Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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V. L. Hansen (Ed.)

Differential Geometry

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Preface

During the period 29 July - 9 August 1985 a Nordic Summer School in mathematics was held at the Technical University of Denmark in Lyngby. The subject of the summer school was "Differential Geometry with Applications". The school was attended by 81 participants and had 8 main lecturers. In addition to the main lectures, there was a series of seminars by the participants.

The main reason for choosing differential geometry as the subject for the 1985 Nordic Summer School in mathematics was that the last two decades have witnessed a new strong interaction between mathematics and physics originating in the study of differential geometry of fibre bundles in mathematics and field theories in physics. For many physicists the goal is to find a theory combining all known forces of nature, and it appears that the so-called gauge theories, which in the language of differential geometry correspond to connections in fibre bundles, may provide an answer.

Any reasonable field theory is based on a variational principle, and the extremals for the associated differential geometric variational problem are the solutions searched for. Normally, this leads to problems concerning nonlinear (partial differential) equations on manifolds.

The purpose of the summer school was to present some of the important mathematical tools, methods, and results necessary for doing research within differential geometry and its above mentioned applications in theoretical physics.

The overall structure of the final program for the summer school contained 3 main themes (sections):

Gauge theory/moduli: General variational methods and other global analytic methods to describe spaces of solutions to nonlinear problems from differential geometric physics.
Main lecturers: Jean Pierre Bourguignon and Cliff Taubes.

Twistor methods: Methods based on the theory of harmonic and holomorphic maps for constructing specific solutions to nonlinear equations arising from differential geometric variational problems.
Main lecturers: Francis Burstall and John Rawnsley.

Global differential geometry: This section contained some of the foundations for the applications in the two preceding ones, in particular elements of the theory of complex manifolds (especially Kähler manifolds). However, it also contained topics of independent interest,

e.g. comparison theorems in Riemannian geometry, and partial differential equations arising in differential geometry.
Main lecturers: Robert Greene, Karsten Grove and Jerry Kazdan.

This volume contains the manuscripts for the main lectures given at the summer school, except for the lectures by Cliff Taubes.

The opening address was delivered by Jim Eells, who spoke on the physics of classical field theories. This was further supplemented in a second lecture by Eells and a lecture by Peter Braam, who spoke on the quantum aspects. On our request, Peter Braam has kindly written a manuscript, which opens this volume and serves as an introduction to the section on gauge theory.

The program for the summer school was planned in close collaboration with Karsten Grove. Also Jim Eells, John Rawnsley and Jean Pierre Bourguignon were extremely helpful advisors. In connection with the practical arrangements, Flemming Damhus Pedersen was of immense importance. Splendid work was also done by our secretarial staff, not least Lone Aagesen, who was attached to the project from the outset. In the preparation of this volume, Steen Markvorsen has been of great help to me.

The summer school was made possible through donations from Nordiska Forskarkurser, the Danish Natural Science Research Council, Otto Mønsteds Fond and the Danish Mathematical Society.

Lyngby, March 1987 Vagn Lundsgaard Hansen

Differential Geometry with Applications

Nordic Summer School 29. July - 9. August 1985

The Technical University of Denmark, Lyngby

List of participants

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Lars Smedegaard Andersen, University of Århus, Denmark.
Amir Assadi, University of Virginia, USA.

Gunnar Berg, University of Uppsala, Sweden.
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Jonas Bjerg, Technical University of Denmark.
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Cliff Taubes, University of California, USA.
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Anatol Winter, Technical University of Denmark.
Carl Aage D. Winther, Technical University of Denmark.
Christopher M. Wood, University of Southampton, England.
John C. Wood, University of Leeds, England.

Bent Ørsted, University of Odense, Denmark.

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U. Abresch	Lower curvature bounds, Toponogov's theorem and bounded topology.
A. Assadi	Some remarks on curvature, symmetry and differentiable structure.
J. Dupont	Dilogarithms as characteristic classes.
J. Gravesen	Spaces of holomorphic mappings and instantons.
H. Haahti	On linear connections given by projection valued maps.
O. Hijazi	A conformal lower bound for the smaller eigenvalue of the Dirac operator and Killing spinors.
Y. Itokawa	Around the sphere theorem - a survey.
R. Kobayashi	Moduli space of generalized K3 surfaces with a Kähler-Moišezon metric.
P.B. Kronheimer	Magnetic monopoles and multi-Taub-NUT metrics.
S. Markvorsen	Heat kernel comparison on minimal submanifolds.
E. Määttä	On connection preserving maps between vector bundles over Banach and Hilbert manifolds.
H. Pedersen	Einstein metrics, spinning top motion and monopoles.
Y.S. Poon	Self-duality on $P^2 \# P^2$.
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G. Thorbergsson	Tight immersions of 4-dimensional manifolds.
M. Ville	Pinching the sectional curvature in dimension 4.
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C.M. Wood	Harmonic sections.

