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Edited by

J. SIEMONS

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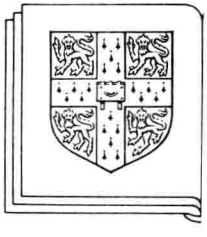

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PREFACE

Since its beginning in 1969 the British Combinatorial Conference has grown into an established international meeting. This year the twelfth conference is being held in Norwich under the auspices of the School of Mathematics at the University of East Anglia. Participants come from a great number of countries worldwide and represent a multitude of interests in combinatorial theory.

This volume contains the contributions of the principal speakers.They were invited to prepare a survey paper for this book and to deliver a lecture in an area of their expertise. In this way it is hoped to make available a valuable source of reference to the current state of art in combinatorics. The speakers have produced their papers well in advance so that they are now all available in time for the conference.

This book has been produced to a tight schedule. I am grateful to the authors for their cooperation and to the referees for their assistance and comments about the papers. The British Combinatorial Conference is largely self-financing but on behalf of the committee I would like to thank the London Mathematical Society, Norwich Union and Peat Marwick McLintock for their financial support.

Johannes Siemons
Norwich April 1989

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On the theory of designs

E. F. ASSMUS, JR.

INTRODUCTION

Several years ago I was asked a seemingly innocuous question: *What are the minimal-weight vectors of the code of an affine plane?* I thought the answer would be that they were, just as in the projective case, simply the scalar multiples of the lines; indeed, that may be true and the question is still open. I managed to prove this (for arbitrary affine planes) only for those of prime order.

The question is deeper than it at first seems. If, for example, one could prove that the minimal-weight vectors of the code of an arbitrary affine plane were simply the scalar multiples of the lines, one would have a proof of the fact [**15**] that a projective plane of order ten has no ovals; indeed, one would prove that no projective plane of order congruent to two modulo four, except the Fano plane, could have an oval. (The minimal-weight vectors of the code of a *desarguesian* affine plane are the scalar multiples of the lines of the plane but the only known proof relies heavily on algebraic coding theory.)

These considerations led J. D. Key (who asked the original question) and me to what seems to be a fruitful approach to affine and projective planes and to what we hope will be a fruitful approach to the theory of designs. The purpose of this paper is to explain these matters. Much of the work we have done has already appeared and thus the present paper will rely heavily on four joint papers : *Arcs and ovals in hermitian and Ree unitals*, *Affine and projective planes*, *Baer subplanes, ovals and unitals*, and *Translation planes and derivation sets*. A *brief* summary of some of the material contained in these four papers can be found in [**1**].

In the first section the original definition of a symmetric design is given, the congruence constraint explained, and some remarks on the layout of tables of designs made. (These remarks are expanded on in the Appendix.) The second section defines the hull of a design and the third explains the use of this notion in the investigation of affine planes. The Hamada-Sachar Conjecture and translation planes

are then discussed and, following that, derivations are put into the current setting. Finally a few concluding remarks are made.

THE CONGRUENCE CONDITION FOR A SYMMETRIC DESIGN

Tables of possible parameter sets of designs, together with a token design when existence is known, appear from time to time. The layout of these tables usually takes a form more suitable for the design of experiments than for theoretical analysis. Although I have no intention of producing yet another table I do want to suggest a different layout that may throw more light on the theoretical aspects of design theory. I do this here only for so-called symmetric designs but the Appendix shows how to carry out a similar approach for more general designs.

A symmetric (v, k, λ)-design, where v, k, and λ are integers with $1 < k < v - 1$, consists of two disjoint v-sets, $\mathcal{P}$ and $\mathcal{B}$, called points and blocks, with a regular, bipartite graph of valency k imposed, $\mathcal{P}$ and $\mathcal{B}$ being the two parts. The graph satisfies the further condition that given any two distinct points there are precisely λ paths of length two from one point to the other. It follows that the same property holds for any two distinct blocks (hence the unfortunate term "symmetric design"). Thus, one has no reason to call the elements of $\mathcal{P}$ points and those of $\mathcal{B}$ blocks for it might as well have been the reverse. Moreover, the complementary bipartite graph (eliminate the given edges (P, B) and introduce, instead, those (Q, C) which weren't originally edges) is also a symmetric design—with parameters

$$(v, v - k, \frac{1}{\lambda}(k - \lambda)(k - \lambda - 1)).$$

So symmetric designs come in pairs, a design and its complement, and in fours if one wishes (as I do) to distinguish $(\mathcal{P}, \mathcal{B})$ from $(\mathcal{B}, \mathcal{P})$, its *dual*.

