PROBLEMS AND PROPOSITIONS IN ANALYSIS

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PREFACE

Solving problems is an essential activity in the study of mathematics. Instructors pose problems to define scope and content of knowledge expected of their students; mathematical competitions and written qualifying examinations are designed to test the participant's ability and ingenuity in solving unusual problems. Moreover, it is a familiar fact of mathematical instruction that a single good problem can awaken a dormant mind more readily than highly polished lectures do.

This book contains problems with solutions and the reader is invited to produce additional solutions. To ensure a wide appeal I have concentrated on basic matters of real analysis and have consulted problem sections in various mathematical journals and the collected works of some great mathematicians.

Dr. John Abramowich aroused my interest to write this book and Dr. Edward L. Cohen has encouraged me throughout the project; I am grateful to both these personal friends. I am pleased to express my gratitude to Mrs. Wendy M. Coutts, my technical typist, for her fine work and to the administration of the University of Ottawa for the generous support that I have enjoyed in connection with this and two other book writing projects. My warmest thanks are due to my family.

Gabriel Klambauer

CONTENTS

CHAPTER 1

ARITHMETIC AND COMBINATORICS

PROBLEM 1. Let A and B denote positive integers such that $A > B$. Suppose, moreover, that A and B expressed in the decimal system have more than half of their digits on the left-hand side in common. Show that

$$\sqrt[p]{A} - \sqrt[p]{B} < \frac{1}{p}$$

holds for $p = 2,3,4,\ldots$

Solution. Since

$$\frac{x^p - y^p}{x - y} = x^{p-1} + x^{p-2}y + \cdots + y^{p-1} > py^{p-1}$$

for $y < x$, we obtain, on setting $x^p = A$ and $y^p = B$,

$$\sqrt[p]{A} - \sqrt[p]{B} < \frac{1}{p}\sqrt[p]{\frac{(A - B)^p}{B^{p-1}}}.$$

Let k be the number of digits of $A - B$. Then B has at least $2k + 1$ digits and so $A - B < 10^k$ and $B > 10^{2k}$. Thus

$$\frac{(A - B)^p}{B^{p-1}} < \frac{10^{pk}}{10^{(2p-2)k}} = \frac{1}{10^{(p-2)k}} \leq 1$$

because p is at least equal to 2. Thus $\sqrt[p]{A} - \sqrt[p]{B} < 1/p$.

PROBLEM 2. Show that, for $n = 1,2,3,\ldots,$

$$\left\{1 + \frac{1}{n}\right\}^n < \left\{1 + \frac{1}{n+1}\right\}^{n+1} \quad \text{and} \quad \left\{1 + \frac{1}{n}\right\}^{n+1} > \left\{1 + \frac{1}{n+1}\right\}^{n+2}.$$

Solution. Since, for $0 \le a < b$,

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n \quad \text{or} \quad b^n[(n+1)a - nb] < a^{n+1},$$

setting $a = 1 + 1/(n+1)$ and $b = 1 + 1/n$ we obtain the first inequality. Note also, taking $a = 1$ and $b = 1 + 1/(2n)$, we get

$$\left\{1 + \frac{1}{2n}\right\}^n \frac{1}{2} < 1 \quad \text{or} \quad \left\{1 + \frac{1}{2n}\right\}^{2n} < 4.$$

To verify the second inequality, we observe that, for $0 \le a < b$,

$$\frac{b^{n+1} - a^{n+1}}{b - a} > (n+1)a^n;$$

taking $a = 1 + 1/(n+1)$ and $b = 1 + 1/n$ yields

$$\left\{1 + \frac{1}{n}\right\}^{n+1} > \left\{1 + \frac{1}{n+1}\right\}^{n+2} \left[\frac{n^3 + 4n^2 + 4n + 1}{n(n+2)^2}\right].$$

But the term in square brackets is at least 1.

