = (x,\xi)$. Thus $(x_1,...,x_{n-1},y)$ are the standard coordinates on $\mathbb{C}^n$, and $(\xi_1,...,\xi_{n-1})$ are the coordinates on $T^*\mathbb{C}^{n-1}$ dual to $(x_1,...,x_{n-1})$. Suppose that $M \subset \mathbb{C}^n$ is the graph of the function $y = f(x)$. Using $(x_1,...,x_{n-1})$ as coordinates on M, the Gauss map γ has a Lagrangian factorization

(1.9)
$$\gamma: \mathbb{C}^{n-1} \to T^*\mathbb{C}^{n-1} \to (\mathbb{C}^{n-1})^*,$$
$$(x) \mapsto (x, \frac{\partial f}{\partial x}) \mapsto (\frac{\partial f}{\partial x}).$$

The conic Lagrangian map Γ of example (1.1) takes the following form in the coordinates $(x,y,-t\xi,t)$:

(1.10)
$$\Gamma: (x,t) \mapsto (x, f(x), -t\frac{\partial f}{\partial x}, t) \mapsto (-t\frac{\partial f}{\partial x}, t).$$

The Legendrian map Γ' of example (1.3) takes the following form in the coordinates (x,ξ,z), with $z = x\xi - y$:

(1.11)
$$\Gamma': (x) \mapsto (x, \frac{\partial f}{\partial x}, x\frac{\partial f}{\partial x} - f(x)) \mapsto (\frac{\partial f}{\partial x}, x\frac{\partial f}{\partial x} - f(x)).$$

It can be shown that the Lagrangian structure (1.9) on the Gauss map γ depends (even up to isomorphism) on the choice of coordinates, whereas the conic Lagrangian structure (1.10) on Γ and the Legendrian structure (1.11) on Γ' are independent of coordinates, by definition. The Lagrangian structure (1.9) on γ is obtained from the conic Lagrangian structure (1.10) on Γ by *reduction*. More precisely, the symplectic manifold $T^*\mathbb{C}^{n-1}$ with symplectic form $\omega = dx \wedge d\xi$ is obtained from $T^*\mathbb{C}^n - \mathcal{Z}$ by reduction, or symplectic section and projection (cf. [15, p. 11], [3, p. 289]). For $T^*\mathbb{C}^{n-1}$ is the orbit space of the flow of the Hamiltonian field $X_t = -\partial/\partial y$ on the hypersurface $\{t = 1\}$. This reduction depends on the choice of the Hamiltonian function t, that is, on the choice of contact form α on $\mathbb{P}T^*\mathbb{C}^n = (T^*\mathbb{C}^n - \mathcal{Z})/\mathbb{C}^*$.

Next we consider $\mathbb{Z}/2$-symmetry. Let ι be the generator of $\mathbb{Z}/2$; if ν is a $\mathbb{Z}/2$ action we abbreviate ν_ι to ι.

(1.12) <u>Proposition</u>. Let $\pi: \mathcal{E} \to \mathcal{B}$ be an odd conic Lagrangian fibration germ at the $\mathbb{C}^*$-orbit $\mathcal{O} \subset \mathcal{E}$. There are local coordinates (p,y,q,t) as in (1.8) such that the action of $\mathbb{Z}/2$ is $\iota(p,y,q,t) = (-p,-y,q,-t)$.

<u>Proof</u>. First we show there is an *equivariant* 1-form α which defines the canonical contact structure on the $\mathbb{C}^*$-orbit space of $\mathcal{E}$. By hypothesis, $\mathcal{O}$ is a $\mathbb{C}^*$-orbit of $\mathcal{E}$ with $\iota\mathcal{O} = \mathcal{O}$. Since $\iota(t\cdot x) = t\cdot\iota(x)$ for $t \in \mathbb{C}^*$, there is a character $\chi: \mathbb{Z}/2 \to \mathbb{C}^*$ such that $\iota(x) = \chi(\iota)\cdot x$ for all $x \in \mathcal{O}$. Let $\varepsilon = \chi(\iota) = \pm 1$. Consider the orbit map $\rho: \mathcal{E} \to E$, and let $e = \rho(\mathcal{O})$. Define a function ψ from $\mathcal{O}$ to the set of contact forms on T_eE as follows. Let β be the homogeneous ι-invariant 1-form on $\mathcal{E}$ such that $d\beta = \Omega$ and $\beta(u) = 0$ for u tangent to a $\mathbb{C}^*$-orbit. If $x \in \mathcal{O}$, $v \in T_x\mathcal{E}$, and $w = \rho_* v$, then $\langle\psi(x),w\rangle = \beta_x(v)$. The function ψ is well-defined, and ψ is a bijection equivariant with respect to the actions of $\mathbb{C}^*$ and $\mathbb{Z}/2$. Therefore if a is a contact 1-form on T_eE, then $\iota^* a = \varepsilon a$. Let η be a 1-form on E defining the contact structure, and set $\alpha = \frac{1}{2}(\eta + \varepsilon\iota^*\eta)$. Then $\alpha_e = a \neq 0$, so α also defines the

contact structure in a neighborhood of e, and $\iota^*\alpha = \varepsilon\alpha$.

