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Min-Teh Cheng Xing-Wei Zhou
Dong-Gao Deng (Eds.)

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FOREWORD

This is the second volume of the articles submitted by the participants in a program on Harmonic Analysis held by the Nankai Institute of Mathematics from March 1 to June 30, 1988. The first volume was a monograph due to Prof. Guy David entitled "Wavelets, Calderon-Zygmund Operators and Singular Integrals on Curves and Surfaces", which contains a series of lectures given by him in the Nankai Institute of Mathematics in June, 1988.*

We wish to thank all the participants for their cooperation and those who offered courses in the March, 1988 as the preliminaries to the graduate students for the lectures. We are also very thankful to Prof. Rong Wu from Nankai University for her fine arrangement for this volume. Our thanks are also due to Ms. He Li from the Nankai Institute of Mathematics for her careful typewriting.

For the editors,

M.T.Cheng

X.W. Zhou

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April, 1990

*Guy David "Wavelets and Singular Integrals on Curves and Surfaces", LNM 1465, 1991

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Nankai lecture in $\overline{\partial}$-Neumann problem

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This article is based on a series of lectures presented by the author at Nankai Institute of Mathematics from May 16, 1988 to June 8, 1988. The article which follows is an attempt to give an exposition of some of the recent progress in the $\overline{\partial}$- Neumann problem:

Given a bounded domain $\Omega \subseteq \mathbb{C}^{n+1}$ with smooth boundary $\partial\Omega$ and let f be a $(0,1)$-form, find another (0.1)-form u such that

$$\Box u = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f$$

$$u \in \mathrm{dom}(\overline{\partial}^*), \quad \overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*).$$

In some respects, this field is not that new since in the early 1960's the L^2-estimates for the $\overline{\partial}$-Neumann problem were being investigated by Kohn [FoK] and others. However, for many of the problems in this area that deeply depend on the techniques of Fourier analysis and the real break through to the finite type domains in $\mathbb{C}^2$ happened in 1988.

The exposition is divided into two parts. In the first, we introduce the technique to construct the parametrix of the $\overline{\partial}$-Neumann problem. For the elliptic part of the problem we actually construct the kernel to solve the boundary value problem. The author believes this is the first time that this part of the kernel has been computed precisely. For the subelliptic part of the problem we construct the parametrix as a product of couple operators. The solvability of the $\overline{\partial}$-Neumann problem is equivalent to the problem of inverting the Calderón operator $\Box^+$. We may also consider the $\overline{\partial}$-Neumann problem on the complementary domain $\Omega^- = \mathbb{C}^{n+1} \setminus \overline{\Omega}$. This problem also gives rise to a first order pseudo-differential operator which we call $\Box^-$. The miracle is the following:

$\Box^- \circ \Box^+ = -\Box_b +$ first order terms only involves "good" vector fields + zero order terms.

Hence the inverse of $\Box^+$ should be $(\Box_b)^{-1} \cdot \Box^-$. When $n \geqq 2$, $\overline{\Omega}$ is strongly pseudoconvex, we give the parametrix of the $\overline{\partial}$-Neumann problem. These method based on the results of Greiner-Stein [GS], Chang [C], etc. When $n = 1$, $\overline{\Omega}$ is pseudoconvex but finite type m, we also give the full parametrix for the $\overline{\partial}$-Neumann problem which is a new result obtained by Chang-Nagel-Stain [CNS1] recently.

In the second part of this paper, we discuss the regularity properties for the Neumann operator, especially the operator of Poisson type. The proof of the theorem involves lot of techniques about singular integrals, pseudo-differential operators and oscillatory integrals. The author thinks it is worthwhile to discuss this theorem, and then let the audience see how real analysis plays the role in several complex variables. The reader can also read the results in Greiner-Stein [GS], Phong [P2] and Chang [C].

It gives me a great pleasure to thank Mr. Zhang Wen Ping for preparing the lecture notes which greatly simplified the task of writing this paper. The author would also like to thank Professor M.T. Cheng and Professor T.G. Deng, and Nankai Institute of Mathematics for the very warm hospitality he received.

Much of this paper is an exposition of collaborated work with Alex Nagel, Eli Stein, and Steve Krantz, as well as the mathematics the author learned from Eli Stein and D.H. Phong. The author is particularly grateful to them for their many years of teaching, stimulation, and encouragement.