PROBLEM 3. Show that, for $n = 1,2,3,\ldots,$

$$\left\{1 + \frac{1}{n}\right\}^n < 3.$$

Solution. Since

$$\left\{1 + \frac{1}{n}\right\}^n = 1 + \binom{n}{1}\frac{1}{n} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \binom{n}{n}\frac{1}{n^n},$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \quad \text{with} \quad k! = 1\cdot2\cdot3\cdots(k-1)k,$$

and, for $2 \le k \le n$,

$$\binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \le \frac{1}{2\cdot3\cdots k} \le \frac{1}{2^{k-1}},$$

we have, for $n \ge 2$,

$$\left\{1 + \frac{1}{n}\right\}^n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

PROBLEM 4. For $n = 3,4,5,\ldots$, show that

$$\sqrt{n} < \sqrt[n]{n!} < \frac{n+1}{2}.$$

Solution. We begin by showing that $(n!)^2 > n^n$ for $n = 3,4,5,\ldots$ Consider

$$(n!)^2 = [1\cdot n][2(n-1)][3(n-2)]\cdots[(n-1)2][n\cdot 1].$$

But the first and the last factors in square brackets are equal and are less than the other factors in square brackets because, for $n - k > 1$ and $k > 0$ we have

$$(k+1)(n-k) = k(n-k) + (n-k) > k\cdot 1 + (n-k) = n.$$

Thus $(n!)^2 > n^n$ which is equivalent with $\sqrt{n} < \sqrt[n]{n!}$.

To verify that

$$n! < \left\{\frac{n+1}{2}\right\}^n \quad \text{for } n = 2,3,4,\ldots,$$

we first note that

$$\left\{\frac{n+2}{n+1}\right\}^{n+1} = \left\{1 + \frac{1}{n+1}\right\}^{n+1} > 2 \quad \text{for } n = 1,2,3,\ldots$$

by the first inequality in Problem 2. Thus

$$\frac{\left\{\frac{n+2}{2}\right\}^{n+1}}{\left\{\frac{n+1}{2}\right\}^n} = \left\{\frac{n+2}{n+1}\right\}^{n+1} \frac{n+1}{2} > n+1$$

or

$$\frac{(n+1)^{n+1}}{2^n} < \left\{\frac{n+2}{2}\right\}^{n+1} \quad \text{for } n = 1,2,3,\ldots$$

We now proceed by induction: if

$$n! < \left\{\frac{n+1}{2}\right\}^n$$

holds, then

$$n!(n + 1) < \frac{(n + 1)^{n+1}}{2^n}$$

follows. But we have shown already that

$$\frac{(n + 1)^{n+1}}{2^n} < \left\{\frac{n + 2}{2}\right\}^{n+1} \qquad \text{for } n = 1,2,3,\ldots$$

Hence

$$(n + 1)! < \left\{\frac{n + 2}{2}\right\}^{n+1}.$$

However

$$n! < \left\{\frac{n + 1}{2}\right\}^n$$

is obviously true for n = 2.

Remark. For a generalization of the result in Problem 4 see Problem 5 in Chapter 2.

PROBLEM 5. Show that

$$\sqrt{2} > \sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} > \cdots > \sqrt[n]{n} > \sqrt[n+1]{n + 1} > \cdots$$

Solution. We have

$$\frac{\sqrt[n+1]{n + 1}}{\sqrt[n]{n}} = \sqrt[n(n+1)]{\frac{(n + 1)^n}{n^{n+1}}} = \sqrt[n(n+1)]{\left(1 + \frac{1}{n}\right)^n \frac{1}{n}} < \sqrt[n(n+1)]{\frac{3}{n}}$$

(see Problem 3). But, for $n \geq 3$, we have $3/n \leq 1$. Therefore

$$\frac{\sqrt[n+1]{n + 1}}{\sqrt[n]{n}} < 1 \qquad \text{for } n \geq 3.$$

PROBLEM 6. Show that

$$2 > \sqrt{3} > \sqrt[3]{4} > \sqrt[4]{5} > \cdots > \sqrt[n-1]{n} > \sqrt[n]{n + 1} > \cdots$$

Solution. We have to show that

$$\frac{\sqrt[n]{n+1}}{\sqrt[n-1]{n}} < 1 \qquad \text{for } n = 2,3,4,\ldots$$

But we have, by Problem 3,

$$\sqrt[n(n-1)]{\frac{(n+1)^{n-1}}{n^n}} = \sqrt[n(n-1)]{\left[1+\frac{1}{n}\right]^{n-1}\frac{1}{n}} = \sqrt[n(n-1)]{\left[1+\frac{1}{n}\right]^{n}\frac{1}{n+1}}$$

$$= \sqrt[n(n-1)]{\frac{3}{n+1}} < 1.$$

PROBLEM 7. Show that the number M which in the decimal system is expressed by means of 91 unities is a composite number, that is, $M = K \cdot L$, where K and L are integers different from 1.