Given a (v, k, λ) design on $(\mathcal{P}, \mathcal{B})$ one can associate to each $B \in \mathcal{B}$ the k-subset of $\mathcal{P}$ given by $\{P \in \mathcal{P} | (P, B) \text{ is an edge}\}$ and these v k-subsets of $\mathcal{P}$ are (again) called blocks. Because of the path condition, any two blocks meet in λ points and any two points are contained in λ blocks. Moreover, the path condition immediately implies that

$$\lambda(v - 1) = k(k - 1), \qquad *$$

the congruence condition relating the parameters.

The most important parameter of a symmetric design has not yet appeared. It is the *order*, namely $k - \lambda$, of the design and it is denoted by n. Observe that * implies that $n(n - 1)$, clearly congruent to $k(k - 1)$ modulo λ, is congruent to 0 modulo λ and, writing $n(n - 1) = \lambda\mu$, the parameters are

$$(2n + \lambda + \mu, n + \lambda, \lambda),$$

with the parameters of the complementary design being

$$(2n + \lambda + \mu, n + \mu, \mu).$$

This suggests listing the possible parameter sets for symmetric designs via the order n, using the divisors, λ, of $n(n-1)$ in turn, possibly eliminating the complementary divisor. Here's how such a table would begin:

Order	Parameter sets
2	(7, 3, 1), (7, 4, 2)
3	(13, 4, 1), (13, 9, 6)
	(11, 5, 2), (11, 6, 3)
4	(21, 5, 1), (21, 16, 12)
	(16, 6, 2), (16, 10, 6)
	(15, 7, 3), (15, 8, 4)

For each order the parameters of the possible projective plane and its complement appear first and the parameters of the Hadamard designs (i.e., designs with $\lambda = n-1$ or n) last. Moreover, the organization suggested permits one easily to examine the parameter sets for any particular order. Order twelve is interesting. Here, omitting complements, one gets

$$(157, 13, 1),$$
$$(92, 14, 2),$$
$$(71, 15, 3),$$
$$(61, 16, 4),$$
$$(52, 18, 6),$$
$$(47, 23, 11).$$

The Bruck-Ryser-Chowla Theorem rules out $\lambda = 2$ and $\lambda = 6$, but Beker and Haemers [7] have constructed a design with $\lambda = 3$ and van Trung [18] one with $\lambda = 4$. Of course, the Hadamard design exists (presumably, an enormous number) and hence only the plane of order twelve is in doubt.

Were one to list parameter sets by increasing order and sieve with the Bruck-Ryser-Chowla Theorem, then $(157, 13, 1)$ would be only the second question mark, the first being the plane of order ten. Put another way, all symmetric designs that could exist do exist through order twelve, excepting, possibly, the projective planes of orders ten and twelve. It has been recently reported that the computer calculation undertaken by Clement Lam et al has been completed and that the

result is that there is no projective plane of order ten; the other two symmetric designs of order ten that could exist, do exist; they are the Hadamard design and the design with parameters $(41, 16, 6)$.

The designs of orders two and three are unique and the precise number of designs is known through order six. Roughly speaking, for a given order, existence gets easier and more designs exist (if they can at all) as λ grows (up to, of course, $n-1$). For order four, for example, the plane is unique, there are three biplanes, and five triplanes. And, more generally, it is conjectured that there are Hadamard designs for every possible order.

THE CODE AND THE HULL OF A DESIGN

To bring algebraic coding theory into play notational compromises must be made. The concern is with arbitrary designs (for a definition see the Appendix): $|\mathcal{P}|$, the cardinality of the point set, will be denoted by N (rather than v) and the cardinality of a block will, generally, be denoted by d (rather than k). The number of blocks through two points— λ— and the difference between the number of blocks through one point and the number through two—the order n of the design—will be denoted in the standard way.

So, given $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, a design of order n, and any field F, let $F^{\mathcal{P}}$ be the vector space of all functions from $\mathcal{P}$ to F with, of course, point-wise addition and scalar multiplication. For $v \in F^{\mathcal{P}}$, denote the value of v at the point P by v_P and, for X a subset of $\mathcal{P}$, denote by v^X the characteristic function of X. Thus $v_P^X = 1$ if $P \in X$ and 0 otherwise.

