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Phase Transitions and Hysteresis

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Phase Transitions and Hysteresis

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INTRODUCTION

This volume contains the notes of the courses held at a C.I.M.E. school in Montecatini in July 1993. This was intended as an introduction to physical, analytical and numerical aspects of two classes of phenomena of high applicative interest: phase transitions and hysteresis effects.

About one century ago the macroscopic description of solid-liquid transitions led to the formulation of the (by now classical) *Stefan problem.* This is a boundary value problem for a parabolic partial differential equation. Here the thermal field and the evolution of the interface between the two phases are coupled and unknown: this is a typical example of *free boundary problem.* This model and its many generalizations have then been applied to a multitude of physical phenomena. Mathematical aspects have also been intensively studied in the last thirty years or so; see the monographs of Rubinstein [5] and Meirmanov [4], e.g., and the proceedings of the conferences on free boundary problems which have regularly been held for almost two decades. Relevant results on generalizations of the Stefan problem are dealt with in Kenmochi's and Rodrigues's contributions.

The status of hysteresis modelling is quite different.

Hysteresis can be defined as a *rate independent memory effect.* This is a property of some constitutive laws, which relate an input variable u and an output variable w. *Memory* means that at any instant t, $w(t)$ is determined by the previous evolution of u, and not just by $u(t)$. *Rate independence* means that the curves described in $\mathbf{R}^2$ by the couple (u, w) *(loops,* typically) are invariant for changes of the input rate, such as changes of the frequence, e.g..

Plasticity, ferromagnetism, ferroelectricity are among the most typical examples of hysteresis phenomena. More recently, also so called pseudo-elastic alloys were discovered, where hysteresis appears also as *shape memory;* see Müller's report. Several models have been devised by physicists and engineers to describe hysteresis; in particular, plasticity has a long tradition of mathematical studies. However, no systematic analysis of hysteresis appeared, until in 1970 a group of Russian mathematicians introduced the concept of *hysteresis operator,* and started a detailed investigation of its properties. Krasnosel'skiĭ and Pokrovskiĭ were the most active pioneers in this field, and their research is presented in the monograph [2]. Hysteresis operators are dealt with in Brokate's notes.

In the early 1980's other mathematicians began to study hysteresis phenomena, especially in connection with applications. A monograph of Mayergoyz [3] and the proceedings volume [7] appeared; at this moment the books [1] and [8] are in preparation.

There is a strict relation between phase transitions and hysteresis. For instance, in single-phase systems *supercooling* and *superheating* effects prior to phase nucleation are rate independent, and accordingly can be labelled as hysteresis phenomena. Here surface tension also plays an important role.

Here is a more mathematical example, which illustrates how hysteresis and free boundary problems can be related. The weak formulation of the classical Stefan problem problem involves the *Heaviside graph.* A hysteresis relation is easily obtained by replacing the critical value 0 by two thresholds a, b (with $a < 0 < b$), for downward and upward jumps, respectively. Results have been obtained for this problem, see [6]. This model applies to ferromagnetism more properly than to solid-liquid phase transitions.

Connections between phase transitions and hysteresis appear also by Verdi's contribution, which addresses the numerical treatment of both phenomena.

The school was and this volume is an attempt to cast a bridge between hysteresis and free boundary problems. The reasons for such an interaction are in the phenomena we deal with; but sometimes the mathematical world is moved by different dynamics.

Augusto Visintin

Bibliography

[1] M. Brokate, J. Sprekels: Hysteresis phenomena in phase transitions (Forthcoming monograph). Springer-Verlag

[2] M.A. Krasnosel'skiĭ, A.V. Pokrovskiĭ: Systems with hysteresis. Springer, Berlin 1989 (Russian ed.- Nauka, Moscow 1983).

[3] I.D. Mayergoyz: Mathematical models of hysteresis. Springer, New York 1991.

[4] A.M. Meirmanov: The Stefan problem. De Gruyter, Berlin 1992. (Russian edition: Nauka, Moscow 1986.)

[5] L.I. Rubinstein: The Stefan problem. Trans. Math. Monographs 27. American Mathematical Society, Providence, RI 1971.

