Numerical Methods in Fluid Dynamics

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Preface

This monograph is based on a graduate course, Mechanical Engineering 266, which was developed over a number of years at the University of California—Berkeley. Shorter versions of the course were given at the University of Paris VI in 1969, and at the University of Paris XI in 1972. The course was originally presented as the last of a three quarter sequence on Compressible Flow Theory, with emphasis on the treatment of non-linear problems by numerical techniques. This is reflected in the material of the first half of the book, covering several techniques for handling non-linear wave interaction and other problems in Gas Dynamics. The techniques have their origins in the Method of Characteristics (in both two and three dimensions). Besides reviewing the method itself the more recent techniques derived from it, firstly by Godunov and his group, and secondly by Rusanov and his co-workers, are described. Both these approaches are applicable to steady flows calculated as asymptotic states of unsteady flows and treat elliptic problems as limiting forms of unsteady hyperbolic problems. They are therefore applicable to low speed as well as to high speed flow problems.

The second half of the book covers the treatment of a variety of steady flow problems, including effects of both viscosity and compressibility, by the Method of Integral Relations, Telenin's Method, and the Method of Lines. The objective of all these methods is to eliminate finite difference calculations in one or more coordinate directions by using interpolation formulae, especially polynomials, to represent the unknowns in selected directions. These methods were used originally to solve flow problems connected with re-entry aerodynamics but have subsequently been applied over the whole speed range. They are, in principle, applicable to a broad range of problems governed by elliptic, mixed elliptic-hyperbolic, and parabolic partial differential equations.

Many of the applications described in the book result from research, carried out at Berkeley, sponsored by the Air Force Office of Scientific Research, NASA Ames Research Center and the Office of Naval Research. The support of all these agencies is gratefully acknowledged.

The less familiar methods discussed in the book are illustrated by solutions to model problems worked out by graduate students enrolled in the class and their contributions are recognized in the text. In addition,

several students assisted in checking the equations, especially A. Falade, W. K. Chan, and K. S. Chang. The manuscript was carefully typed by Mrs. Arlene Martin and I am indebted to her for completing this exacting task with cheerful forbearance.

My original venture into the numerical field was encouraged by Sydney Goldstein who pointed out to me the importance of this approach to problems in Fluid Dynamics many years ago, when non-linear effects first assumed significance. In the present enterprise Victor Rusanov was very helpful not only in providing material for Chapter 3, but also in obtaining less accessible papers for Chapters 2 and 6. Oleg Belotserkovskii has been a steady source of information on the Method of Integral Relations and many of the applications of the method to inviscid problems originate from his group.

I am grateful to Dr. W. Beiglböck for including the monograph in this new Springer series. I wish to express my appreciation to Mrs. Oelschläger and the editorial staff of Springer-Verlag for their assistance and courteous cooperation in the production of the book. Finally, I wish to thank my wife, Eileen, for her patient support during the writing of the manuscript.

Berkeley, California
January 21, 1977 Maurice Holt

Contents

CHAPTER 1

General Introduction

Brief Review of Concepts of Numerical Analysis

1.1 Introduction

At the present time the majority of unsolved problems in Fluid Dynamics are governed by non-linear partial differential equations and can only be treated by a numerical approach. As a consequence, specialists in Fluid Dynamics have recently devoted increasing attention to numerical, as opposed to analytical, techniques. Of course, there is no point in developing a novel numerical method unless it can be applied to actual problems of interest. In the early days of research on numerical analysis the capacity of computing machines was too restricted to permit many applications to be carried out. Today this situation has changed; the machines now available are sufficiently advanced to deal with an almost limitless range of problems; all that is needed is to discover effective numerical methods to attack them.

Although the major advances in construction and development of actual computing machines have taken place in the United States, many of the principal advances in Numerical Methods were made in the Soviet Union. This is especially true of methods applicable to problems in Fluid Dynamics and here the methods can be divided into two categories. The first depend purely on finite difference techniques, while in the second, the number of independent variables in the numerical scheme is reduced by supposing that the unknowns are polynomials or trigonometric functions of one, or more, of these variables.