Now denote by

$$C_F(\mathcal{D})$$

the subspace of $F^{\mathcal{P}}$ generated by all v^B where B is a block of $\mathcal{D}$ and call this subspace the *code of $\mathcal{D}$ over F*. If the dimension of $C_F(\mathcal{D})$ is k, then it is an (N, k)-code in the language of algebraic coding theory, N being the *length* of the code.

Moreover, its minimum weight is at most d. Here, the *weight* of a vector $v \in F^{\mathcal{P}}$, $wgt(v)$, is $|\{P \in \mathcal{P} | v_P \neq 0\}|$ or, in other words, the cardinality of the support of the function v. The *minimum weight* of any subspace C of $F^{\mathcal{P}}$ is

$$\text{Min}\{wgt(c) | c \in C, c \neq 0\}.$$

Equip $F^{\mathcal{P}}$ with the standard inner product: $[v, w] = \sum_{Q \in \mathcal{P}} v_Q w_Q$. For C a subspace of $F^{\mathcal{P}}$, $C^{\perp}$ denotes the subspace orthogonal to C :

$$C^{\perp} = \{v \in F^{\mathcal{P}} | [v, c] = 0 \text{ for all } c \in C\}.$$

Finally, set

$$Hull_F(\mathcal{D}) = C_F(\mathcal{D}) \cap C_F(\mathcal{D})^\perp$$

and call this subspace the *hull of* $\mathcal{D}$ *over* F.

EXAMPLE: If $\mathcal{D} = AG_2(\mathbf{F}_4)$, the affine plane of order four, then $Hull_{\mathbf{F}_2}(\mathcal{D})$ is the $(16,5)$ extended binary Reed-Muller code and $Hull_F(\mathcal{D}) = \{0\}$ for any field F of characteristic other than two.

To obtain non-trivial hulls one must choose fields with characteristic dividing n [17]. The choice will always be $F = \mathbf{F}_p$, p a prime; $Hull_F(\mathcal{D})$ will be denoted by $Hull_p(\mathcal{D})$ and referred to as the *hull at* p *of* $\mathcal{D}$. Similarly, $C_p(\mathcal{D})$ will replace $C_F(\mathcal{D})$ and the subscript will be omitted if p is clear from the context.

If σ is any permutation of $\mathcal{P}$ (i.e. if $\sigma \in Sym(\mathcal{P})$) then σ acts naturally on $F^{\mathcal{P}}$ via

$$(v\sigma)_P = v_{\sigma(P)}.$$

(Observe that σ acts on $\mathcal{P}$ on the left and on $F^{\mathcal{P}}$ on the right.) Clearly, if σ is an automorphism of $\mathcal{D}$ (which means that $\sigma(B)$ is a block whenever B is) then σ leaves the subspace $C(\mathcal{D})$ invariant. It can very easily happen—and in non-trivial ways—that the subgroup of $Sym(\mathcal{P})$ leaving $C(\mathcal{D})$ invariant is larger than the subgroup leaving $\mathcal{D}$ invariant. Moreover, the subgroup leaving the hull invariant may be still larger.

EXAMPLE: If $\mathcal{H}$ is the unital on 28 points given by the unitary group $U_3(\mathbf{F}_3)$ then the symplectic group $Sp_6(\mathbf{F}_2)$ leaves $C_2(\mathcal{H})$ invariant. If $\mathcal{R}$ is the Ree unital on 28 points the group leaving $C_2(\mathcal{R})$ invariant is the small Ree group $P\Gamma L_3(\mathbf{F}_8)$ while the group leaving the hull invariant is again the symplectic group.

This example was decisive in the genesis of the notion of the hull of a design, for these two unitals have isomorphic hulls despite the fact that they are not isomorphic as designs. See [2] for the details.

One of the reasons that it was not previously observed that the hull of a design is as important as the code of a design was the fact that, *for symmetric designs*, the hull is intimately related to the code. We end this section with a result that substantiates this assertion.