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Quantum field theory: the bridge between mathematics and the physical world

by

Peter Braam

§ 0 Introduction.

These notes, based on a talk on the subject, should serve one purpose only: make mathematicians enthusiastic for quantum field theory. Thus they have been written in a heuristic rather than a completely rigorous style, in fact many topics mentioned in § 2 have yet to be rigorously formulated.

We have stressed the central role that symplectic geometry plays in the theory, and the analogy between ordinary quantum mechanics and the quantization of field theories. Therefore, in section 1 classical mechanics is rather quickly repeated, and in section 2 quantization is treated. We would like to thank Dr. P. v. Baal for long discussions on the subject.

§ 1 Mechanics and momentum maps.

The importance of mechanics lies not only in its physical origin, but also in the fact that mechanical ideas are widespread in physical quantum theories and in mathematics. To describe a mechanical system one can choose a certain setting, each with its own advantages. Here the following two are used:

1) Lagrangian formulation

Motion is a path, $c(t)$, in a configuration manifold M, and the path which is actually followed by our mechanical system is a critical point of the action functional, which is a map from the space of paths to $\mathbb{R}$, defined by:

$$S(c) = \int L(c(t), \dot{c}(t))dt \qquad 1\text{-}1$$

where $L : TM \to \mathbb{R}$ is the Lagrangean. Now L is not an arbitrary element of $C^{\infty}(TM)$ and we require it to be of the form

$$L(m, X_m) = g_m(X_m, X_m) - V(m) \qquad 1\text{-}2$$

where $g(\cdot,\cdot)$ is a Riemannian metric on M, called kinetic energy, and $V : M \to \mathbb{R}$ is the potential. The critical paths $c(t)$ are solutions of the Euler Lagrange equations:

$$\frac{\partial}{\partial t}\left\{ \frac{\partial L}{\partial \dot{q}_j}(c(t), \dot{c}(t))\right\} = \frac{\partial L}{\partial q_j}(c(t), \dot{c}(t)) \quad j = 1, \dots, n \qquad 1\text{-}3.$$

Here $(q_1, \dots, q_n)$ are local coordinates around $c(t)$ on M.

The physical meaning of the action functional may seem mysterious at this stage, but quantum mechanics sheds some light on this, see §2, and Manin [9], Ch. 3.

Before we proceed to give the Hamiltonian or symplectic formulation, we discuss an example:

Celestial mechanics. Consider n particles in $\mathbb{R}^3$ with masses m_i. The configuration space is $M = \mathbb{R}^{3n}$ and let q_{ki} be coordinates on M with $k = 1,2,3$, $i = 1, \dots, n$. Then:

$$\begin{aligned} L(q, \dot{q}) &= \sum_i \frac{1}{2} m_i \|\dot{q}_i\|^2 + \sum_{i<j} G\, m_i\, m_j \|q_i - q_j\|^{-1} \\ &= \sum_{k,i} \frac{1}{2} m_i\, q_{ki}^2 + \sum_{i<j} G\, m_i\, m_j \left(\sum_{k=1}^{3} (q_{ki} - q_{kj})^2 \right)^{-1/2} \end{aligned} \qquad 1\text{-}4$$

($G > 0$, the gravitational constant). The first term on the rhs of (1-4) clearly represents the kinetic energy, whereas the second is the gravitational potential. Most facts in mechanics were discovered solely through consideration of the equations of celestial mechanics:

$$\frac{d}{dt} m_i \dot{q}_{ki} = m_i \ddot{q}_{ki} = -\sum_{i<j} \frac{(q_{ki} - q_{kj})\, G\, m_i\, m_j}{\left(\sum_{\ell} (q_{\ell i} - q_{\ell j})^2 \right)^{3/2}}, \quad i = 1, \dots, n,$$

the Euler-Lagrange equations of (1-4).