Finally, as a beginner in mathematics as well as being Chinese, the author would like to give his deep appreciation to Professor S.S. Chern who is the founder of Nankai Institute of Mathematics which provided the best facilities to mathematicians in China. The author would also like to express his respect to Professor Chern for his effort to promote the mathematical level of China.

1. The Parametrix for the $\overline{\partial}$- Neumann Problem

Let us first consider the problem on the model domain

$$D = \{(z_1, \cdots, z_n, z_{n+1}) = (z', z_{n+1}) \in \mathbb{C}^{n+1}, Im z_{n+1} > |z'|^2\}.$$

Its boundary is the "paraboloid"

$$\partial D = \{Im z_{n+1} = |z_1|^2 + \cdots + |z_n|^2\}.$$

As usual, we can consider a real coordinate patch U on D near the obundary point $(0,0)$. Let $(z', t; \rho) \in \overline{U}$ where $z' = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $t = Re z_{n+1}$, and $\rho = Im z_{n+1} - |z'|^2$ is the "height" function defined on $\overline{U}$. Under this consideration, we may treat ∂D as the Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}^1$ under the group law:

$$(z', t) \cdot (w', s) = (z' + w', t + s + 2Im(z' \cdot \overline{w'})).$$

Hence we may consider a basis for $T^{(1,0)}(\overline{U})$ as follows by using this coordinate system:

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t}, \ j = 1, 2, \cdots, n, \text{ and } T = \frac{\partial}{\partial t}$$

$$Z_{n+1} = \frac{1}{\sqrt{2}} \left[\frac{\partial}{\partial \rho} + i\frac{\partial}{\partial t}\right].$$

It is easy to see that $\{Z_1, \cdots, Z_n, T\}$ is a basis for the Lie algebra $\mathcal{H}^n$ and $\{Z_1, \cdots, Z_n\}$ is a basis for the subbundle $T^{(1,0)}(\partial D)$ of the complex tangent bundle $T^{\mathbb{C}}(\partial D)$. The $(1,0)$-forms defined on $\overline{U}$ which is dual to this basis is

$$\omega_j = dz_j, \quad j = 1, 2, \cdots, n$$

$$\omega_{n+1} = \sqrt{2} d\rho.$$

Now we define a Hermitian metrix $ds^2 = \sum_{j=1}^{n+1} \omega_j \otimes \overline{\omega}_j$ (so call a Levi metric) on $\overline{U}$. The article by Folland-Stein $[FS]$ and the book by Beals-Greiner $[BG]$ are good references for the analysis on the Heisenberg group. They also explained the relationship between it and the several complex variables.

Under the assumption that $\overline{U}$ equipped with a Levi metric, let us consider the $\overline{\partial}$-Neumann problem as follows. Given a $(0,1)$-forms $f = \sum_{j=1}^{n+1} f_j\overline{\omega}_j$, find another $(0,1)$-form $u = \sum_{j=1}^{n+1} u_j\overline{\omega}_j$ such that

$$\square u = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f \tag{1.1}$$

with two boundary conditions

$$u \in \operatorname{dom}(\overline{\partial}^*) \text{ and } \overline{\partial}u \in \operatorname{dom}(\overline{\partial}^*). \tag{1.2}$$

We can rewrite (1.1) and (1.2) in terms of the vector fields $\{Z_1, \cdots, Z_{n+1}\}$ as follows,

$$\square u = \textstyle\sum_{j=1}^{n} (\square^* u_j)\overline{\omega}_j + \square^{\#}(u_{n+1})\overline{\omega}_{n+1} \tag{1.1'}$$

and

$$u \in \operatorname{dom}(\overline{\partial}^*) \Longleftrightarrow u_{n+1}|_{\rho=0} \text{and } \overline{\partial}u \in \operatorname{dom}(\overline{\partial}^*) \Longleftrightarrow \overline{Z}_{n+1}u_j|_{\rho=0} = 0. \tag{1.2'}$$

Here

$$\square^* = \square_b - Z_{n+1}\overline{Z}_{n+1}, \quad \square_b = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j) - i(n-2)T$$

$$\square^{\#} = \square_b^{\dagger} - Z_{n+1}\overline{Z}_{n+1}, \quad \square_b^{\dagger} = \square_b - 2iT.$$