THEOREM 1. *Let* $\mathcal{D}$ *be a symmetric design and* $\mathcal{D}_{comp}$ *the complementary design. Let* p *be a prime dividing* n*, their common order. Then, if* p *does not divide* d*, the block size of* $\mathcal{D}$*,* $Hull_p(\mathcal{D}) = C_p(\mathcal{D}_{comp})$*, it is of codimension one in* $C_p(\mathcal{D})$*, and it consists of those vectors in* $C_p(\mathcal{D})$ *with* $\sum_{P \in \mathcal{P}} v_P = 0$*. If* p *does divide* d*, then* $C_p(\mathcal{D}) \subseteq C_p(\mathcal{D})^\perp$ *and* $Hull_p(\mathcal{D}) = C_p(\mathcal{D})$.

PROOF: Let $\mathcal{D}$'s parameters be (N, d, λ). Now $d = \lambda + n$, $N = 2n + \lambda + \mu$

where $n(n-1) = \lambda\mu$ and $\mathcal{D}_{comp}$'s parameters are $(N, n+\mu, \mu)$. Clearly, if p does not divide d, it does not divide λ and, since it must divide $n(n-1)$, it must divide μ. Hence p divides $n+\mu$, the block size of $\mathcal{D}_{comp}$. This shows that $C(\mathcal{D}_{comp}) \subset C(\mathcal{D}_{comp})^{\perp}$. Reversing the rôles of $\mathcal{D}_{comp}$ and $\mathcal{D}$ yields the last assertion of the Theorem. Continuing the argument, we now have, since p does not divide d, that

$$d^{-1} \sum_{B \in \mathcal{B}} v^B = J,$$

where J is the *all-one* vector. It follows, since $J \in C(\mathcal{D})$, that $C(\mathcal{D}_{comp}) \subset C(\mathcal{D})$. Moreover, $C(\mathcal{D}) = C(\mathcal{D}_{comp}) \oplus \mathbf{F}_p J$, the sum being direct because $[J, J] = N$, a non-zero element of $\mathbf{F}_p$. Thus $C(\mathcal{D}_{comp})$ is of codimension one in $C(\mathcal{D})$ and, because $\sum_{P \in \mathcal{P}} v_P = 0$ whenever $v \in C(\mathcal{D}_{comp})$,

$$C(\mathcal{D}_{comp}) = \{v \in C(\mathcal{D}) | \sum_{P \in \mathcal{P}} v_P = 0\} = Hull(\mathcal{D}).$$

REMARKS:

(1) Since $(J - v^B) - (J - v^C) = v^C - v^B$, it follows easily, when p does not divide d, that $Hull(\mathcal{D})$ is generated by the vectors of the form $v^C - v^B$, B and C being blocks of $\mathcal{D}$.

(2) It can happen that p divides both λ and μ and hence all the parameters. (This occurs, for example, for the $(16, 6, 2)$ designs.) In this case it is easy to see that $C(\mathcal{D}) = C(\mathcal{D}_{comp})$ if and only if J is in both codes and then

$$Hull(\mathcal{D}) = C(\mathcal{D}) = C(\mathcal{D}_{comp}) = Hull(\mathcal{D}_{comp}).$$

If J is in neither code then $C(\mathcal{D}) \cap C(\mathcal{D}_{comp})$ is of codimension one in the code of each design. Moreover, this intersection is generated by vectors of the form $v^B - v^C$ where B and C are both blocks of $\mathcal{D}$ (or both blocks of $\mathcal{D}_{comp}$). I do not have an example of this phenomenon. It would be very interesting to know—even for $p = 2$—precisely when J is in the code of the design.For example, were one to show that that J were in the code of a design with parameters $(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1})$ and dimension $2m+2$ one would have the following improvement of a result of Dillon and Schatz [**11**, Corollary 1].

A symmetric design has parameters $(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1})$ *and code of dimension* $2m+2$ *if and only if it can be obtained from the Reed-Muller code of length* 2^{2m} *and a difference set in the elementary abelian 2-group of order* 2^{2m} *as the vectors of minimal weight in the code spanned by the Reed-Muller code and the difference set.*

Were this result true it would characterize the designs of minimal 2-rank with the given parameters and it could be viewed as a theorem of Hamada-type (see Theorem 5 below) but one should be aware of the fact [**12**] that the number of such designs goes to infinity with m —in contrast with planes where, conjecturally, the design of minimum rank is unique.