[6] A. Visintin: A phase transition problem with delay. Control and Cybernetics, 11 (1982), 5–18.

[7] A. Visintin (ed.): Models of hysteresis. Proceedings of a meeting held in Trento in 1991. Longman, Harlow 1993.

[8] A. Visintin: Differential models of hysteresis (Forthcoming monograph). Springer-Verlag.

TABLE OF CONTENTS

Hysteresis Operators

Martin Brokate *

Contents

1. Introduction

Hysteresis phenomena appear in many branches of science. They usually arise because the underlying process admits more than one stable equilibrium state for certain (or for all) values of the process parameters. For example, let

$$F(x, v) = 0 \tag{1.1}$$

describe the equilibria x of a process in dependence of the parameter v. If the solution set of (1.1) forms a curve like the one in Figure 1, and if the part connecting the points A and B consists of unstable equilibria while the others are stable, then a variation $v = v(t)$ of the parameter in time leads to a relay-type hysteresis relationship $x = x(v)$ as indicated by the arrows in the figure. Nonconvex potentials and nonlinear dynamical systems give rise to numerous variants of this situation, and the tools of nonlinear PDE analysis, bifurcation and singularity (catastrophe) theory provide a lot of structural results. While hysteresis occurs regularly here, it does so rather as an accessory than as an organizing concept, and consequently its role is not emphasized (see e.g. [47], [39]).

Much more complicated hysteresis effects occur in continuum mechanics. Let us consider longitudinal vibrations of a (one-dimensional) rod. Newton's law coupled with the constitutive stress-strain relation, i.e.

$$\partial_{tt} u = \partial_x \sigma\,, \quad \sigma = \mathcal{W}[\varepsilon]\,, \quad \varepsilon = \partial_x u\,, \tag{1.2}$$

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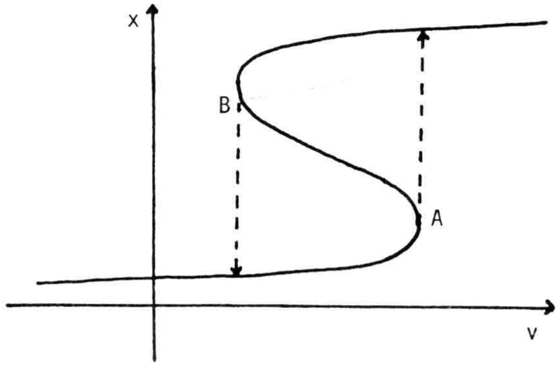

Figure 1: Hysteresis in parameter dependent processes.

together with certain initial and boundary conditions determine the displacement u, the stress σ and the strain ε as functions of time t and space x. Within the elastic range, Hooke's law

$$\mathcal{W}[\varepsilon] = E\varepsilon \tag{1.3}$$

holds, where E denotes the modulus of elasticity. Beyond the elastic limit, many materials exhibit plastic behaviour. Even its simplest description, namely the elastic-perfectly plastic model of Figure 2 with a fixed yield stress $|\sigma| = r$ and pure plastic flow, admits a

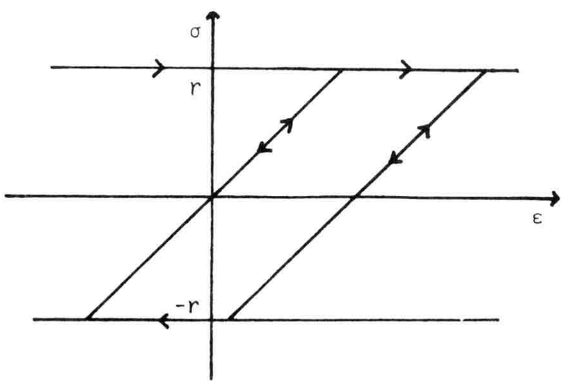

Figure 2: The elastic-perfectly plastic element.