In the first part of the monograph we shall be concerned mainly with problems in Gas Dynamics. In many of these problems viscosity is unimportant and the equations of motion reduce to a system of partial differential equations of the first order. If the motion is unsteady, this system is always hyperbolic. If the motion is steady the classification of the system depends on the magnitude of the fluid speed, being hyperbolic if it is supersonic and elliptic if subsonic. Thus the problems of Gas Dynamics are of three types, firstly, elliptic for steady flow at low speeds; secondly, hyperbolic for steady supersonic flow and all unsteady flow; and finally, of mixed type, when the flow is steady and subsonic in one region while being supersonic elsewhere. In all discussions of finite difference methods given in this course we shall always regard a steady flow as the asymptotic state of an unsteady flow and shall therefore only consider hyperbolic systems.

We shall describe two recent finite difference methods. The first is due to GODUNOV, originally presented in 1960 and revised in 1970 (see Refs. Chapt. 2). The second method was developed principally by RUSANOV, also in two stages. The original formulation was presented in 1964 in collaboration with BABENKO, VOSKRESENSKII and LIUBIMOV and is familiarly known as the BVLR method (BABENKO et al., Refs. Chapt. 3). This method was extended to three dimensional unsteady flow in 1970 by RUSANOV and LYUBIMOV (Refs. Chapt. 3). Both the Godunov and BVLR methods have their origins in the method of characteristics. In principle one could solve all nonstationary problems of Gas Dynamics by a method of characteristics and there are at present several research workers in the field who rely exclusively on this method. However, the method does not lend itself easily to machine computations. The main difficulty here results from the fact that the system of characteristic coordinates is not rectangular but curvilinear and, frequently, the angle between coordinate lines of opposing families is very small. It is, of course, always preferable to work with a rectangular coordinate system if possible.

The starting point in Godunov's method is the solution of the problem of piston motion in a cylinder. This is a classical problem. When the piston speed is constant the solution is well known; a shock wave propagates into the undisturbed gas, moving ahead of the piston with a larger but still constant speed. The value of this speed, for a perfect gas, is defined by a simple quadratic formula. When the piston motion is nonuniform the problem can be solved numerically by a method of characteristics employing Riemann invariants.

GODUNOV proposes to solve the general piston problem as follows: the region between the piston face and the shock wave is divided into a number of cells of small length (in general the cells are of equal length). If the velocity distribution along the cylinder is given at one time we can calculate the mean value of the velocity (and also the value of the other dependent variables) in each cell. The actual distribution can then be replaced by a sequence of constant values, one per cell. Then, across the boundary between two adjacent cells the values of the velocity, pressure and density are in general discontinuous. To determine the corresponding values at a slightly later instant it is necessary to solve a problem of breakdown at a diaphragm. This problem has an analytical solution defined by algebraic formulae. Thus, at the later instant, the values of the unknowns on each cell boundary are defined (they are in fact constant) and new cell values are determined as the means of values on the left and right boundaries. GODUNOV applies this process at successive times.

To solve a more general problem, for example, that of flow of supersonic gas past a cylinder, the field of flow is divided into strips, the boundaries of which are parallel to the axis of symmetry. The spacing of the strips is constant and each strip is treated as a channel. However, it is now necessary to take account of diaphragm breakdown between adjacent strips as well as between cell boundaries along each strip.

The BVLR method is purely a finite difference method. Three independent variables are considered; namely, the time (or a space variable which plays the role of time) and two coordinates. To advance the calculation in time we must connect the values in two planes representing conditions at successive time intervals by certain finite difference relations along characteristic lines. In the BVLR method these relations are replaced by equivalent conditions along lines running in the time or coordinate directions. Furthermore, an important part of the calculation is the determination of the shape of shock which encloses the given body. To this end, the boundary conditions at the body surface must be connected with conditions satisfied at the shock wave. To carry this out RUSANOV et al. use an extension of the double sweep method originally proposed by GEL'FAND and LOKUTSIEVSKII (see Refs. Chapt. 3).