Solution. Since

$$M = 1 + 10 + 10^2 + \cdots + 10^{90} = \frac{10^{91} - 1}{10 - 1}$$

and

$$\frac{10^{91} - 1}{10 - 1} = \frac{(10^7)^{13} - 1}{10^7 - 1}\,\frac{10^7 - 1}{10 - 1} = \frac{(10^{13})^7 - 1}{10^{13} - 1}\,\frac{10^{13} - 1}{10 - 1}$$

we note that the claim is true because

$$\frac{(10^7)^{13} - 1}{10^7 - 1},\quad \frac{10^7 - 1}{10 - 1} \quad \text{and} \quad \frac{(10^{13})^7 - 1}{10^{13} - 1},\quad \frac{10^{13} - 1}{10 - 1}$$

are integers as can be seen from the identity

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + y^{n-1} \qquad \text{for } x \neq y.$$

PROBLEM 8. Let n be a positive integer. Show that n^k ($k \geq 2$ an integer) can be represented as a sum of n successive odd numbers.

Solution. We have to verify that for n and k as given we can find an integer s such that

$$(2s + 1) + (2s + 3) + \cdots + (2s + 2n - 1) = n^k.$$

But the expression on the left-hand side of the last equation equals $(2s + n)n$. It therefore remains to prove that it is possible to find an integer s such that

$$(2s + n)n = n^k, \quad \text{that is,} \quad s = \frac{n(n^{k-2} - 1)}{2}.$$

But n can be either even or odd. In both cases s will be an integer, however.

PROBLEM 9. Show that a sum of positive integers in the decimal system is divisible by 9 if and only if the sum of all digits of those numbers is divisible by 9.

Solution. First we observe that the difference between a positive integer a and the sum s of its digits is divisible by 9; this is clear by noting that

$$a = C_0 + C_1 \cdot 10 + C_2 \cdot 10^2 + C_3 \cdot 10^3 + \cdots$$

$$s = C_0 + C_1 + C_2 + C_3 + \cdots$$

and thus

$$a - s = C_1 \cdot 9 + C_2 \cdot 99 + C_3 \cdot 999 + \cdots.$$

Next, let $a_1, a_2, \ldots, a_n$ denote positive integers and $s_1, s_2, \ldots, s_n$ the respective sums of their digits. In the identity

$$a_1 + a_2 + \cdots + a_n = [(a_1 - s_1) + (a_2 - s_2) + \cdots + (a_n - s_n)] + (s_1 + s_2 + \cdots + s_n)$$

the component on the right-hand side in square brackets is divisible by 9 because $a_k - s_k$ (for $k = 1,2,\ldots,n$) is divisible by 9, as we know already. Consequently, $a_1 + a_2 + \cdots + a_n$ is divisible by 9 if and only if we have that $s_1 + s_2 + \cdots + s_n$ is divisible by 9.

PROBLEM 10. Let a and h be real numbers, and n some positive integer. We introduce the notation:

$$a^{n|h} = a(a - h)(a - 2h)\cdots[a - (n - 1)h];$$

when n = 0 we define $a^{0|h} = 1$. Thus, in particular, $a^{n|0} = a^n$, $a^{1|h} = a$. Moreover, we set

$$\binom{n}{0} = 1, \quad \binom{n}{k} = \frac{n!}{k!(n - k)!} \quad \text{for } k = 1,2,\dots,n.$$

Prove that

$$(a + b)^{n|h} = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)|h} b^{k|h}.$$

The foregoing result is called the *Factorial Binomial Theorem*; it contains the ordinary *Binomial Theorem* as a special case (when h = 0).