continuum of possible stable states $\{(\sigma, \varepsilon) : \sigma \in [-r, r]\}$ for every value of ε. It will be explained in detail below how Figure 2 gives rise to a *hysteresis operator* $\mathcal{E}_r$ acting on a space of functions; accordingly, the constitutive equation

$$\sigma = \mathcal{E}_r[\varepsilon] \tag{1.4}$$

represents an equation between functions rather than between certain values of stress and strain. In 1928, Prandtl [34] constructed a more elaborate model as the continuous parallel combination of elastic-perfectly plastic elements; in operator notation, it has the form

$$\mathcal{W}[\varepsilon] = \int_0^\infty p(r)\mathcal{E}_r[\varepsilon]\,dr\,. \tag{1.5}$$

For certain types of steel at least, Prandtl's model (1.5)[1] yields a reasonable approximation in cyclic plasticity, where one investigates the so-called stabilized elastic-plastic behaviour, namely the behaviour after a large number of load cycles. We will see below how the density function p is related to the stabilized $\sigma-\varepsilon-$curve for initial loading (i.e. starting from $\sigma = \varepsilon = 0$), as well as to the shape of the hysteresis loops.

Prandtl's model allows smaller hysteresis loops embedded into larger ones with an arbitrary depth of nesting; moreover, all outer loops which have not yet been closed may affect the future behaviour. In this manner, the possible stable states of the material not only form a continuum like the interval $[-r, r]$ in the case of the elastic-perfectly plastic element $\mathcal{E}_r$, but require a potentially infinite dimensional memory representation. The same is true for the model developed by Preisach [35] in 1935 in order to describe the hysteresis loops traced out by magnetic field and magnetization in ferromagnetic materials. The operator form of the Preisach model reads

$$\mathcal{W}[v] = \int_0^{+\infty}\int_{-\infty}^{+\infty} \omega(r,s)\,\mathcal{R}_{s-r,\,s+r}[v]\,ds\,dr\,. \tag{1.6}$$

Here, ω denotes some density function like p does in Prandtl's model, and $\mathcal{R}_{x,y}$ denotes the relay with thresholds $x < y$, switching to the value $+1$ when the input attains the value y from below, and to -1 when it attains x from above. In addition, an initial value has to be prescribed for each relay.

All models mentioned so far have one aspect in common: They are *rate independent*. This means that the input-output behaviour does not depend on the speed (or frequency) of the input in the sense that, if the input frequency is doubled while its form is retained, the same is true for the output. This excludes from consideration relaxation effects like viscosity, creep or diffusion, which typically depend on the time scale. We will not enter a discussion of the relative significance of rate independent versus relaxation effects except for the remark that it varies greatly with the application. In the present notes, we will deal exclusively with rate independent models.

Continuum mechanics is not scalar, but takes place in $\mathbf{R}^3$. Although in many special situations a reduction of dimension is possible, the material laws usually are inherently multidimensional. Consequently, one expects that the memory structure has to take into account that the inputs are vectors (or tensors). However, on a level of memory complexity comparable to Prandtl's or Preisach's model, only very few mathematical investigations have been carried out so far – one has to admit, on the other hand, that the experimental basis, which could guide the selection of models, too is much less developed in the true vector case than for situations where one scalar quantity dominates the situation (for example, the tangential stress at the boundary of an interior hole of a two dimensional body). Since it fits well here, we will discuss one particular model which emphasizes memory structure, namely the continuous version of the model due to Mróz [33].

[1] It is also often named after Ishlinskiĭ [16].

The field of partial differential equations presents a particularly challenging area, because of both its difficulty and its importance for the continuum mechanics applications. We will present some results due Hilpert, Krejčí and Visintin; according to the spirit of these notes, we concentrate on the relevant properties of the hysteresis operators.

The approach to hysteresis described in these notes constitutes a mathematical technique whose goal is to analyze systems with hysteresis. A hysteresis operator results from a translation of a hysteresis diagram into a mathematical object, but it does not contribute to an explanation why the hysteresis is there at all. For that reason, hysteresis operators are said to be part of a *phenomenological approach* to hysteresis, and they offer themselves as a natural mathematical tool for a lot of problems in engineering. Nevertheless, there are also connections to the foundation of mechanics, since a hysteresis operator represents a mechanism for the dissipation of energy, if it satisfies an appropriate inequality. We will not explicitly discuss this aspect; it is, however, implicitly present in the analysis of PDE's with hysteresis in the last two sections.