To make the coverage of finite difference methods for hyperbolic equations complete, a chapter is included on the method of characteristics in three dimensions. In this the different versions of the method are described and particular attention is given to two of these; namely, the bicharacteristics method of BUTLER and the near characteristics method of SAUER.

The later chapters are devoted to techniques based partly on polynomial or other series representations in one (or more) of the independent variables. The first of these, the Method of Integral Relations, was introduced by DORODNITSYN in 1950, principally as a means to solve the problem of high speed flow past a blunt nosed body (see Refs. Chapt. 5). The application of this method carried out by BELOTSERKOVSKII provided the first solution to the problem of reentry of a space vehicle in the earth's atmosphere. In 1960 DORODNITSYN extended the method to apply to viscous flow, especially to boundary layer and wake problems. The method has been used widely in the United States and many applications will be discussed. The method consists in writing the equations of motion in divergence form and then integrating them with respect to one of the independent variables from one side of the disturbed flow field to the other. To calculate the integrals it is assumed that the integrands are polynomial or trigonometric functions of the variable of integration. These functions contain unknown coefficients which satisfy a reduced system of ordinary or partial differential equations.

The second group of methods using function fitting comes under the headings of Telenin's method and the Method of Lines. The Method of Lines has a comparatively long history as an essentially Soviet technique for solving linear partial differential equations—this is reviewed by LISKOVETS (see Chapt. 6 Refs.). Telenin's method was developed in collaboration with GILINSKI, TINYAKOV and LEBEDEV from 1964 onwards. In both methods the unknowns are represented as polynomials or trigonometric functions of one of the independent variables but it is no longer required to integrate the equations of motion with respect to this variable. TELENIN and his collaborators applied this method to the blunt body problem in both two and three dimensions, considering a wide range of body shapes. At Berkeley Telenin's method was applied to the Supersonic Yawed Cone Problem by NDEFO and HOLT and by FLETCHER and HOLT. It was applied by CHATTOT to solve the transonic double wedge flow problem in the hodograph plane.

The Method of Lines differs from Telenin's method in using local polynomial fitting rather than fitting over a whole coordinate range. It has been applied to the Yawed Cone Problem by JONES, SOUTH, and by FLETCHER (see Chapt. 6 Refs.).

In the remainder of this chapter we shall give a brief review of boundary value and initial value problems, followed by a discussion of one dimensional unsteady flow needed as an introduction to the Godunov schemes. We then deal with the method of characteristics for two dimensional steady supersonic flow as necessary background for Chapt. 4. The chapter concludes with an outline of the basic concepts of finite difference methods, such as stability, consistency and convergence. This draws on unpublished course notes of CHORIN and MILLER used at the University of California.

1.2 Boundary Value Problems and Initial Value Problems

The partial differential equations to be solved in problems in Fluid Mechanics are of three main types, elliptic, hyperbolic and parabolic. Steady inviscid flow in an incompressible fluid, or at subsonic speeds in a compressible fluid is governed by elliptic equations. When viscous boundary layer effects are included in such problems the addition of a diffusion type term converts these equations to parabolic form. Problems of steady supersonic or unsteady inviscid flow of a compressible fluid require the solution of hyperbolic equations. Transonic flow problems are governed by equations of mixed type, elliptic in subsonic regions and hyperbolic in supersonic regions. Although many finite difference methods have been developed for such problems they will not be discussed in the present monograph. Other methods for transonic problems will be discussed in Chapts. 5 and 6.

The three types of equations can be identified by their simplest forms, namely, Laplace's equations, the Wave equation, and the Heat Conduction equation. Elliptic problems are associated with values of the unknowns (or their normal derivatives) given on a closed curve and require the solution of boundary value problems. The solution at a general point of an elliptic problem depends on the data at every point of the boundary and a change in values at one boundary point changes the whole solution. In hyperbolic problems either one of the independent variables is the time or it has time like character and their solution requires the specification of values of unknowns on an open line or surface at some initial time. In this case the solution at a general point only depends on data on a part of the line. Parabolic problems require, in addition, that data be prescribed on certain fixed boundaries at all times. Both problems are classified as initial value problems.