Now let $\pi': E \to B$ be the dehomogenization of $\pi: \mathcal{E} \to \mathcal{B}$. By averaging, we define local coordinates (p,q,z) in which ι acts *linearly* and such that $\alpha = dz - pdq$. Namely, let (λ,μ,ζ) be local coordinates on E centered at e such that $\alpha = d\zeta - \lambda d\mu$ and $\pi'(\lambda,\mu,\zeta) = (\mu,\zeta)$. Identify a neighborhood of e with the tangent space T_eE using the coordinates (λ,μ,ζ), and let ι_* be the linear involution of T_eE induced by ι. Then set

$$(p,q,z) = \tfrac{1}{2}\Big((\lambda,\mu,\zeta) + \iota_*(\iota(\lambda,\mu,\zeta))\Big).$$

Finally we use the fact that the conic Lagrangian fibration germ $\mathcal{E} \to \mathcal{B}$ is *odd*. Since ι acts linearly in the coordinates (p,q,z), and $\iota^*\alpha = \varepsilon\alpha$ with $\alpha = dz - pdq$, the hypersurface $V = \{z = 0\}$ of E is ι-invariant, and ι acts on the coordinate z by multiplication by ε. Consider the symplectic form $\omega = dp \wedge dq$ on V. We have $\iota^*\omega = \varepsilon\omega$, since $\omega = d(\alpha|_V)$. If $\varepsilon = +1$, the action of ι on V is *symplectic*. On the other hand, if we introduce Darboux coordinates $(p,y,-tq,t) \to (-tq,t)$ on $\mathcal{E} \to \mathcal{B}$ as in (1.8), then q gives coordinates for the $\mathbb{C}^*$-orbit space of $\mathcal{B}$, so ι acts trivially on q. Thus ι must act trivially on V, which implies that ι acts trivially on E, the $\mathbb{C}^*$-orbit space of $\mathcal{E}$, contrary to hypothesis. Therefore $\varepsilon = -1$, and $\iota^*\omega = -\omega$, i.e., the action of ι on V is *antisymplectic*. Since ι acts trivially on q, by a linear symplectic change of coordinates on (V,ω) we obtain Darboux coordinates (p,q) such that $\iota(p,q) = (-p,q)$. Thus we have coordinates $(p,y,-tq,t) \mapsto (-tq,t)$ as in (1.8) for the germ $\mathcal{E} \to \mathcal{B}$, with $\mathbb{Z}/2$ action $\iota(p,y,-tq,t) = (-p,-y,tq,-t)$. □

2. Generating families

In this section we carry out the classification of infinitesimally stable conic G-Lagrangian germs $\mathcal{L}$ in case either G is trivial or $G = \mathbb{Z}/2$ and $\mathcal{L}$ is odd. For simplicity we will state and prove results in the odd case only. Statements (2.1) through (2.12) below are also true for G trivial; the proofs are simplifications of those given for the odd case.

Let ν be an action of the finite group G on the germ $(\mathbb{C}^n,0)$, and let ν_* be the action induced on $T\mathbb{C}^n$ by ν. Let $H \subset T_0\mathbb{C}^n$ be a hyperplane such that $\nu_*H = H$. The projection $\pi: (\mathbb{P}T^*\mathbb{C}^n,H) \to (\mathbb{C}^n,0)$ is a G-Legendrian fibration germ. The action induced by ν_* on the quotient $T_0\mathbb{C}^n/H$ is multiplication by a character χ_ν. In the following discussion, we fix ν and H, and we let $\chi = \chi_\nu$.

A (G,ν)-*family* is a pair $(\mathcal{F},\eta)$, where $\mathcal{F}: (\mathbb{C}^{k+n},0) \to \mathbb{C}$ is a germ and η is a G-action on $(\mathbb{C}^{k+n},0)$, such that

(1) The projection $\rho: (\mathbb{C}^{k+n},0) \to (\mathbb{C}^n,0)$, $\rho(x,\lambda) = \lambda$, is a G-map, and