So the problem (1.1'), (1.2') breaks into two parts

$$\begin{cases} \square^* u_j = f_j & \text{in } U \\ \overline{Z}_{n+1}u_j = 0 & \text{when } \rho = 0 \end{cases}, \quad j = 1, 2, \cdots, n \tag{1.3}$$

$$\begin{cases} \square^{\#} u_{n+1} = f_{n+1} & \text{in } U \\ u_{n+1} = 0 & \text{when } \rho = 0. \end{cases} \tag{1.4}$$

We will solve the problems (1.3) and (1.4) by different methods. Now we solve the problem (1.4) first.

(I) The elliptic part

Recall that

$$\square^* = \left\{\sum_{j=1}^{n} -\frac{1}{2}(Z_j\overline{Z}_j + \overline{Z}_jZ_j) - \frac{1}{2}\frac{\partial^2}{\partial t^2} - inT\right\} - \frac{1}{2}\frac{\partial^2}{\partial \rho^2}.$$

Let

$$A_\alpha = 2\left\{-\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j) - \frac{1}{2}\frac{\partial^2}{\partial t^2} - i\alpha T\right\}$$

then

$$\square^* = \frac{1}{2}A_\alpha - \frac{1}{2}\frac{\partial^2}{\partial\rho^2}.$$

From the equation (1.4) we need to solve the following system:

$$\begin{cases} \left[\frac{1}{2}A_\alpha - \frac{1}{2}\frac{\partial^2}{\partial\rho^2}\right]u = f(z',t;\rho) & \text{in } D = \{z',t;\rho) \in \mathbb{C}^n \times \mathbb{R}^1 \times \mathbb{R}^+ : \rho > 0\} \\ u = 0 & \text{on } \partial D = \{z',t;o) \in \mathbb{C}^n \times \mathbb{R}^1 \times \{0\}\}. \end{cases}$$

Taking the Fourier transform in ρ-variable, we have

$$\begin{aligned}(\mathcal{F}_{\rho\to\nu}u)(z',t;\nu) =& \frac{2(\mathcal{F}_{\rho\to\nu}f)(z',t;\nu)}{A_\alpha + \nu^2} \\ \therefore u(z',t;\rho) =& \frac{1}{2\pi}\iint \frac{e^{i(\rho-\tilde{\rho})\nu}}{A_\alpha+\nu^2}f(z',t;\tilde{\rho})d\tilde{\rho}d\nu \\ =& \int \frac{e^{-|\rho-\tilde{\rho}|A_\alpha^{1/2}}}{A_\alpha^{1/2}}f(z',t;\tilde{\rho})d\tilde{\rho}.\end{aligned}$$

Plugging in the boundary condition, we have

$$\begin{aligned}u(z',t;\rho) =& \int_0^\infty \frac{e^{-|\rho+\tilde{\rho}|A_\alpha^{1/2}}}{A_\alpha^{1/2}}f(z',t;\tilde{\rho})d\tilde{\rho} \\ &- \int_0^\infty \frac{e^{-(\rho-\tilde{\rho})A_\alpha^{1/2}}}{A_\alpha^{1/2}}f(z',t;\tilde{\rho})d\tilde{\rho}\end{aligned}$$

Since we want to find the kernel $G^{\#}$ of Green's operator for the operator $\frac{1}{2}A_\alpha - \frac{1}{2}\frac{\partial^2}{\partial\rho^2}$, we let

$$f(z',t;\tilde{\rho}) = \delta_{z'}(w') \otimes \delta_t(s) \otimes \delta_{\tilde{\rho}}(\mu).$$

Then

$$\begin{aligned}G^{\#}((z',t;\rho),(w',s;\mu)) =& \frac{e^{-|\rho-\mu|A_\alpha^{1/2}}}{A_\alpha^{1/2}}(\delta_{z'}(w')\otimes\delta_t(s)) \\ &- \frac{e^{-(\rho+\mu)A_\alpha^{1/2}}}{A_\alpha^{1/2}}(\delta_{z'}(w')\otimes\delta_t(s)).\end{aligned}$$

Let $\sigma(A_\alpha)$ denote the symbol of A_α and let $\Delta = \left[z\sum_{j=1}^{n}|\sigma(Z_j)|^2 + \tau^2\right]^{1/2}$ with $\tau = \sigma\left[\frac{\partial}{i\partial t}\right]$. Then we have $\sigma(A_\alpha) = \Delta^2 + 2\alpha\tau$. In our case, $\alpha = n$ or $n-2$.