We consider now the solution of Laplace's equation in two dimensions in which unknowns are prescribed on a closed curve. The analytical solution of this Dirichlet problem requires finding the conformal mapping transforming the boundary into the unit circle, equivalent to finding the Green's function for the problem. For a complicated boundary curve this is a difficult task and it may often be easier to solve the problem numerically. If we seek a solution by Finite Differences we divide the region bounded by the curve into a network and set up five point difference equations for the unknown at each interior point, supplementing these with difference equations derived from boundary points. These difference equations are coupled and, although in principle their solution is unique, in practice either an iterative or matrix inversion method must be used to find it.

An alternative method to the iterative process of solving Laplace's equation for such problems is to replace the steady boundary value problem by an unsteady initial value problem. In this boundary values are fixed on the closed curve at all times and values of the unknowns are estimated at the interior network points at some initial time. Values of unknowns at interior points at a later time are then found from difference equations in time and space variables. The solutions of these are determined from only local network values at the new and old times and are found directly without any iteration. The process is carried out at successive instants until the difference between the current value of the unknown at a general point and its corresponding value at the previous instant is less than some assigned small quantity. In other words, the solution of the steady state plane potential problem is found as the limiting approach to steady state of the unsteady initial value problem in two space variables.

The equivalent unsteady problem can either be formulated in parabolic or hyperbolic form, depending on whether a first or second order time derivative is added to Laplace's equation.

In discussing solutions of problems in Fluid Mechanics by Finite Difference Methods we shall treat Boundary Value Problems by this technique of unsteady approach to the steady state, normally using equations in wave propagation rather than parabolic form. Thus we shall only discuss finite difference methods for time dependent problems.

We first consider the simplest equations of motion in one-dimensional gas dynamics, defining characteristic lines, Riemann invariants and the role they play in finite difference methods, including the Method of Characteristics. We then generalize to equations of unsteady flow in two space variables so that we can deal with problems of compressible inviscid motion in plane or axi-symmetric flow. We describe techniques of solving these equations directly by the Method of Characteristics (in three independent variables) and then consider finite difference methods based on treating two dimensional flows as interdependent layers of one dimensional flows. Two methods of the latter type have been successfully developed over the past decade or so in the USSR, the first due to Godunov (1961, 1970), and the second by Rusanov and others, usually known as the BVLR method (Babenko et al., 1964; Rusanov and Lyubimov, 1970).

1.3 One Dimensional Unsteady Flow Characteristics

The equations of motion of unsteady flow in one dimension are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\varrho}\frac{\partial p}{\partial x} = 0 \tag{1.3.1}$$

$$\frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial u}{\partial x} = 0 \tag{1.3.2}$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0 \tag{1.3.3}$$

Here t is the time measured from some initial instant; x is the distance in the direction of motion; u, p. ϱ and S are velocity, pressure, density, and entropy, respectively.

If we define the speed of sound a by

$$a^2 = \left(\frac{\partial p}{\partial \varrho}\right)_S \tag{1.3.4}$$

Eqs. (1.3.1) and (1.3.2) may be combined to yield

$$\left(\frac{\partial u}{\partial t} \pm \frac{1}{\varrho a}\frac{\partial p}{\partial t}\right) + (u \pm a)\left(\frac{\partial u}{\partial x} \pm \frac{1}{\varrho a}\frac{\partial p}{\partial x}\right) = 0 \tag{1.3.5}$$

Eqs. (1.3.5) and (1.3.3) are called the equations of one dimensional unsteady flow in characteristic form and have special properties connected with the families of lines defined by

$$\frac{dx}{dt} = u \pm a \tag{1.3.6}$$

$$\frac{dx}{dt} = u \tag{1.3.7}$$

called *characteristic lines* or *characteristics*.

Eqs. (1.3.3) and (1.3.5) state that the original equations of motion (1.3.1)–(1.3.3) can be replaced by equivalent equations in characteristic form, namely, that

$$du \pm \frac{1}{\varrho a} dp = 0 \tag{1.3.8}$$

$$dS = 0 \tag{1.3.9}$$

along the directions (1.3.6) and (1.3.7), respectively.