Solution. We prove the claim by induction on n. When n = 0 both sides of the formula reduce to 1, and therefore the claim is true in that case. Now suppose that the claim is true for some integer n ≥ 0, that is,

$$(a + b)^{n|h} = a^{n|h} + \binom{n}{1} a^{(n-1)|h} b^{1|h} + \binom{n}{2} a^{(n-2)|h} b^{2|h} + \cdots + b^{n|h} \tag{E.1}$$

is valid. We must then show that the claim is also true for n + 1. To do this, we multiply both sides of equation (E.1) by a + b - nh. On the left-hand side we obtain $(a + b)^{(n+1)|h}$, as can be seen directly from the definition. On the right-hand side we obtain a sum whose k th term (where k runs from 0 to n) is

$$\binom{n}{k} a^{(n-k)|h} b^{k|h} (a + b - nh)$$

$$= \binom{n}{k} a^{(n-k)|h} b^{k|h} [a - (n-k)h] + \binom{n}{k} a^{(n-k)|h} b^{k|h} (b - kh)$$

$$= \binom{n}{k} a^{(n-k+1)|h} b^{k|h} + \binom{n}{k} a^{(n-k)|h} b^{(k+1)|h}.$$

Summing over k and using the relation

$$\binom{n}{k - 1} + \binom{n}{k} = \binom{n + 1}{k}$$

we obtain

$$\binom{n}{0} a^{(n+1)|h} + \binom{n}{1} a^{n|h} b^{1|h} + \cdots + \binom{n}{n} a^{1|h} b^{n|h}$$

$$+ \binom{n}{0} a^{n|h} b^{1|h} + \cdots + \binom{n}{n-1} a^{1|h} b^{n|h} + \binom{n}{n} b^{(n+1)|h}$$

$$= a^{(n+1)|h} + \binom{n+1}{1} a^{n|h} b^{1|h} + \binom{n+1}{2} a^{(n-1)|h} b^{2|h}$$

$$+ \cdots + b^{(n+1)|h}. \qquad \text{(E.2)}$$

We have therefore shown that $(a + b)^{(n+1)|h}$ is equal to the expression in (E.2), which is precisely the statement of the claim for $n + 1$.

PROBLEM 11. Use the result in Problem 10 to evaluate the following sums:

(a) $\binom{n}{0}\binom{m}{j} + \binom{n}{1}\binom{m}{j-1} + \binom{n}{2}\binom{m}{j-2} + \cdots + \binom{n}{j}\binom{m}{0}$,

(b) $\binom{m}{0}\binom{n}{j} - \binom{m+1}{1}\binom{n}{j-1} + \binom{m+2}{2}\binom{n}{j-2} - \cdots + (-1)^j \binom{m+j}{j}\binom{n}{0}$;

here $j \le \min\{m,n\}$ in Part (a), and $j \le n$ in Part (b).

Solution. Since

$$\binom{n}{k} = \frac{n(n - 1)\cdots(n - k + 1)}{k!} = \frac{n^{k|1}}{k!}$$

we obtain

$$\binom{n}{k}\binom{m}{j-k} = \frac{n^{k|1}}{k!}\frac{m^{(j-k)|1}}{(j - k)!} = \frac{1}{j!}\frac{j!}{k!(j - k)!} n^{k|1} m^{(j-k)|1}$$

$$= \frac{1}{j!}\binom{j}{k} n^{k|1} m^{(j-k)|1}.$$

Hence the sum to be evaluated in Part (a) equals

$$\frac{1}{j!}\left\{\binom{j}{0} m^{j|1} + \binom{j}{1} m^{(j-1)|1} n^{1|1} + \binom{j}{2} m^{(j-2)|1} n^{2|1}\right.$$

$$\left. + \cdots + \binom{j}{j} n^{j|1}\right\}$$

$$= \binom{m + n}{j}.$$

To evaluate the sum in Part (b), we observe that

$$(-1)^k\binom{m+k}{k}\binom{n}{j-k} = (-1)^k \frac{(m+k)^{k|1}}{k!}\,\frac{n^{(j-k)|1}}{(j-k)!}$$

$$= \frac{1}{j!}\binom{n}{k}\,(-1)^k(m+k)^{k|1}\,n^{(j-k)|1}.$$

But

$$(-1)^k(m+k)^{k|1} = (-1)^k(m+k)(m+k-1)\cdots(m+1)$$

$$= (-m-1)(-m-2)\cdots(-m-k+1)(-m-k)$$

$$= (-m-1)^{k|1}.$$

Thus

$$(-1)^k\binom{m+k}{k}\binom{n}{j-k} = \frac{1}{j!}\binom{j}{k}\,(-m-1)^{k|1}\,n^{(j-k)|1}$$

and so our sum equals

$$\frac{1}{j!}\left\{\binom{j}{0}n^{j|1} + \binom{j}{1}n^{(j-1)|1}(-m-1)^{1|1} + \binom{j}{2}n^{(j-2)|1}(-m-1)^{2|1}\right.$$

$$\left. + \cdots + \binom{j}{j}(-m-1)^{j|1}\right\}$$

$$= \frac{(n-m-1)^{j|1}}{j!} = \frac{(n-m-1)(n-m-2)\cdots(n-m-j)}{j!}.$$

If $n - m - 1 \geq j$, this equals $\binom{n-m-1}{j}$.