These notes are lecture notes. We will not attempt to review, or even cite, all the relevant literature on the subject. For some time, the basic references have been the monograph of Krasnosel'skii and Pokrovskii [17] and the survey of Visintin [45]; now there is also the survey of Macki, Nistri and Zecca [27]. There will probably soon arrive the monograph of Visintin [46]. Concerning the special topic of optimal control of ODE systems with hysteresis, we also refer to [1]. We also will omit or abridge proofs on several occasions. They are to be found either in the references given or in the forthcoming monograph of Sprekels and the author.

2. Scalar Hysteresis Operators

Given a hysteresis diagram in the $v-w$–plane and an input function $v:[0,T]\to\mathbf{R}$, $T>0$, we want to choose an output function $w:[0,T]\to\mathbf{R}$ such that $(v(t),w(t))$ moves along the curves in the diagram. For such a procedure it is natural to require that the function v is piecewise monotone. Let us denote by $Map[0,T]$ the set of all real-valued functions on $[0,T]$, and by $M_{pm}[0,T]$ and $C_{pm}[0,T]$ the subset of all (respectively, continuous) piecewise monotone functions on $[0,T]$.

Definition 2.1 We say that an operator $\mathcal{W}:C_{pm}[0,T]\to Map[0,T]$ is a *hysteresis operator*, if it is rate independent and has the Volterra property. Rate independence means that

$$\mathcal{W}[v]\circ\varphi=\mathcal{W}[v\circ\varphi] \tag{2.1}$$

holds for all $v\in C_{pm}[0,T]$ and all continuous monotone time transformations $\varphi:[0,T]\to[0,T]$ satisfying $\varphi(0)=0$ and $\varphi(T)=T$. □

The rate independence implies that only the local extremal values of the input function v can have an influence on the memory of the process; consequently, we may replace input functions $v\in C_{pm}[0,T]$ by input strings $(v_0,\ldots,v_N)$ with $v_i\in\mathbf{R}$. Let us denote by S the set of all finite strings of real numbers,

$$S=\{(v_0,\ldots,v_N):N\in\mathbf{N}_0,\ v_i\in\mathbf{R},\ 0\le i\le N\},\quad \mathbf{N}_0:=\mathbf{N}\cup\{0\}, \tag{2.2}$$

and by S_H the set of *alternating strings*

$$S_H = \{(v_0, \ldots, v_N) : v_0 \neq v_1 \text{ if } N \geq 1\,,\ (v_{i+1} - v_i)(v_i - v_{i-1}) < 0\,,\ 0 < i < N\}\,. \quad (2.3)$$

For any hysteresis operator $\mathcal{W}$, we define its *final value mapping* $\mathcal{W}_f : S_H \to \mathbf{R}$ by

$$\mathcal{W}_f(v_0, \ldots, v_N) = \mathcal{W}[v](T)\,, \quad (2.4)$$

where $v \in C_{pm}[0,T]$ is any input function having a monotonicity partition $0 = t_0 < \ldots < t_N = T$ such that $v(t_i) = v_i$, $0 \leq i \leq N$. Conversely, any mapping $\mathcal{W}_f : S_H \to \mathbf{R}$ yields a hysteresis operator $\mathcal{W}$ if we set

$$\mathcal{W}[v](t) = \mathcal{W}_f(v(t_0), \ldots, v(t_k))\,, \quad (2.5)$$

where $0 = t_0 < \ldots < t_k = t$ is a monotonicity partition of $v_{|[0,t]}$ such that the string $(v(t_0), \ldots, v(t_k))$ is alternating. One may check from the definitions that the formulas (2.4) and (2.5) establish a bijective correspondence between the set of all hysteresis operators and the set of all real-valued mappings on S_H. Since we can use (2.4) to define $\mathcal{W}_f$ on all of S, we can interpret any hysteresis operator $\mathcal{W}$ as a mapping $\mathcal{W} : S \to S$ if we set

$$\mathcal{W}(v_0, \ldots, v_N) = (\mathcal{W}_f(v_0), \mathcal{W}_f(v_0, v_1), \ldots, \mathcal{W}_f(v_0, \ldots, v_N))\,. \quad (2.6)$$