Eqs. (1.3.8) and (1.3.9) are in "inner" differential form, i.e., they only contain derivatives along the corresponding directions (1.3.6) and (1.3.7). The characteristics therefore have the property that derivatives normal to them may be discontinuous. Eqs. (1.3.8) and (1.3.9) form the basis for solving problems of one dimensional gas dynamics in finite difference form, called the Method of Characteristics.

For isentropic flow of a perfect gas, (1.3.9) disappears and (1.3.8) simplify. We can now introduce the variable

$$\sigma = \int \frac{dp}{\varrho a} \tag{1.3.10}$$

and (1.3.8) can be integrated to give

$$\begin{aligned} u + \sigma &= \alpha \\ u - \sigma &= \beta \end{aligned} \tag{1.3.11}$$

where α and β are constant along lines (1.3.6), respectively, and are called *Riemann invariants.*

In general Eqs. (1.3.11) and (1.3.6) state that Riemann invariants α and β are propagated without change along plus and minus characteristic directions, respectively. In problems where disturbances are propagated in one direction only either α or β is constant throughout the whole flow region. Exact solutions can easily be found in such flows, which are called simple waves. Simple waves also can be used in certain flows with propagation in both directions.

We now cite two problems with simple wave solutions resulting from the sudden breakdown of a discontinuity. In both cases the undisturbed gas obeys perfect gas behavior with constant specific heat ratio γ and is uniform.

Problem 1

A semi infinite column of gas is bounded at its right end by a diaphragm to the right of which is a vacuum. At time $t=0$ the diaphragm is suddenly ruptured. Determine the way in which gas escapes from the column.

Solution

No fundamental length or time enter the problem so the solution is a function of x/t only, where x is the space coordinate measured from the original position of the diaphragm and t is the time measured from the instant of rupture. If γ is the specific heat ratio, and suffix zero refers to undisturbed conditions, it can easily be shown that the solution is

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} + a_0\right) \tag{1.3.12}$$

$$a = \frac{2}{\gamma+1}\left(a_0 - \frac{\gamma-1}{2}\frac{x}{t}\right) \tag{1.3.13}$$

This gives a linear variation with x/t for both u and a, and has the property that at the original position of diaphragm $x=0$, $u=a=2a_0/(\gamma+1)$ for all time. This is the simplest of the breakdown solutions used in Godunov's method.

Problem 2

A diaphragm separates two semi infinite columns of gas, initially at rest, with pressure p_1, density ϱ_1 on the left and pressure p_5, density ϱ_5 on the right. If $p_1 > p_5$, determine the motion after the diaphragm is instantaneously ruptured.

Solution

This is the shock tube problem. A shock wave is propagated to the right while a centered expansion wave is propagated to the left. The expansion and compression regions are separated by a contact discontinuity which acts like a uniform piston moving to the right.

The significant flow regions after breakdown are shown in Fig. 1.1.

1	2	3	4	5
undisturbed	expansion fan	uniform region	uniform region	undisturbed

Head | Tail | C.D. | Shock

Fig. 1.1 Shock tube problem

The problem is solved iteratively. One approach is to assume a value for p_4 and hence for $\xi = p_4/p_5$, the pressure ratio across the shock. This then determines the velocity of the contact discontinuity. We then solve in the expansion regions 2 and 3 regarding the contact discontinuity as a withdrawing piston and obtain a value for the velocity of sound a_1, at the head of the wave. This value is compared with the undisturbed value $a_1 = (\gamma p_1/\varrho_1)^{1/2}$ and if the two are unequal, another value of p_4 must be prescribed and the cycle repeated.

In the full breakdown solutions the semi-infinite columns of gas 1 and 2 are not initially at rest but are moving with *different* uniform velocities. These can also be solved iteratively. The full formulae are given in Sect. 2.2.