Before we go further, let us review the following theorem about the Kohn Nirenberg formula for the composition of two left- invariant pseudo-differential operators on the Heisenberg group:

Theorem:

Let a_1 and a_2 be symbols of ψdos $Op(a_1)$ and $Op(a_2)$ respectively, both of them depending on the symbols of the Heisenberg vector fields with constant coefficients. Suppose $Op(a) = Op(a_1)\cdot Op(a_2)$, then the symbol a of $Op(a)$ is also such an operator. We have the following asymptotic expansion for a:

$$a \approx \sum_{\beta} \frac{\tau^{|\beta|}}{\beta! i^{|\beta|}} \left\{ \left[\frac{\partial}{\partial \xi}\right]^{\beta} a_1(x,\xi) \cdot \left[\frac{\partial}{\partial \xi}\right]^{\prime\beta} a_2(x,\xi) \right\} \tag{1.5}$$

where

$$\left[\frac{\partial}{\partial \xi}\right]' = \left[\frac{\partial}{\partial \xi_1}\right]' \cdots \left[\frac{\partial}{\partial \xi_{2n}}\right]' \text{ is defined by}$$

$$\left[\frac{\partial}{\partial \xi_j}\right]' = -\frac{\partial}{\partial \xi_{j+n}} \text{ if } 1 \le j \le n$$

and

$$\left[\frac{\partial}{\partial \xi_{j+n}}\right]' = \frac{\partial}{\partial \xi_j} \text{ if } 1 \le j \le n.$$

The proof can be found in Chang [C].

Using this formula (1.5), it is easy to calculate the asymptotic expansion of the following symbols:

(i) $\sigma(A_\alpha^{1/2}) = \Delta + \frac{\alpha\tau}{\Delta} + \cdots$ (Since $\sum_\beta \frac{\tau^{|\beta|}}{\beta! i^{|\beta|}} \left\{ \left[\frac{\partial}{\partial \xi}\right]^{\beta} \Delta \cdot \left[\frac{\partial}{\partial \xi}\right]^{\prime\beta} \Delta \right\} \equiv 0$).

(ii) $\sigma\left[e^{-\rho A_\alpha^{1/2}}\right] = e^{-\rho\Delta}\left[1 - \frac{\alpha\rho\tau}{\Delta} + \cdots\right]$

(iii) $\sigma\left[\frac{e^{-\rho A_\alpha^{1/2}}}{A_\alpha^{1/2}}\right] = \frac{e^{-\rho\Delta}}{\Delta}\left[1 - \left[\frac{\alpha\rho\tau}{\Delta} + \frac{\alpha\tau}{\Delta^2}\right] + \cdots\right].$

Hence we have

$$\begin{aligned} &\frac{e^{-\rho A_\alpha^{1/2}}}{A_\alpha^{1/2}}(\delta_{z'}(w') \otimes \delta_t(s)) \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{2n+1}} \sigma\left[\frac{e^{-\rho A_\alpha^{1/2}}}{A_\alpha^{1/2}}\right] (\delta_{z'}(w') \otimes \delta_t(s))^\wedge e^{i[\sum_{j=1}^{2n} x_j\xi_j + t\tau]} d\xi\, d\tau. \end{aligned} \tag{1.6}$$

Here we use $z_j = x_j + i x_{j+n}$ and $w_j = y_j + i y_{j+n}$ for $j = 1, \cdots, n$. Then

$$\begin{aligned} (1.6) =& \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{2n+1}} \left[\frac{e^{-\rho\Delta}}{\Delta}\right] e^{i[\sum_{j=1}^{2n}(x_j - y_j)\xi_j + (t-s)\tau]} d\xi\, d\tau \\ &- \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{2n+1}} e^{-\rho\Delta} \left[\frac{\alpha\rho}{\Delta^2} + \frac{\alpha 1}{\Delta^3}\right] e^{i[\sum_{j=1}^{2n}(x_j - y_j)\xi_j + (t-s)\tau]} d\xi\, d\tau \\ &+ \text{ terms with weaker singularity } = I + II + t.w.w.s. \end{aligned}$$