To make a clear formal distinction, we will write $\mathcal{W}[v]$ for functions and $\mathcal{W}(s)$ for strings $s = (v_0, \ldots, v_N)$. We note also that (2.5) makes sense for inputs $v \in M_{pm}[0,T]$. In this manner, we obtain a canonical extension for any hysteresis operator from $C_{pm}[0,T]$ to $M_{pm}[0,T]$.

All scalar hysteresis operators mentioned during the introduction have a common memory structure. Its description involves the hysteresis operator which describes the mechanical play and is called the *play operator*, see Figure 3.

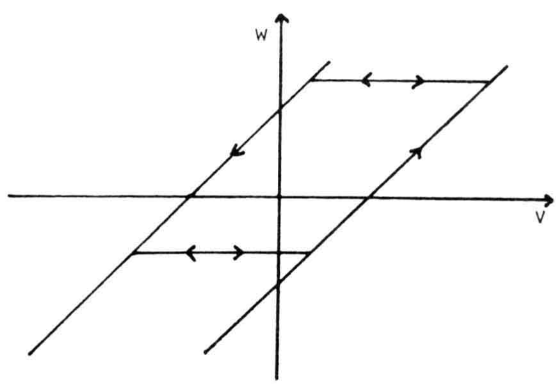

Figure 3: The play operator.

Definition 2.2 Let $r \geq 0$ be given. We define the play operator $\mathcal{F}_r[\,\cdot\,; w_{-1}]$ for the initial value[2] $w_{-1} \in \mathbf{R}$,

$$w(t) = \mathcal{F}_r[v\,; w_{-1}](t)\,, \tag{2.7}$$

by its corresponding final value mapping $\mathcal{F}_{r,f} : S_H \to \mathbf{R}$ given recursively by

$$\begin{aligned} \mathcal{F}_{r,f}(v_0) &= f_r(v_0, w_{-1})\,, \\ \mathcal{F}_{r,f}(v_0, \ldots, v_N) &= f_r(v_N, \mathcal{F}_{r,f}(v_0, \ldots, v_{N-1}))\,, \end{aligned} \tag{2.8}$$

where

$$f_r(v, w) = \max\{v - r\,, \min\{v + r, w\}\}\,. \tag{2.9}$$

If we do not state the choice of the initial value explicitly, we assume it to be zero. Accordingly, we write $\mathcal{F}_r[v]$ instead of $\mathcal{F}_r[v\,; 0]$. □

The final value mapping $\mathcal{R}_{x,y,f}$ of the relay $\mathcal{R}_{x,y}$ with thresholds $x < y$ and initial value $w_{-1}(x,y)$ has the form

$$\mathcal{R}_{x,y,f}(v_0, \ldots, v_N) = \begin{cases} 1\,, & v_N \geq y \\ -1\,, & v_N \leq x \\ \mathcal{R}_{x,y,f}(v_0, \ldots, v_{N-1})\,, & x < v_N < y\,,\ N \geq 1\,, \\ w_{-1}(x,y)\,, & x < v_N < y\,,\ N = 0\,. \end{cases} \tag{2.10}$$

If we do not state the choice of the initial value explicitly, we assume $w_{-1}(x,y) = 1$ if $x + y \leq 0$ and $w_{-1}(x,y) = -1$ otherwise.

The play operator $\mathcal{F}_r$ incorporates the memory of all relays $\mathcal{R}_{x,y}$ with $|x - y| = 2r$.