Observe that since $\binom{m+k}{k} = \binom{m+k}{m}$ we can rewrite this identity in the form

$$\binom{m}{m}\binom{n}{j} - \binom{m+1}{m}\binom{n}{j-1} + \binom{m+2}{m}\binom{n}{j-2} - \cdots + (-1)^j\binom{m+j}{m}\binom{n}{0}$$

$$= \frac{(n-m-1)(n-m-2)\cdots(n-m-j)}{j!}.$$

PROBLEM 12. Let m and j be positive integers and $j \leq m$. Put

$$(m,j) = \frac{(1-x^m)(1-x^{m-1})\cdots(1-x^{m-j+1})}{(1-x)(1-x^2)\cdots(1-x^j)}.$$

Show the following results (due to Gauss):

(i) $(m,j) = (m,m-j)$;

(ii) $(m,j+1) = (m-1,j+1) + x^{n-j-1}(m-1,j)$;

(iii) $(m,j+1) = (j,j) + x(j+1,j) + x^2(j+2,j) + \cdots + x^{m-j-1}(m-1,j)$;

(iv) (m,j) is a polynomial in x;

(v) $1 - (m,1) + (m,2) - (m,3) + \cdots + (-1)^m(m,m)$

$$= \begin{cases} (1-x)(1-x^2)\cdots(1-x^{m-1}) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Solution. Part (i) is clear from the fact that (m,j) equals

$$\frac{(1-x)(1-x^2)\cdots(1-x^{m-j})(1-x^{m-j+1})\cdots(1-x^{m-1})(1-x^m)}{(1-x)(1-x^2)\cdots(1-x^j)(1-x)(1-x^2)\cdots(1-x^{m-j})}.$$

To obtain Part (ii) we note that

$$(m,j+1) = (m-1,j+1)\,\frac{1-x^m}{1-x^{m-j-1}} = (m-1,j+1)\left[1 + x^{m-j-1}\,\frac{1-x^{j+1}}{1-x^{m-j-1}}\right]$$

$$= (m-1,j+1) + x^{m-j-1}(m-1,j).$$

To prove Part (iii) we make use of the result in Part (ii) and get

$$(m,j+1) = (m-1,j+1) + x^{m-j-1}(m-1,j),$$

$$(m-1,j+1) = (m-2,j+1) + x^{m-j-2}(m-2,j),$$

$$\cdots$$

$$(j+2,j+1) = (j+1,j+1) + x(j+1,j),$$

$$(j+1,j+1) = (j,j).$$

Adding these equalities termwise, we find

$$(m,j+1) = (j,j) + x(j+1,j) + \cdots + x^{m-j-1}(m-1,j).$$

To verify Part (iv) we observe that

$$(m,1) = \frac{1-x^m}{1-x} = 1 + x + x^2 + \cdots + x^{m-1}$$

and so $(m,1)$ is a polynomial in x for any positive integer m. Assuming that

(m,j) is a polynomial in x for $k \le j$, we get by Part (iii) that $(m,j+1)$ is also a polynomial in x and so the claim follows by induction.

We introduce the notation

$$f(x,m) = 1 - (m,1) + (m,2) - (m,3) + \cdots + (-1)^m(m,m)$$

to prove Part (v). Since

$$\begin{aligned}
1 &= 1,\\
(m,1) &= (m-1,1) + x^{m-1},\\
(m,2) &= (m-1,2) + x^{m-2}(m-1,1),\\
(m,3) &= (m-1,3) + x^{m-3}(m-1,2),\\
&\cdots\\
(m,m-1) &= (m-1,m-1) + x(m-1,m-2),\\
(m,m) &= (m-1,m-1),
\end{aligned}$$

we get, upon multiplying these equalities successively by ± 1 and adding,

$$f(x,m) = (1 - x^{m-1}) - (m-1,1)(1 - x^{m-2}) + (m-1,2)(1 - x^{m-3})$$
$$+ \cdots + (-1)^{m-2}(m-1,m-2)(1 - x).$$