Since $\Delta = \{\frac{1}{2}\sum_{j=1}^{n}[(\xi_j + 2x_{j+n}\tau)^2 + (\xi_{n+j} - 2x_j\tau)^2] + \tau^2\}^{\frac{1}{2}}$, after change variables, we hve the following:

$$I = C_n \frac{1}{[2|z' - w'|^2 + (t - s + 2Im(z' \cdot \overline{w'}))^2 + \rho^2]^n} \text{ with } C_n = \frac{2^{n-1}\Gamma(n)}{\pi^{n+1}}$$

and

$$II = \frac{-C_n}{2(n-1)} \frac{1}{[2|z' - w'|^2 + (t - s + 2Imz' \cdot \overline{w'})^2 + \rho^2]^{n-1}}$$

Thus,

$$\begin{aligned} &G^{\#}((z', t; \rho), (w', s; \mu)) \\ =&C_n \frac{1}{[2|z' - w'|^2 + (t - s + 2Imz' \cdot \overline{w'})^2 + (\rho - \mu)^2]^n} \\ &- C_n \frac{1}{[2|z' - w'|^2 + (t - s + 2Imz' \cdot \overline{w'})^2 + (\rho + \mu)^2]^n} \\ +&t.w.w.s. \end{aligned}$$

So the principal part of the solution for the problem (1.4) just involves two fractional integrals. From the theory of standard singular integral operators, the solution $u^1_{n+1} = N_0(f_{n+1}) = f_{n+1} * N_0$ will gain two in all directions, i.e.,

$$G^{\#} : L^p_k(\overline{D}) \to L^p_{k+2}(\overline{D}) \text{ for } 1 < p < \infty \text{ and } k \in \mathbb{Z}^+.$$

Now we turn to construct the parametrix of the problem (1.3).

<u>(II) The sub-elliptic part</u>

The method in this part are basically follow the idea in Greiner- Stein [GS]. In the paper by Chang-Nagel-Stein [CNS1] has a more elegant way to look at this method. First of all we want to construct the Calderón operator $\Box^*$ for the $\overline{\partial}$-Neumann problem. The construction depends on the construction of the Dirichlet to Neumann operator. Under the assumption that $\overline{U}$ is equipped with a Levi metric, the matrix-valued operator $\Box^+$ becomes:

$$\begin{aligned} \Box^* &= \left\{ \left[-\frac{1}{2} \sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j) - \frac{1}{2} \frac{\partial^2}{\partial t^2} - i(n-2)T \right] - \frac{1}{2} \frac{\partial^2}{\partial \rho^2} \right\} I_n \\ &= \left\{ -\frac{1}{2} \frac{\partial^2}{\partial \rho^2} - \frac{1}{2} \left[\sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j) + \frac{\partial^2}{\partial t^2} \right] - i(n-2)T \right\} I_n \\ &= \left\{ -\frac{1}{2} \frac{\partial^2}{\partial \rho^2} - A + B \right\} I_n. \end{aligned}$$

Here I_n is the $n \times n$ identity matrix. Hence we may treat $\Box^*$ as a scalar operator.

We have the following theorem, the proof can be found in [CNS2].

<u>Theorem</u>:

The Calderón operator $\Box^+$ is

$$\Box^+ = -(-A)^{\frac{1}{2}} - (-A)^{\frac{1}{2}} B - iT.$$

Once we have this operator, the $\overline{\partial}$-Neumann problem (1.3) is equivalent to the problem of solving

$$\Box^+(u_j)\overline{\omega}_j = -Rest(\overline{Z}_{n+1} G^+(f_j)\overline{\omega}_j)$$

i.e., the $\overline{\partial}$-Neumann problem is equivalent to the problem of inverting the operator $\Box^+$. If we look at the symbol of $\Box^+$.