Lemma 2.3 *For each $r > 0$ and each $s \in \mathbf{R}$ there holds*

$$\mathcal{R}_{s-r,s+r,f}(v_0, \ldots, v_N) = \begin{cases} 1\,, & \mathcal{F}_{r,f}(v_0, \ldots, v_N) > s\,, \\ -1\,, & \mathcal{F}_{r,f}(v_0, \ldots, v_N) < s\,, \end{cases} \tag{2.11}$$

for every $N \geq 0$ and every string $(v_0, \ldots, v_N) \in S$.

Proof. We set $w_N = \mathcal{F}_{r,f}(v_0, \ldots, v_N)$, $\rho_N = \mathcal{R}_{s-r,s+r,f}(v_0, \ldots, v_N)$ and use induction on N. For $N = 0$, the assertion follows from the definitions. We provide the induction step $N - 1 \to N$. Assume that $v_N > v_{N-1}$. By (2.8) and (2.9) we have that

$$w_N = \max\{w_{N-1}, v_N - r\}\,. \tag{2.12}$$

According to the right hand side of (2.11), we distinguish two cases:

- If $s < w_N$, then we have either $w_N = w_{N-1}$ and $\rho_{N-1} = 1$, or $w_N = v_N - r$ and $s + r < v_N$. In both cases, $\rho_N = 1$ follows from (2.10).
- Assume that $s > w_N$. Then we have $s > w_{N-1}$ and hence $\rho_{N-1} = -1$; on the other hand, $s + r > v_N$. Together, this implies that $\rho_N = -1$.

[2] w_{-1} represents the internal state *before* $v(0)$ is applied at time $t = 0$.

□

If we insert (2.11) into the defining formula (1.6) of the Preisach operator $\mathcal{W}$, we get

$$\mathcal{W}[v](t) = \int_0^{+\infty} \int_{-\infty}^{\mathcal{F}_r[v](t)} \omega(r,s)\,ds\,dr \;-\; \int_0^{+\infty} \int_{\mathcal{F}_r[v](t)}^{+\infty} \omega(r,s)\,ds\,dr \tag{2.13}$$

for any input $v \in C_{pm}[0,T]$. Therefore, the Preisach operator can be expressed in terms of the play operator as

$$\mathcal{W}[v](t) = \int_0^{\infty} q(r,\, \mathcal{F}_r[v](t))\,dr + q_{00}\,, \tag{2.14}$$

where

$$\begin{aligned} q(r,s) &= 2\int_0^s \omega(r,\sigma)\,d\sigma\,, && (2.15)\\ q_{00} &= \int_0^{+\infty}\left(\int_{-\infty}^{0} \omega(r,\sigma)\,d\sigma \;-\; \int_0^{+\infty} \omega(r,\sigma)\,d\sigma\right) dr\,. && (2.16) \end{aligned}$$

Note that $q_{00} = 0$ if $\omega(r,s) = \omega(r,-s)$ for all r and s.

Next on the list is the elastic-plastic element $\mathcal{E}_r$. The following definition formalizes the picture in Figure 2.

Definition 2.4 Let $r \geq 0$ be given. We define the hysteresis operator $\mathcal{E}_r[\,\cdot\,;\, w_{-1}]$ for the initial value $w_{-1} \in \mathbf{R}$ by its final value mapping $\mathcal{E}_{r,f} : S_H \to \mathbf{R}$ given recursively by

$$\begin{aligned} \mathcal{E}_{r,f}(v_0) &= e_r(v_0 - w_{-1})\,,\\ \mathcal{E}_{r,f}(v_0,\ldots,v_N) &= e_r(v_N - v_{N-1} + \mathcal{E}_{r,f}(v_0,\ldots,v_{N-1}))\,, && (2.17) \end{aligned}$$

where

$$e_r(v) = \min\{r, \max\{-r, v\}\}\,. \tag{2.18}$$

Again, we assume $w_{-1} = 0$ if not stated otherwise, and write $\mathcal{E}_r[v]$ instead of $\mathcal{E}_r[v\,;\,0]$.

