# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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# Differential Function Fields and Moduli of Algebraic Varieties

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Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 13N05, 14D20, 14L30

ISBN 3-540-17194-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17194-0 Springer-Verlag New York Berlin Heidelberg


Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

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## INTRODUCTION.

Our background consists of two theories each having quite classical roots namely:

A) The theory of algebraic differential equations (ADE's) with no movable singularity and

B) The Galois theory of ADE's.

The first theory was initiated by Fuchs, Poincaré, Painlevé [Poin] [Pa] and has been given modern treatments through the work of several people (for a foliation-theoretic approach see Gérard-Sec [GS] and Jouanolou [$J_1$] while for a differential algebraic approach in the one dimensional case see Matsuda [Mtd]). The second theory goes back to Picard and Vessiot and reached a very elegant and general form through the work of Kolchin [$Kol_n$] $1 \leq n \leq 3$.

The primary goal of this research monograph is to relate the two theories above; this will turn out to be profitable for both of them.

To establish the link between A) and B) the first step is to develop a higher dimensional differential algebraic version of A). None of the methods used in [GS], [$J_1$], [Mtd] seems suitable for this purpose: [GS] and [$J_1$] are too "analytic" while [Mtd] is too related to the one-dimensional case. Our approach will be quite different and will lead us beyond our "primary goal", to what we called a "differential descent theory". This theory has an interest in itself and should be viewed as an "infinitesimal" analog of Shimura-Matsusaka theory of fields of moduli [$Sh_2$], [Mtk]. Our proofs in this step will be combinations of moduli-theoretic methods (deformations of polarized algebraic varieties and compact analytic spaces) and differential algebraic methods (logarithmic derivatives on algebraic groups).

The second step in our approach will be Galois-theoretic. We shall use results proved in the first step plus Kolchin's differential Galois theory to describe in detail the interaction between A) and B). Proofs will also involve an analysis of K'/K-forms of quasi-homogenous projective varieties and some geometry of automorphisms of surfaces and abelian varieties.

The book is organized as follows.

In Chapter I we introduce our main objects and review some definitions and basic facts from differential algebra. A certain familiarity with the material in [$Kol_1$] and [Mtd] would be preferable but **is not** indispensable. An account of Kolchin's Galois theory is included.

Chapters II and III are new; they deal with the first and second steps described above respectively.

In Chapter IV we discuss the link between our theory and the classical analytic setting. Most facts presented in this Chapter are "well known to the experts" but there seems to be no suitable reference for them.

Internal references will be given by (X,y,z) or just (X,y) where X is the number of the chapter and y is the number of the paragraph; Within the same chapter we shall sometimes write (y,z) instead of (X,y,z).

Now we would like to explain in some detail our main applications; for simplicity we shall restrict ourselves to the "analytic case". So start with a region R in $\mathbb{C}^m$, let $w_1,\dots,w_m$ be coordinates in $\mathbb{C}^m$, put $\delta_j=\partial/\partial w_j$, consider the field of all meromorphic functions on R and $K\subset F$ subfields of it containing $\mathbb{C}$ such that K is relatively algebraically closed in F, F is finitely generated over K and $\delta_j(K)\subset \subset K$, $\delta_j(F)\subset F$ for $1\leq j\leq m$. Denote by $Gal_\Delta(F/K)$ the group of all K-automorphisms of F which commute with $\delta_1,\dots,\delta_m$ and call it the $\Delta$-Galois group of F/K. We shall mainly be interested here in the following three properties:

(WN) F/K is called weakly normal if $F^{Gal_{\Delta}(F/K)}=K$.

(SN) F/K is called strongly normal if there exists a connected algebraic group G over $\mathbb{C}$ and a principal homogenous space W/K for G such that W is a model for F/K and the action of G on W induces an isomorphism $Gal_{\Delta}(F/K)\simeq G(\mathbb{C})$ (=set of $\mathbb{C}$-points of G).

(NMS) F/K is said to have no movable singularity if it has a projective model V such that $\delta_j(\mathcal{O}_V)\subset\mathcal{O}_V$ for $1\leq j\leq m$.