$$\sigma(\Box^+) = (\tau - \Delta) + \frac{1}{2}(2-n)\frac{\tau}{\Delta}$$

where $\Delta = (2\sum_{j=1}^n |\sigma(Z_j)|^2 + \tau^2)^{\frac{1}{2}}$. The $\Box^+$ is a first order pseudodifferential operator double characteristic on half the line bundle $\sum$ on the cotangent bundle $T^*(\partial U)$. So far we have dealt exclusively with the $\overline{\partial}$-Neumann problem on the domain U. However, we may also consider the $\overline{\partial}$-Neumann problem on the complementary domain $U^- = \mathbb{C}^{n+1}\setminus\overline{U}$. This problem also gives rise to a first order pseudo-differential operator which we call $\Box^-$. Then a calculation similar to the one above gives us

$$\sigma(\Box^-) = (\tau + \Delta) - \frac{1}{2}(2-n)\frac{\tau}{\Delta}.$$

We can see $\Box^-$ is characteristic on the other half of $\sum$ but elliptic on the characteristics of $\Box^+$. But the important phenomenon is that

$$\Box^- \circ \Box^+ = -\Box_b + \text{ zero order term}$$

and

$$\Box^+ \circ \Box^- = -\Box_b + \text{ zero order term.}$$

Here $\Box_b = -\frac{1}{2}\sum_{j=1}^n (Z_j\overline{Z}_j + \overline{Z}_j Z_j) - i(n-2)T$ is the complex sub-Laplacian on the (0.1)-forms defined on the boundary ∂D. From the result of Folland-Stein [FS], when $n \geq 2$, $\Box_b$ has an inverse K_{n-2} given by a convolution operator on the Heisenberg group with kernel $\kappa_{n-2}(z', t)$:

$$\kappa_{n-2}(z', t) = \frac{\Gamma(n-1)}{2^{2-2n}\pi^{n+1}} \frac{1}{(|z'| - it)^{n-1}(|z'| + it)^1}.$$

Putting these results together, we have

$$u_j\overline{\omega}_j = G^*(f_j)\overline{\omega}_j + P(K_{n-2}\Box^- \text{Rest } \overline{Z}_{n+1}G^+(f_j))\overline{\omega}_j + S_{-\infty}(f_j) \tag{1.7}$$

where $S_{-\infty}$ is a smoothing operator and P is a pseudo- differential operator of Poisson type (we will discuss such operators in the following section).
Here

$$u_j = G^*(f_j) + P([u_j]_b)$$

is a solution for the elliptic boundary value problem:

$$\begin{cases} \Box^* u_j = f_j & \text{in } U \\ u_j|_{\partial U} = [u_j]_b & \text{on } \partial U \end{cases}$$

From the discussion above, we can write down the paramatrix for the problem (1.3) from (1.7). For $n \geq 2$, let $\mathcal{N}_j$ be the jth component of the fundamental solution (Neumann operator N), then

$$\mathcal{N}_j = G^* + P(\kappa_{n-2}\Box^- \text{Rest}\overline{Z}_{n+1}G^+) + \text{Smoothing operator.} \tag{1.8}$$

From the results of Folland-Stein [FS], Greiner-Stain [GS] and Rothschild-Stein [RS], we know that this result not only true for the model domain $D = \{Imz_{n+1} > |z'|^2\}$ but also true for any bounded strongly pseudo-convex domain $\Omega \subset \mathbb{C}^{n+1}$ ($n \geqq 2$) with smooth boundary $\partial\Omega$. We may transfer κ_{n-2} the inverse of $\Box_b$ from a small neighborhood $x \in U$ of the Heisenberg group to a small neighborhood of $\underline{y} \in V \subset \partial\Omega$ by using an admissible Heisenberg coordinate system $\circledS(x, y)$. We may cover $\overline{\Omega}$ by coordinate sets $U_1, \cdots, U_m$ and for each j we fix an orthonormal frame $\{Z_1^j, \cdots, Z_{n+1}^j\}$ of $T^{(1,0)}(\overline{U}_j)$ and an admissible Heisenberg coordinate system $\circledS_j$ with respect to the frame. Let ϕ_j be nonnegative C^∞ functions such that $\{\phi_j^2\}$ is a partition of unity subordinate to the cover $\{U_j\}$. We define our parametrix, or first approximation $N_{\circledS}((z', t, \rho), (w', s, \mu))$ to the Neumann operator by