(3) The Theorem gives the relationship between the codes of a symmetric design and its complement. The relationship between the codes of a symmetric design and its dual has not been thoroughly investigated; obviously the dimensions are equal, but, beyond that, only fragmentary results are known at present. An interesting example of what can happen is given by the two triplanes of order four which are duals of one another: the weight distributions of the two codes are identical but they are not isomorphic (see the discussion preceding Theorem 4).

THE HULL OF AN AFFINE PLANE

The importance of the hull in design theory is best illustrated when the design is an affine plane. Given such a plane π it determines a unique projective plane Π and a line of that plane L_∞ called the line at infinity. Both π and Π have the same order, n say, and, as designs, their parameters are

$$(n^2, n, 1) \text{ and } (n^2 + n + 1, n + 1, 1).$$

If the point sets of Π and π are $\mathcal{P}$ and $\mathcal{A}$, respectively, then $\mathcal{A}$ is simply $\mathcal{P} \setminus L_\infty$. There is a natural linear transformation from $F^{\mathcal{P}}$ to $F^{\mathcal{A}}$ given by simply restricting the functions on $\mathcal{P}$ to $\mathcal{A}$. This transformation relates the codes, the hulls, and their orthogonals. Precisely, we have the following result:

PROPOSITION 1. *Let π be an affine plane of order n and Π its projective completion with L_∞ the line at infinity. Then, for a prime p dividing n, both $C_p(\pi)$ and $Hull_p(\pi)^\perp$ are, respectively, the images of $C_p(\Pi)$ and $Hull_p(\Pi)^\perp$ under the natural transformation of $F^{\mathcal{P}}$ onto $F^{\mathcal{A}}$. Further, $Hull_p(\pi)$ is the image of the subcode $\{c \in Hull_p(\Pi) \mid c_Q = 0 \text{ for } Q \text{ on } L_\infty\}$, dim $Hull_p(\pi) =$ dim $C_p(\pi) - n$, and $Hull_p(\pi)$ is generated by all $v^\ell - v^m$ where ℓ and m are two parallel lines of π.*

A proof of this result can be found in [**3**]. One important point here is that the orthogonal of the hull of Π has minimum weight $n + 1$ and the minimal-weight vectors are *all* of the form αv^L where L is a line of Π and α a non-zero element of $\mathbf{F}_p$, but the orthogonal of the hull of π (although it has minimum weight n, as expected) has minimal-weight vectors that are not necessarily, in fact not usually,

of the form αv^{ℓ} where ℓ is a line of π. They *are* of the form αv^{X}, however, where X sometimes has a nice geometric interpretation. Vectors of the form αv^{X} are called *constant* vectors and are referred to, by an abuse of language, as *scalar multiples of* X.

A second important point is the fact that, of the codes involved, the affine hull is the one of smallest dimension and it is generated by the differences of parallel lines and hence easily computable. The nature of its minimal-weight vectors is of crucial importance: for example, were it always true that the minimum weight of the affine hull were $2n$ and that the minimal-weight vectors were precisely the scalar multiples of the above generators, then one would recover the projective plane from the affine hull for p odd and the theory we are sketching would probably be useless. For desarguesian affine (or projective) planes it *is* true that the minimum weight of the hull is $2n$ and that there are no unexpected minimal-weight vectors; moreover, to establish these results one makes heavy use of algebraic coding theory. More generally, the hull of an affine translation plane has minimum weight $2n$ (a fact that follows easily from Proposition 3 below) but it is not, in general, true that there are only the expected minimal-weight vectors; for example, the hull of the non-desarguesian affine translation plane of order nine has unexpected minimal-weight vectors and the plane cannot be recovered from the hull. It is to be noted that an arbitrary affine plane of prime order can be recovered from its hull; perhaps this could be viewed as evidence that such a plane is desarguesian although my own view is that it is too early to speculate. The following result, whose proof can be found in [**3**], details part of the story:

THEOREM 2. *Let π be an affine plane of order n and p a prime dividing n. Then the minimum weight of $Hull_p(\pi)^{\perp}$ is n and all minimal-weight vectors are constant. Moreover,*

(1) *If $n = p$, then the minimal-weight vectors of $Hull_p(\pi)^{\perp}$ are precisely the scalar multiples of the lines of π and the hull uniquely determines the plane.*
(2) *If $n = p^2$, then the minimal-weight vectors of $Hull_p(\pi)^{\perp}$ are scalar multiples of either lines or Baer subplanes of π.*

In the desarguesian case the minimal-weight vectors of $C_p(AG_2(\mathbf{F}_q))$, where $q = p^s$, are scalar multiples of the lines of $AG_2(\mathbf{F}_q)$ and the minimal-weight vectors of the hull are precisely the scalar multiples of vectors of the form $v^{\ell} - v^{m}$ where ℓ and m are two parallel lines.