The first definition is due to Kolchin [$Kol_2$] and is the first (and weakest) concept of normality one could think of but not much could be proved about it in general (cf. [$Kol_2$]). The second concept is also due to Kolchin [$Kol_1$], [$Kol_2$] (cf. also Białynicki-Birula [BB]); Kolchin's definition is in fact different from the one given above and the equivalence between the two definitions is a non-trivial fact (cf. [BB] or [$Kol_1$] p.430). One should say that strong normality has classical roots going back to Ehresmann's connections in principal bundles [NW]. It is related, as Kolchin's theory [$Kol_3$] shows, to the problem of "linearizing" algebraic differential equations by means of abelian functions. A lot of beautiful properties could be proved for strongly normal extensions (cf. [$Kol_n$] $1\leq n\leq 3$): a "Galois correspondence" holds for such extensions and moreover strongly normal extensions with commutative group can be described explicitely in terms of "special values" of certain automorphic functions as it happens in classical class field theory, see (IV.1). The third definition is inspired from Matsuda's book [Mtd] where the case m=tr.deg.F/K=1 was treated; it has however classical roots too going back essentially to Fuchs and Poincaré [Poin]. We should emphasize that in definition of (NMS) $\delta_j$ cannot be interpreted as vector fields on V since they do not vanish on K (except of course the case $K=\mathbb{C}$ which corresponds to the case of differential equations with constant coefficients; this will appear in our setting as the trivial case !).

One of our main results will be the following:

THEOREM (III.3.1) (SN) is equivalent to (WN) + (NMS)

Using this "geometric characterisation" of strong normality we shall prove:

THEOREM (III.4.1) (SN) is equivalent to (WN) in each of the following cases:

1) tr.deg.F/K=1 ("curve" case)

2) tr.deg.F/K=2 and $\varkappa(F/K)\geqslant 0$ ("non-ruled surface" case) and

3) tr.deg.F/K=q(F/K) and $\varkappa(F/K)\geqslant 0$ (essentially the case of abelian varieties). Here $\varkappa$=Kodaira dimension and q=irregularity.

Note that case 1) for genus 0 is due to Kolchin [$\mathrm{Kol}_2$] and was one of the starting points of our investigation; note also that condition $\varkappa(F/K)\geqslant 0$ in 2) cannot be removed as shown by an example of Kolchin (I.3.5).

Theorems above help one to get a better understanding of strong normailty. One the other hand one can prove:

THEOREM (III.2.1) If F/K has (NMS) then there exists an extension E/F such that E/K' is (SN) where K' is the algebraic closure of K in E.

Since strongly normal extensions have by Kolchin's theory an explicit description in terms of abelian functions and solutions of linear differential equations [$\mathrm{Kol}_3$](see also (IV.1)) we are led to a differential algebraic solution of Poincaré's problem [Poin] of describing the "new transcendental functions" which may appear by integrating (systems of higher order) ADE's with "no movable singularity". We would like to note that the point of view of differential algebra is here much more precise than the classical point of view of analytic foliations,as explained in (IV.2).

The author wishes to express his sincere gratitude to Professor N.Radu for introducing him to differential algebra and to L.Bădescu and C.Bănică for continous encouragement and helpful discussions.

February 18, 1986 Alexandru Buium

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CONTENTS

## CHAPTER I. PRELIMINARIES

### 1. Terminology. Basic objects.

(1.1) Throughout the paper rings will be assumed commutative with 1-element and containing the field $\mathbb{Q}$ of rationals. All schemes will be over $\mathbb{Q}$.

If $A$ is an integral domain $Q(A)$ will denote its field of quotients; if $X$ is an integral scheme $Q(X)$ will denote its field of rational functions.

By a variety $V$ over a field $K$ we will always understand a quasi-projective geometrically integral scheme over $K$; if $p \in V$ the residue field at $p$ will be denoted by $K(p)$.

If $A \longrightarrow B$ is a ring homomorphism and $M$ is a B-module we denote as usual by $\mathrm{Der}_A(B,M)$ the B-module of A-derivations of $B$ into $M$. If $A=\mathbb{Q}$ the subscript $A$ will be omitted. We will also write $\mathrm{Der}_A(B)$ instead of $\mathrm{Der}_A(B,B)$ and $\mathrm{Der}(B)$ instead of $\mathrm{Der}(B,B)$. As well known the functor $\mathrm{Der}_A(B,-)$ is representable by $\Omega_{B/A}$ = module of differentials [Ha] p.172. Now if $X \longrightarrow S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module we denote as usual by $\Omega_{X/S}$ the $\mathcal{O}_X$-module of relative differentials [Ha] p.175 and put

$$\underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{F}) := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_{X/S},\mathcal{F})$$

$$\mathrm{Der}_S(\mathcal{O}_X,\mathcal{F}) := \mathrm{Hom}_X(\Omega_{X/S},\mathcal{F}) = H^0(X, \underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{F}))$$