1.4 Steady Supersonic Plane or Axi-Symmetric Flow. Equations of Motion in Characteristic Form

The equations of motion for steady irrotational flow in a plane or with axial symmetry are

$$u_y - v_x = 0 \tag{1.4.1}$$

$$(a^2-u^2)u_x - 2uvu_y + (a^2-v^2)v_y + ja^2 v/y = 0 \tag{1.4.2}$$

Here x, y are Cartesian coordinates (plane flow)
Cylindrical coordinates (axi-symmetric flow with x along the axis of symmetry)
(u, v) are velocity components in the directions (x, y)

a = speed of sound, $j=0$ plane
$=1$ axial symmetry.

We wish to write (1.4.1) and (1.4.2) in characteristic form. This means that we seek a new set of coordinates $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ with the following property: When (1.4.1) and (1.4.2) are referred to ξ and η instead of x and y as independent variables, the first equation contains only ξ derivatives and the second equation only η derivatives. Each equation therefore only contains derivatives along the coordinate direction in question (inner derivatives) and derivatives in directions oblique or normal to the coordinate (outer derivatives) are absent.

To determine (ξ, η) we investigate the following problem. Given a curve $x = x(s)$, $y = y(s)$ and values of u, v along the curve $u = u(s)$, $v = v(s)$, under what conditions will (1.4.1) and (1.4.2) determine outer derivatives of u and v?

Without loss of generality we suppose that the given curve is nowhere parallel to the y axis. Then it is sufficient to investigate the conditions under which (1.4.1) and (1.4.2) determine u_y, v_y on the line. We denote the slope of the line by $m = (y_s/x_s)$.

The inner derivatives of u, v on the line are given by

$$u_s = u_x x_s + u_y y_s \tag{1.4.3}$$

$$v_s = v_x x_s + v_y y_s \tag{1.4.4}$$

where $u_s = (du/ds)$, etc., $u_x = (\partial u/\partial x)$, etc.

We can solve (1.4.3) and (1.4.4) for u_x, v_x on the line, in terms of inner derivatives and u_y, v_y. We have

$$u_x = \frac{u_s}{x_s} - m u_y \tag{1.4.5}$$

$$v_x = \frac{v_s}{x_s} - m v_y \tag{1.4.6}$$

Substitute for u_x, v_x in (1.4.1) and (1.4.2) and rearrange as simultaneous equations for u_y, v_y. Then we obtain,

$$u_y + m v_y = v_x / x_s \tag{1.4.7}$$

$$-\{m(a^2-u^2)+2uv\} u_y + (a^2-v^2) v_y = -(a^2-u^2)\frac{u_s}{x_s} - j\frac{a^2 v}{y} \tag{1.4.8}$$

Eqs. (1.4.7) and (1.4.8) are simultaneous algebraic equations to determine u_y, v_y. The matrix of the pair is

$$\begin{pmatrix} 1 & m & v_s/x_s \\ -\{m(a^2-u^2)+2uv\} & (a^2-v^2) & -(a^2-u^2)u_s/x_s - ja^2 v/y \end{pmatrix}$$

Denote the leading determinant of the matrix by Δ, and the determinant formed from the first and last columns by Δ_1.

Then

$$\Delta = \begin{vmatrix} 1 & m \\ -\{m(a^2-u^2)+2uv\} & (a^2-v^2) \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} 1 & v_s/x_s \\ -\{m(a^2-u^2)+2uv\} & -(a^2-u^2)u_s/x_s - ja^2 v/y \end{vmatrix}$$

In connection with solving (1.4.7) and (1.4.8) for u_y, v_y the following three possibilities arise,

(i) $\Delta \neq 0$.

In this case u_y, v_y are determined uniquely by the data on the line and by the equations of motion.

(ii) $\Delta = 0, \Delta_1 \neq 0$.

In this case (1.4.7) and (1.4.8) give no solutions for u_y, v_y.

(iii) $\Delta = 0, \Delta_1 = 0$.

In this case solutions for u_y, v_y are finite but are not unique. In fact there is a single infinity of pairs of (u_y, v_y) satisfying the linear relation (1.4.7) (or (1.4.8) which is now the same equation).

Conditions (iii) correspond to the case of interest. The condition $\Delta = 0$ determines the directions of the given line for which u_y, v_y are not uniquely