$$\begin{aligned} & N_{\circledS}((z', t; \rho), (w', s; \mu)) \\ & = \sum_{j=1}^{m} \Bigg\{ \sum_{k=1}^{n} \phi_j(z', t; \rho) \mathcal{N}_k(\circledS_j((z', t; \rho), (w', s; \mu))) \\ & \cdot \phi_j(w', s; \mu) \omega_k^j(z', t; \rho) \Lambda \overline{\omega}_k^j(w', s; \mu) \\ & \oplus \phi_j(z', t; \rho) G^{\#}(\circledS_j((z', t; \rho), (w', s; \mu))) \\ & \cdot \phi_j(w', s; \mu) \omega_{n+1}^j(z', t; \rho) \Lambda \overline{\omega}_{n+1}^j(w', s; \mu) \Bigg\}. \end{aligned}$$

Here $\mathcal{N}_k$ is given by (1.8) and G^+ is the fundamental solution for the problem

(1.4) Here $\left\{\omega_k^j\right\}_{k=1}^{n+1}$ is the basis of $\Lambda^{(1,0)}(U_j)$ dual to $\left\{Z_k^j\right\}_{k=1}^{n+1}$.

For the case $n = 1$ (i.e., $\mathbb{C}^2$ case). We are not only have the result for the model domain D (hence the result for a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^2$) but also have results for all pseudoconvex finite type domains in $\mathbb{C}^2$. Suppose Ω is a bounded, pseudoconvex domain of finite type m. (In particular, when Ω is strongly pseudoconvex domain, then $m = 2$). We recall the results of Chang- Nagel-Stein [CNS1] as follows: First we consider

$$N_a(f) = P(\Gamma_+ K \Box^- \mathrm{Rest}(\overline{Z}_2 - s) G^+(f)) + G^+(f) \tag{1.9}$$

where P is the Poisson operator, Γ_+ is a pseudo- differential operator of order zero whose principal symbol equals 1 on the set

$$\{\sigma([-(T)^2 - \overline{Z}_1 Z_1]^{1/2}) < \frac{1}{4}\sigma(-iT)\},$$

and whose principal symbol equals 0 on the set

$$\{\sigma([-(T)^2 - \overline{Z}_1 Z_1]^{1/2}) < \frac{1}{2}\sigma(-iT)\},$$

K is a parametrix of $\square_b$. s is a smooth function given by $\bar{\partial}\bar{\omega}_1 = s\bar{\omega}_1 \wedge \bar{\omega}_2$. The we have the following theorem:

Theorem:

the operator N_a given by formula (1.9) is an approximation to $\mathbf{N}$ in the following sense. Fix $-\omega < k' < k < \infty$. Then there are error operators ε_1 and ε_2, so that

$$N(f) = N_a(f) + \varepsilon_1(f) + \varepsilon_2(u).$$

Here ε_1 is of the same kind as the operator N_a, except that at least one of the following replacements have been made: Γ_+ has been replaced by a pseudodifferential operator of order $\leqq -1$, K has been replaced by a NIS operator smoothing of degree ≥ 3; and $\square^-$ has been replaced by a pseudodifferential operator of order $\leqq 0$. Moreover, the operator ε_2 maps $L^2_{k'}$ to L^2_k.

Here the class of NIS operators of smoothing degree k (NIS means nonisotropic smoothing are sutdied by [NRSW] and [CNS1]. Here we just write down its definition.

Definition

T is a NIS operator of smoothing degree k, if there exists a family $T_\varepsilon(f)(x) = \int_{\partial\Omega} T_\varepsilon(x, y)f(y)d\sigma(y)$ so that (i) $T_\varepsilon(f) \to T(f)$ in C^∞, as $\varepsilon \to 0$, whenever $f \in C^\infty(\partial\Omega)$; (ii) each $T_\varepsilon(x,y)$ is in $C^\infty(\partial\Omega \times \partial\Omega)$; and (iii) the following two conditions hold uniformly in ε. (In writing these conditions we shall omit the subscript ε.)

(a)
$$|X_x^I X_y^J G(x,y)| \leq C_{\ell_1,\ell_2} \frac{\rho(x,y)^{k-\ell_1-\ell_2}}{V(x,y)}$$

where $|I| = \ell_1$, $|J| = \ell_2$, $0 \leqq \ell_1, \ell_2 < \infty$. Here we have used the abbreviation $X^I = X_{i_1} X_{i_2} \cdots X_{i_{\ell_1}}$, where $X_{i_j} = ReZ_1$ or ImZ_1. The subscript x in X_x^I indicates the variable to be differentiated.