Remark that if $S=\mathrm{Spec}\ A$, $U=\mathrm{Spec}\ B$ is an open subset of $X$ and $\mathcal{F}|_U = \widetilde{M}$ for some B-module $M$ then

$$H^0(U, \underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{F})) = \mathrm{Der}_A(B,M)$$

We shall write $\underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X)$ instead of $\underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{O}_X)$ and $\mathrm{Der}_S(\mathcal{O}_X)$ instead of $\mathrm{Der}_S(\mathcal{O}_X,\mathcal{O}_X)$; note that $\underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X)$ is not in general a quasi-coherent sheaf because $\Omega_{X/S}$ is not in general coherent. In our applications this situation will often occur. If $S=\mathrm{Spec}\,\mathbb{Q}$ the subscripts $\mathcal{O}_S$ and $S$ will be omitted. If $S=\mathrm{Spec}\,K$ with $K$ a field and $V$ is a variety over $K$ then elements of $\mathrm{Der}_K(\mathcal{O}_V)$ will be called (global) vector fields on $V$.

We shall several times deal with a basic well known short exact sequence which we now recall. Let

$$X \xrightarrow{f} S=\mathrm{Spec}\,A \longrightarrow T=\mathrm{Spec}\,k$$

be morphisms of integral schemes with $f$ dominant, $k$ a field and $\Omega_{A/k}$ a flat $A$-module. Then there is an exact sequence

$$0 \longrightarrow f^*\Omega_{S/T} \xrightarrow{u} \Omega_{X/T} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

Indeed we have to prove that $u$ is injective; we may suppose $X=\mathrm{Spec}\,B$. Then we have a commutative diagram

$$\begin{array}{ccc}
\Omega_{A/k}\otimes_A B & \xrightarrow{\quad u \quad} & \Omega_{B/k} \\
\cap & & \\
\Omega_{A/k}\otimes_A Q(B) & & \downarrow \\
\| & & \\
\Omega_{Q(A)/k}\otimes_{Q(A)} Q(B) & \xrightarrow{\quad v \quad} & \Omega_{Q(B)/k}
\end{array}$$

with $v$ injective by separability and we are done. Note that applying $\mathrm{Hom}_X(-,\mathcal{F})$ to the above exact sequence ($\mathcal{F}$ being quasi-coherent on $X$) we get an exact sequence of $A$-modules

$$\mathrm{Der}_T(\mathcal{O}_X,\mathcal{F}) \longrightarrow \mathrm{Der}_T(\mathcal{O}_S,f_*\mathcal{F}) \xrightarrow{\rho} \mathrm{Ext}^1(\Omega_{X/S},\mathcal{F})$$

The map $\rho$ will be called the Kodaira-Spencer map associated to $X,f$ $S,T$ and $\mathcal{F}$.

(1.2) Now start with an arbitrary set $\Delta$ which we call the set of differential operators. Unlike in Kolchin's book [Kol$_1$] we do not suppose that $\Delta$ is finite; this is because some of our main applications (II.1.3),(II.3.8),(II.3.10) will involve infinite sets of (non-commuting) derivations. By a $\Delta$-ring we mean a ring A together with a map

$$\begin{array}{l} \Delta \longrightarrow \mathrm{Der}(A) \\ \delta \longmapsto \delta_A \end{array}$$

When there is no danger of confusion we write $\delta a$ instead of $\delta_A a$ for $a \in A$, $\delta \in \Delta$. Define the ring of constants

$$A^{\Delta} = \{a \in A;\ \delta a = 0 \quad \text{for all} \quad \delta \in \Delta\}$$