But

$$(1 - x^{m-2})(m-1,1) = (1 - x^{m-1})(m-2,1),$$
$$(1 - x^{m-3})(m-1,2) = (1 - x^{m-1})(m-2,2),$$
$$\cdots$$

Therefore

$$f(x,m) = (1 - x^{m-1})\left\{1 - (m-2,1) + (m-2,2) - \cdots + (-1)^{m-2}(m-2,m-2)\right\}$$
$$= (1 - x^{m-1})f(x,m-2).$$

Thus

$$f(x,m) = (1 - x^{m-1})f(x,m-2),$$
$$f(x,m-2) = (1 - x^{m-3})f(x,m-4),$$
$$\cdots$$

We first assume that m is an even number. We get

$$f(x,m) = (1 - x^{m-1})(1 - x^{m-3})\cdots(1 - x^3)f(x,2).$$

But

$$f(x,2) = 1 - (2,1) + (2,2) = 2 - \frac{1 - x^2}{1 - x} = 1 - x.$$

This shows that

$$f(x,m) = (1 - x^{m-1})(1 - x^{m-3})\cdots(1 - x^3)(1 - x)$$

when m is even.

Finally, when m is odd,

$$f(x,m) = (1 - x^{m-1})(1 - x^{m-3})\cdots(1 - x^2)f(x,1).$$

But $f(x,1) = 0$, consequently $f(x,m) = 0$ for any odd number m.

PROBLEM 13. Show the following result (due to Euler):

$$(1 + xz)(1 + x^2z)\cdots(1 + x^nz) = F(n),$$

where

$$F(n) = 1 + \sum_{k=1}^{n} \frac{(1 - x^n)(1 - x^{n-1})\cdots(1 - x^{n-k+1})}{(1 - x)(1 - x^2)\cdots(1 - x^k)} x^{\frac{k(k+1)}{2}} z^k.$$

Solution. A straightforward calculation shows that

$$F(n+1) - F(n) = zx^{n+1}F(n),$$

that is,

$$F(n+1) = (1 + zx^{n+1})F(n).$$

Therefore

$$F(n) = (1 + zx^n)F(n-1),$$

$$F(n-1) = (1 + zx^{n-1})F(n-2),$$

...

$$F(3) = (1 + zx^3)F(2),$$

$$F(2) = (1 + zx^2)F(1),$$

$$F(1) = 1 + zx.$$

However, these equalities imply the desired result

$$F(n) = (1 + xz)(1 + x^2 z)\cdots(1 + x^n z).$$

Remark. In a completely similar way one can show that

$$(1 + xz)(1 + x^3 z)\cdots(1 + x^{2n-1}z)$$

$$= 1 + \sum_{k=1}^{n} \frac{(1 - x^{2n})(1 - x^{2n-2})\cdots(1 - x^{2n-2k+2})}{(1 - x^2)(1 - x^4)\cdots(1 - x^{2k})} x^{k^2} z^k.$$

PROBLEM 14. Let x and a be positive. Find the largest term in the expansion of $(x + a)^n$, where n is a positive integer.

Solution. Let the largest term be

$$T_k = \binom{n}{k} x^{n-k} a^k.$$

This term must not be less than the two neighbouring terms T_{k-1} and T_{k+1}; thus $T_k \geq T_{k-1}$ and $T_k \geq T_{k+1}$. Whence

$$\frac{k}{n - k + 1}\frac{x}{a} \leq 1 \quad \text{and} \quad \frac{n - k}{k + 1}\frac{a}{x} \leq 1.$$

The first of these inequalities yields

$$k \leq \frac{(n + 1)a}{x + a}$$

and from the second inequality we get

$$k \geq \frac{(n + 1)a}{x + a} - 1.$$

We assume first that $\frac{(n + 1)a}{x + a}$ is an integer. Then $\frac{(n + 1)a}{x + a} - 1$ will be an integer also, and since k is an integer satisfying

$$\frac{(n + 1)a}{x + a} - 1 \leq k \leq \frac{(n + 1)a}{x + a},$$

it can attain one of the two values