In general one cannot say a great deal about the dimension of the code of an

affine plane, but the dimension is known if the plane is desarguesian: if $q = p^s$, p a prime, then the dimension of the code of the desarguesian affine plane of order q is

$$\binom{p+1}{2}^s.$$

A conjecture, due independently to Hamada and Sachar, states that the code of any affine plane of order q has at least this dimension, with equality only if it *is* the desarguesian plane. The next Section will discuss this matter further.

What one can say in general is that whenever p but not p^2 divides the order of the plane, π say, then dim $C_p(\pi)$ is $\frac{1}{2}n(n+1)$ and hence, by Proposition 1 and Theorem 2, the classification of planes of prime order p is tantamount to the classification of certain $[p^2, \frac{1}{2}p(p-1)]$-codes.

The Hamada-Sachar Conjecture and Translation Planes

Let π be an affine plane of order n and H its hull at p for some prime p dividing n. Then, as indicated above, $H^\perp$ usually contains many more minimal-weight vectors than simply the scalar multiples of lines. Thus if $n = p^2$, we have the Baer subplanes appearing and, if n is even and $p = 2$, the hyperbolic ovals. (A *hyperbolic oval* of π is a set of $n+2$ points with two at infinity and no three collinear.) Many other configurations arise as well (see [5]).

The collection of supports of these constant vectors (the *support* of a vector v is $\{P \in \mathcal{P} | v_P \neq 0\}$) may very well contain affine planes other than the one with which one started. For example, $Hull_2(AG_2(\mathbf{F}_4))$ is the (16,5) binary Reed-Muller code with weight distribution $1 + 30X^8 + X^{16}$ and $Hull_2(AG_2(\mathbf{F}_4))^\perp$ is the (16,11) binary extended Hamming code with 140 weight-4 vectors. These vectors form a Steiner quadruple system but, as a *two-design*, they contain 112 subdesigns which are affine planes of order four, all of which have the same hull. The 20 vectors forming an affine plane of order four having been chosen, the remaining 120 are Baer subplanes of that plane. Of course, no new affine planes occur in this simple example, but in the next possible case, order nine, interesting things do occur: there are four projective planes of order nine and seven affine planes of order nine; the two translation planes of order nine each have an affine part that yields the other in the way indicated—and the other two projective planes do likewise.

In order to discover and perhaps classify planes of order n one should have at one's disposal linear codes of length $N = n^2$ over $\mathbf{F}_p$ (where p divides n). The minimum weight should be n, and there must be a sufficiently rich structue of

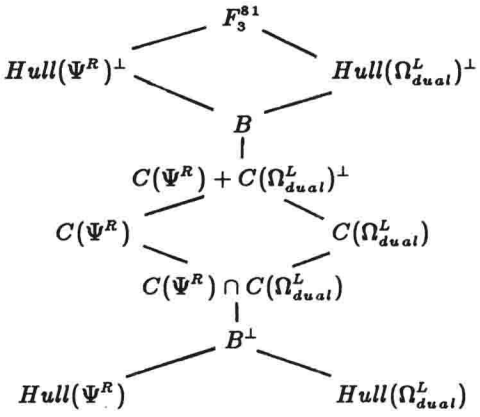
minimal-weight vectors to accommodate $C_p(\pi)$'s for various affine planes π. Of course, given an affine plane π, $B = Hull_p(\pi)^\perp$ is such a code. Fortunately, for $n = p^s$ there are very standard choices for such codes B and, more importantly, these codes have been intensively studied by algebraic coding theorists, in particular, by Philippe Delsarte [8] and Delsarte et al [9]. The results contained in these two papers were crucial to the development of the ideas here presented. In particular, without the dimensions of what are here called the *standard choices* no bounds on the ranks of the incidence matrices of translations planes would be available and, without the results on the minimal-weight vectors of these standard choices, the notion of *tame* would not have surfaced and the weak version of Hamada-Sachar would not have been proved. These papers are rather difficult to read—especially for finite geometers; a brief outline of the results one needs is contained in Appendix I of [3].