We say that A is a partial $\Delta$-ring if $\Delta$ is finite and $[\delta_A, \delta'_A] = 0$ for all $\delta, \delta' \in \Delta$ where $[\, ,\, ]$ denotes the Poisson bracket on Der(A). A will be called an ordinary $\Delta$-ring if $\Delta$ is reduced to one element. An ideal I in A is called a $\Delta$-ideal if $\delta(I) \subset I$ for all $\delta \in \Delta$. By a morphism of $\Delta$-rings (or simply a $\Delta$-morphism) we mean a ring homomorphism $f: A \longrightarrow B$ between $\Delta$-rings such that $f(\delta a) = \delta(f(a))$ for all $a \in A$ and $\delta \in \Delta$. When A,B are fields we say that A,B are $\Delta$-fields, $A \subset B$ (or B/A) is a $\Delta$-field extension, or an isomorphism of A into B or that A is a $\Delta$-subfield of B. Here are some basic facts about $\Delta$-field extensions:

a) If F/K is an algebraic field extension then any derivation on K uniquely extends to F hence if K is a $\Delta$-field there is a unique structure of $\Delta$-field on F such that F/K is a $\Delta$-field extension. Moreover $F^{\Delta}/K^{\Delta}$ is easily seen to be also algebraic.

b) If K is a $\Delta$-field then $K^{\Delta}$ is algebraically closed in K.

c) If F/K is a $\Delta$-field extension then $F^{\Delta}$ and K are linearly disjoint over $K^{\Delta}$ (for a proof see [BB] p.93).

Suppose now $f_1: A \longrightarrow B_1$ and $f_2: A \longrightarrow B_2$ are $\Delta$-morphisms; then there is a structure of $\Delta$-ring on $B_1 \otimes_A B_2$ making $B_1 \otimes_A B_2$ the fibred sum of $B_1$ and $B_2$ in the category of $\Delta$-rings; it is given by the formula $\delta(x\otimes y)=x\otimes\delta y+\delta x\otimes y$. The following particular case will often appear: suppose $K$ is a $\Delta$-field, $C$ is a subfield of $K^\Delta$ and $R$ is a $C$-algebra. Then for any $\delta\in\Delta$ there is a unique derivation $\delta_K^*$ on $R\otimes_C K$ satisfying $\delta_K^*(x\otimes y)=x\otimes\delta y$ for all $x\in R$ and $y\in K$; it will be called the trivial lifting of $\delta_K$ to $R\otimes_C K$.

Finally note that if $A$ is a $\Delta$-ring and $S$ is a multiplicative system in $A$ then there is a unique structure of $\Delta$-ring on $S^{-1}A$ making $A\longrightarrow S^{-1}A$ a $\Delta$-morphism.

Now one can make similar definitions for schemes instead of rings. So by a $\Delta$-scheme we mean a scheme $V$ together with a map

$$\Delta\longrightarrow \mathrm{Der}(\mathcal{O}_V)$$

$$\delta\longmapsto\delta_V$$

It will be called a partial $\Delta$-scheme if $\Delta$ is finite and the $\delta_V$'s are pairwise commuting. Analog definition for ordinary $\Delta$-scheme. A point $p\in V$ will be called a $\Delta$-point if $\delta(m_p)\subset m_p$ where $m_p$ is the maximal ideal of the local ring $\mathcal{O}_{V,p}$. We shall denote by $V_\Delta$ the set of all $\Delta$-points of $V$. A closed subscheme $W$ of $V$ will be called a $\Delta$-subscheme if $\delta(I_W)\subset I_W$ where $I_W$ is the ideal sheaf of $W$. Note that any $\Delta$-subscheme has a natural structure of $\Delta$-scheme. A morphism of $\Delta$-schemes $f:V\longrightarrow W$ (or simply a $\Delta$-morphism) will mean a morphism of schemes such that the corresponding homomorphism $\mathcal{O}_W\longrightarrow f_*\mathcal{O}_V$ induces morphisms of $\Delta$-rings. By a $\Delta$-variety $V$ over a $\Delta$-field $K$ we will mean a morphism of $\Delta$-schemes $f:V\longrightarrow \mathrm{Spec}\,K$ such that $V$ is a variety over $K$. One of the main features of the theory is that for a $\Delta$-variety $V$ the derivations $\delta_V$ are not vector fields on $V$ (since they do not vanish on $K$); see also (IV.2). If $V,W$ are $\Delta$-varieties over $K$ and $u$ is a rational map from $V$ to $W$ we say that $u$ is a $\Delta$-rational map if the morphism $V_o\longrightarrow W$ ($V_o$=locus where $u$ is defined) is