2) Hamiltonian formulation

The advantage of Hamiltons version is that the place (q_i) and speed $(\dot{q}_j)$ variables no longer play a different role. Put differently the theory is mathematically a bit more intrinsic.

Let X be a symplectic manifold, i.e. a manifold X with a so-called symplectic form, $\omega \in \Omega^2(X)$, where ω obeys

a) $\omega_x : T_xX \times T_xX \to \mathbb{R}$ is nondegenerate

b) $d\omega = 0$.

A mechanical system is now described by a Hamiltonian or total energy function $H \in C^\infty(X)$. This Hamiltonian H determines a vectorfield V_H on X , called the Hamilton vectorfield of H , defined as follows:

$$i_{V_H}\omega = dH \ . \qquad (i_{\cdot} \text{ is contraction}) \qquad 1\text{-}5$$

Hamiltons equations now read

$$\dot{x}(t) = V_H(x(t)) \ . \qquad 1\text{-}6$$

If we put $X = T^*M$ then we can recover the Euler Lagrange equations. For convenience we work in coordinates q_i on M . The conjugate momenta:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial g(\dot{q},\dot{q})}{\partial \dot{q}_j}$$

supply coordinates on $T^*M = X$ and it is not hard to see that

$$\omega = \sum_j dp_j \wedge dq_j$$

defines (globally!) a symplectic form on X . Associated to a Lagrangian $L(q,\dot{q})$ is the Hamiltonian

$$H(q,p) = \sum_j p_j \dot{q}_j(p) - L(q, \dot{q}(p))$$

Now the Hamilton vector field is given by:

$$V_H(q,p) = \left(+\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$

so Hamiltons equations read:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \ . \end{aligned} \qquad 1\text{-}7$$

The equivalence of 1-7 and 1-3 follows easily from the definitions of p and H . A more extended exposition of Hamilton's and Lagrange's description of mechanics can be found in Abraham-Marsden [1] and Arnold [3].

Finally we discuss conserved quantities in mechanical systems in the setting of momentum maps, see also Bourguignon's seminar notes. Let G be a connected Lie group acting symplectically on (X,ω) ; this means $g^*\omega = \omega$ $(g \in G)$. Consider the vectorfields induced by the action, i.e. consider the map ($\mathfrak{g}$ is the Lie algebra og G)

$$\tilde{} : \mathfrak{g} \to \Gamma(T\,X): Y \to \tilde{Y} = \left\{ m \to \frac{d}{dt}(\exp t\,Ym)_{|t=0} \right\}$$

Then for $Y \in \mathfrak{g}$, $L_{\tilde{Y}}\,\omega = d(i_{\tilde{Y}}\,\omega) = 0$, so assuming for simplicity that $H^1(X, \mathbb{R}) = 0$, we get:

$$i_{\tilde{Y}}\,\omega = d\,\Phi_Y \quad \text{for some} \quad \Phi_Y \in C^\infty(X)$$

in other words $\tilde{Y}$ is the Hamilton vectorfield of Φ_Y . Now Φ_Y is only determined up to a constant, but with a bit more work we get a map:

$$\Phi : X \to \mathfrak{g}^* \quad \text{s.t.} \quad \langle \Phi, Y \rangle = \Phi_Y \quad , Y \in \mathfrak{g}$$

and: 1-8

$$\Phi_Y(g\cdot x) = \Phi_{Ad_{g^{-1}}Y}(x) \qquad g \in G$$

(Φ is almost unique, see Abraham-Marsden [1] Ch. 4). Φ , first defined by Lie, is called the <u>momentum map</u> of the G action, and Φ determines the complete G action by integration of the Hamilton vectorfields $\tilde{Y} = V_{\Phi_Y}$.

From 1-8 we see that the co-adjoint orbit of $\Phi(x) \in \mathfrak{g}^*$ is constant on the orbit of $x \in X$ (because $\Phi(gx) = {}^tAd_{g^{-1}}\,\Phi(x)$). If G is Abelian we get that $\Phi(x)$ is constant on Gx . In general $\Phi^{-1}\{0\}$ is a G invariant set and

$$\Phi^{-1}\{0\} / G \qquad 1\text{-}9$$

is called the <u>symplectic quotient</u> or <u>Marsden-Weinstein</u> reduction of X : $\Phi^{-1}\{0\} / G$ is symplectic itself.