In order to properly discuss these linear codes and the affine planes connected with them it is best to make a few definitions.

DEFINITION 1. *Let B be an arbitrary code of length n^2 over $\mathbf{F}_p$ (where p divides n) with minimum weight n.*

(1) *An affine plane π of order n is said to be contained in B if $C_p(\pi)$ is code isomorphic to a subcode of B.*
(2) *An affine plane π of order n is said to be linear over B if $C_p(\pi)$ is code isomorphic to a subcode C of B where $B \subseteq C + C^\perp$.*

EXAMPLE: If one takes the Veblen-Wedderburn plane Ψ of order nine and a "real" line R and sets $\psi = \Psi^R$, the affine plane with R its line at infinity, then the vectors in $Hull(\psi)^\perp$ of the form v^S, where S is a line or Baer subplane of ψ, generate an [81,48] ternary code B over which ψ is linear. Moreover, $\omega = \Omega^L_{dual}$ is also linear over B where Ω is the non-desarguesian translation plane of order nine, Ω_{dual} its dual, and L a line through the translation point. Both $Hull\ (\psi)^\perp$ and $Hull(\omega)^\perp$ are [81,50] ternary codes containing B. They are not code-isomorphic since $Hull(\psi)^\perp$ contains 2×306 minimal-weight vectors while $Hull(\omega)^\perp$ contains 2×522. The 2×306 weight-9 vectors are the scalar multiples of the 90 lines and the 216 Baer subplanes of ψ; that ψ has 216 Baer subplanes follows from an easy counting argument and the facts contained in [10]. A similar situation obtains for ω. By properly choosing subcodes of B isomorphic to the codes of ψ and ω one gets the following Hasse diagram.



The code B of the above example is not a *standard choice* and we turn now to such codes. Set $q = p^s$ and let F be an arbitrary subfield of $\mathbf{F}_q$. Let V be a 2-dimensional vector space over $\mathbf{F}_q$ and consider V as a vector space over F. It will, of course, have dimension $2[\mathbf{F}_q : F]$, where $[K : F]$ denotes the degree of K over F. Set $m = [\mathbf{F}_q : F]$ and consider the collection, $\mathbf{L}_F$, of all m-dimensional subspaces over F and their translates under the addition in V, i.e., all the m-dimensional cosets in the affine space of V, $AG_{2m}(F)$.

Now $\mathbf{F}_p^V$ is a q^2-dimensional vector space over $\mathbf{F}_p$. Each $X \in \mathbf{L}_F$ is a subset of V of cardinality q and defines a vector v^X of $\mathbf{F}_p^V$. Let $B(\mathbf{F}_q|F)$ be the subspace of $\mathbf{F}_p^V$ spanned by $\{v^X \mid X \in \mathbf{L}_F\}$. $B(\mathbf{F}_q|F)$ is a code of length q^2 over $\mathbf{F}_p$ with minimum weight q, its minimal-weight vectors are scalar multiples of the vectors v^X, and its dimension is computable. Let $E(\mathbf{F}_q|F)$ be the subcode of $B(\mathbf{F}_q|F)$ generated by vectors of the form $v^X - v^Y$ where X and Y are translates of the same m-dimensional subspace of V, viewed as a vector space over F. Once again, the minimal-weight vectors are scalar multiples of these generators and the dimension is computable.

If F and K are two subfields of $\mathbf{F}_q$, $q = p^s$, with $F \subseteq K$, then clearly, $B(\mathbf{F}_q|K) \subseteq B(\mathbf{F}_q|F)$. At one extreme, $F = \mathbf{F}_q$, $\mathbf{L}_F$ consists of the lines of the affine plane $AG_2(\mathbf{F}_q)$ and $B(\mathbf{F}_q|\mathbf{F}_q) = C_p(AG_2(\mathbf{F}_q))$. At the other extreme, $F = \mathbf{F}_p$, $\mathbf{L}_F$ consists of all s-dimensional subspace and their translates, where V is viewed as a $2s$-dimensional vector space over $\mathbf{F}_p$.

The mapping $F \mapsto B(\mathbf{F}_q|F)$ establishes a Galois correspondence between the subfields of $\mathbf{F}_q$ and certain subcodes of $\mathbf{F}_p^V$; moreover, this correspondence neatly places the translation planes in their proper place according to the size of their