□

Lemma 2.5 *We have*

$$\mathcal{F}_r \;+\; \mathcal{E}_r = id\,. \tag{2.19}$$

More precisely, for every $v \in C_{pm}[0,T]$ *and every* $w_{-1} \in \mathbf{R}$ *there holds*

$$\mathcal{F}_r[v\,;\,w_{-1}] \;+\; \mathcal{E}_r[v\,;\,w_{-1}] = v\,. \tag{2.20}$$

Proof. From the identity

$$v - f_r(v,w) = e_r(v - w)\,, \tag{2.21}$$

which holds for all $v, w \in \mathbf{R}$, one easily computes that

$$\mathcal{F}_{r,f}(v_0,\ldots,v_N) \;+\; \mathcal{E}_{r,f}(v_0,\ldots,v_N) = v_N\,. \tag{2.22}$$

□

As an immediate consequence, the Prandtl operator $\mathcal{W}$ from (1.5) becomes (note that $\mathcal{F}_0 = id$)

$$\begin{aligned}\mathcal{W}[v](t) &= \int_0^\infty p(r)\,\mathcal{E}_r[v](t)\,dr \\ &= \int_0^\infty p(r)\,dr \cdot \mathcal{F}_0[v](t) - \int_0^\infty p(r)\,\mathcal{F}_r[v](t)\,dr\,. \end{aligned} \tag{2.23}$$

The representations (2.11), (2.14), (2.20) and (2.23) show that the values $(\mathcal{F}_r[v](t))_{r\geq 0}$ play a crucial role in determining the output $\mathcal{W}[v](t)$ for all hysteresis operators considered so far in this section. In fact, these values contain the whole memory information at time t needed to determine the future, i.e. to determine $\mathcal{W}[v]_{[t,T]}$ from $v_{[t,T]}$. In order to see this and to understand the memory evolution, let us consider an arbitrary input string $(v_0, \ldots, v_N) \in S$. It successively generates the memory curves $\psi_k : \mathbf{R}_+ \to \mathbf{R}$,

$$\psi_k(r) = \mathcal{F}_{r,f}(v_0, \ldots, v_k)\,, \quad 0 \leq k \leq N\,, \tag{2.24}$$

through the update

$$\begin{aligned}\psi_k(r) &= f_r(v_k, \psi_{k-1}(r)) \\ &= \max\{v_k - r, \min\{v_k + r, \psi_{k-1}(r)\}\}\,, \quad 0 \leq k \leq N\,. \end{aligned} \tag{2.25}$$

Here, the function $\psi_{-1} : \mathbf{R}_+ \to \mathbf{R}$ represents the initial memory values for the whole family $(\mathcal{F}_r)_{r\geq 0}$. Formula (2.25) shows that, as long as $\|\psi'_{k-1}\|_\infty \leq 1$, the graph of the new memory curve ψ_k consists of a straight line segment with slope $\mathrm{sign}(v_{k-1} - v_k)$ originating at the point $(0, v_k)$, and of a portion of the old memory curve ψ_{k-1}, joined at their meeting point. Consequently, if we start with a suitable initial curve ψ_{-1}, the curve ψ_k continuous and consists of at most k straight pieces of finite length with slope alternating between $+1$ and -1, and of a portion of the initial curve extending to infinity to the right. More precisely, we take ψ_{-1} from the set

$$\begin{aligned}\Psi_0 = \{\varphi : \mathbf{R}_+ \to \mathbf{R} \mid\ & |\varphi(r) - \varphi(\bar r)| \leq |r - \bar r|\,, \quad \text{for all } r, \bar r \geq 0\,, \\ & \varphi|_{[\rho,\infty)} = 0\,, \quad \text{for some } \rho \geq 0\}\,, \end{aligned} \tag{2.26}$$

which we call the *set of Preisach memory curves*. It is easy to check that $\psi_k \in \Psi_0$ for all k, if $\psi_{-1} \in \Psi_0$.

In Preisach's original paper [35] we already find a picture of the memory curve and a brief informal description of its evolution. We therefore call *operator of Preisach type* any operator whose memory structure is governed by that memory curve.