<u>Example:</u> Given a Hamiltonian H on (X,ω) whose flow is complete. Here $\mathbb{R}$ acts symplectically by:

$$(t,x) \to \phi(t,x) \in X$$

where $\phi(t,\ )$ is the flow of V_H . Clearly for $1 \in \mathfrak{g} = \mathbb{R}$ we have $\tilde{1} = V_H$ so

$$\Phi : X \to \mathbb{R}^* : x \to H(x)$$

is constant on the $\mathbb{R}$ orbits. Here we recovered <u>conservation of energy</u>.

§ 2 Classical field theories and their quantization.

In §1 classical mechanics was presented. Here our interest lies in the quantization of this, leading to quantum mechanics. Next we introduce classical field theories, more specifically gauge theories. These can be quantized just as classical mechanics, but the original, classical, Hamiltonian system is now infinite dimensional. The theory has not yet found a mathematically mature framework; our considerations will be heuristic.

To quantize classical mechanics we give a scheme of geometric quantization (see Abraham-Marsden [1] 5.4). The arrows $\longleftrightarrow$ mean something like "corresponds to".

	Classical mechanics		Quantum mechanics
(i)	Points $(q,p) \in X = T^*\mathbb{R}^k$	$\longleftrightarrow$	Functions ϕ in Hilbert space $L^2(\mathbb{R}^k)$, called wave functions.
	More precisely in symplectic terms:		
	Points in a symplectic manifold (X, ω)	$\longleftrightarrow$	Sections of a $\mathbb{C}$ - line bundle L with connection ∇, which are cov. constant on the leaves of a Lagrangean foliation of X.

This process is called geometric quantization.

(ii)	Certain functions on X	$\longleftrightarrow$	Self adjoint operators on $L^2(\mathbb{R}^k)$ called observables.
	Example:		
	place functions q_j	$\longleftrightarrow$	$\phi(x) \longrightarrow x_j\,\phi(x)$
	momentum " p_j	$\longleftrightarrow$	$\phi(x) \longrightarrow i\hbar \dfrac{\partial\phi}{\partial x_j}$

Here $\hbar$ is a small positive number called Planck's constant. The limit $\hbar \to 0$ should describe classical mechanics.

(iii) paths $q(t), p(t)$ $\longleftrightarrow$ A function $\phi(x,t)$ on $\mathbb{R}\times\mathbb{R}^k$ s.t. $\phi(-,t) \in L^2(\mathbb{R}^k)$.

initial values $p(0), q(0)$ $\longleftrightarrow$ A function $\phi(-,t) \in L^2(\mathbb{R}^k)$

Hamiltons equations in a potential $V(x)$ $\longleftrightarrow$ a) Schrödingers equation:

$$i\hbar \frac{\partial\phi}{\partial t} = \hbar^2 m \Delta\phi + V(x)\phi$$

b) The propagator or Greens function K defined by:

$$\phi(x,t) = \int_{\mathbb{R}^k} K(x,t;y,0)\,\phi(y,0)\,dy$$

a) and b) are equivalent: see below.

So far our scheme. The idea of quantum mechanics is that physics is described better using the quantum theory than the classical theory. The wave function $\phi(x,t)$ is supposed to describe the system, not really the states of the system; these aren't so directly accessible. Information of a probabilistic kind about the states can be obtained through the observables, e.g.:

(i) Expectation values of coord. q_j at time t is:

$$\int_{\mathbb{R}^k} \phi(x,t)\, x_j\, \overline{\phi(x,t)}\, dx = \langle \phi, Q_j\, \phi \rangle_{L^2} .$$

(ii) Heisenberg's uncertainty relation: (a consequence of the set up)

$$\begin{pmatrix}\text{variance in observed}\\ \text{values of } q_j\end{pmatrix} \cdot \begin{pmatrix}\text{variance in observed}\\ \text{values of } p_j\end{pmatrix} \geq \hbar .$$

Note that in particular (ii) shows a drastic difference with classical physics. Further details of quantum mechanics can be found in Dirac's book [5].

Before we turn to classical field theories we discuss the propagator, following Lee [8] ch. 19. The propagator is of course the heart of quantum mechanics because it describes the dynamics.

First write Schrödinger's equation in the form:

$$\frac{\partial\phi}{\partial t} = \frac{1}{\hbar i} H(\phi) \quad , \quad H(\phi) = \hbar^2 m \Delta\phi + V(x)\phi .$$