Definition 2.6 Let $\mathcal{W} : C_{pm}[0, t_E] \to Map([0, t_E])$ be a hysteresis operator. We say that $\mathcal{W}$ is of *Preisach type* if its final value map $\mathcal{W}_f : S \to \mathbf{R}$ has the form

$$\mathcal{W}_f(v_0, \ldots, v_N) = Q(\psi_f(v_0, \ldots, v_N))\,, \quad (\psi_f(v_0, \ldots, v_N))(r) = \mathcal{F}_{r,f}(v_0, \ldots, v_N)\,, \tag{2.27}$$

for some mapping $Q : \Psi_0 \to \mathbf{R}$, called the output mapping of Q, and some initial condition $\psi_{-1} \in \Psi_0$. Equivalently, we may write (2.27) as

$$\mathcal{W}[v](t) = Q(\psi(t))\,, \quad \psi(t)(r) = \mathcal{F}_r[v\,; \psi_{-1}(r)](t)\,, \quad t \in [0, T]\,, \quad r \geq 0\,. \tag{2.28}$$

If $\psi_{-1} = 0$, we call $\mathcal{W}$ a $\mathcal{P}_0$–operator.

To illustrate Definition 2.6, we specify the mapping Q for the Prandtl model (1.5) and the Preisach model (1.6). For the Prandtl model we have

$$Q(\varphi) = p_0\varphi(0) - \int_0^\infty p(r)\varphi(r)\,dr\,, \quad p_0 = \int_0^\infty p(r)\,dr\,, \tag{2.29}$$

whereas for the Preisach model we get

$$Q(\varphi) = \int_0^\infty q(r,\varphi(r))\,dr + q_{00}\,, \quad q(r,s) = 2\int_0^s \omega(r,\sigma)d\sigma\,, \tag{2.30}$$

where q_{00} is given in (2.16).

Note that the definition of the class of all Prandtl respectively Preisach operators is not quite unique, since one has to specify the class of allowed density functions (or, more generally, measures). No such element of arbitrariness is present in the definition of a $\mathcal{P}_0$-operator.

In the hysteresis diagrams considered so far, all curves are monotone, so the output $\mathcal{W}[v]$ will be monotone on any time interval where the input v is monotone. In that case, $\mathcal{W}$ maps $M_{pm}[0,T]$ into itself.

Definition 2.7 A hysteresis operator $\mathcal{W}$ is called *piecewise (strictly) monotone*, if its *level functions* $l_N : \mathbf{R} \to \mathbf{R}$ defined by

$$l_N(v) = \mathcal{W}_f(v_0, \dots, v_{N-1}, v) \tag{2.31}$$

are (strictly) increasing functions for any $N \in \mathbf{N}_0$ and any $(v_0, \dots, v_{N-1}) \in S_H$. □

The operators $\mathcal{F}_r$, $\mathcal{E}_r$ and $\mathcal{R}_{x,y}$ are obviously piecewise monotone. The Prandtl and the Preisach operator are piecewise monotone, if their densities p respectively ω are nonnegative functions - in both cases, however, the nonnegativity is not necessary and can be relaxed, see Proposition 4.8 for the Prandtl operator and Proposition 4.9 for the Preisach operator. A general hysteresis operator of Preisach type $\mathcal{W}$ is piecewise monotone, if its output mapping Q is *order preserving*, i.e. if $\varphi_1 \le \varphi_2$, to be understood pointwise, implies that $Q(\varphi_1) \le Q(\varphi_2)$. Again, this condition is not necessary, as (2.29) immediately demonstrates.

3. Continuity and Regularity

We start with a general remark. In connection with hysteresis operators, estimates of the sup norm and of the total variation are very natural, because both are compatible with the rate independence property in the sense that

$$\|W[v \circ \varphi]\|_\infty = \|W[v] \circ \varphi\|_\infty = \|W[v]\|_\infty \tag{3.1}$$

as well as

$$\mathrm{Var}\,(W[v \circ \varphi]) = \mathrm{Var}\,(W[v] \circ \varphi) = \mathrm{Var}\,(W[v]) \tag{3.2}$$

hold for the time transformations φ considered in the definition of rate independence.

Since the play operator appears here as the basic scalar hysteresis operator, it is natural to study its continuity and regularity properties first.