(b) For each $\ell \geq 0$, there exists an $N = N_\ell$, so that whenever φ is a smooth (bump) function supported in $B(x_0, \delta)$, then

$$|(X^I T(\varphi))(x_0)| \leqq c_\ell \delta^{k-\ell} \sup_x \sum_{|J| \leqq N_\ell} \delta^{|J|} |X^J \varphi|,$$

whenever $|I| = \ell$.

The $\rho(x,y)$ is the non-isotropic metric studied by Nagel- Stein-Wainger in [NRS]. $V(x,y)$ is the volume of the ball centered at x with diameter $\rho(x,y)$. The detail can be found in [CNS2] and [NRSW], we will not go through it!

Now the regularity properties for the Neumann operator $\mathbf{N}$ reduce to look at the regularity properties for each operator. $G^+, \square^-, K$. Restriction operator and Poisson type operator. The regularity properties for $G^+, \square^-$ and Rest follows from the theory of standard singular integral operator. The regularity properties for the operator K are discussed by [NRSW], Fefferman-Kohn[FK] and Christ [Ch]. So we just need to discuss operators of Poisson type. In section 2, we use a different approach to consider the regularity properties for Poisson type operators.

Remarks

(1) In the case $\Omega = D = \{Im z_2 > |z|^2\}$, we can compute K precisely. Here K is a convolution operator on the Heisenberg group $\mathbb{H}^1$ induced by the kernel $\kappa(z,t)$:

$$\kappa(z,t) = \frac{1}{2\pi^2}\log\left[\frac{|z|^2 - it}{|z|^2 + it}\right]\cdot\frac{1}{|z|^2 - it}. \tag{1.10}$$

It is easy to see that $\kappa(z,t)$ in (1.10) is homogeneous of degree -2 in the Heisenberg sense. From the result of [GS], we know that K is a NIS operator of smoothing degree 2. In the model case, the non-isotropic metric $\rho(x,y) = |x \cdot y^{-1}| \geqq 0$ for $x, y \in \mathbb{H}^n$ and $V(x,y) = |B(x, \rho(x,y))|$.

Hence we know, the parametrix for the problem (1.3) is

$$\mathcal{N}_1 = G^+ + P(\Gamma^+ K \Box^- Rest \overline{Z}_2 G^+)$$

(2) Even though we consider the Levi metric defined on $\overline{\Omega}$, but in fact all these results are true when we consider $\overline{\Omega}$ equipped with an arbitrary Hermitian metric $ds^2 = \sum_{j,k=1}^n a_{jk}\omega_j \otimes \overline{\omega}_k$, where a_{jk} are smooth functions defined on $\overline{\Omega}$. The result can be found in [C] and [CNS2].

2. Operators of Poisson type and the sharp estimates for the Neumann Operator

When we discuss the coercive boundary value problem

$$\begin{aligned} &p(x,D)u = 0 \qquad \text{in } \mathbb{R}^{n+1}_+ \\ &[Q_j(x,D)u]_{\mathbb{R}^n} = g_j, j = 1,2,\cdots,m \end{aligned} \tag{2.1}$$

where $p(x,D)$ is a strongly elliptic differential operator of order $2m$ with coefficients smooth up the boundary, and g_j are given functions on the boundary. Then operator P_j mapping $C^\infty(\mathbb{R}^n)$ to $C^\infty(\overline{\mathbb{R}}^{n+1}_+)$ can be constructed such that if u satisfies (2.1) then

$$u = \sum_{j=1}^m P_j g_j + S_{-\infty} u \tag{2.2}$$

where $S_{-\infty}$ is an infinitely smoothing operator. The operator P_j which play an analogous role to the Poisson Kernel in the case of the Dirichlet problem for the Laplacian.

Now we give the following definition:

<u>Definition</u>:

A function $p(x,t;\xi) \in C^\infty(\mathbb{R}^n \times [0,\varepsilon] \times \mathbb{R}^n)$ is a symbol of Poisson type of order k if it satisfies:

(1) $p(x,t;\xi)$ has compact support in the (x,t) variables

(2) For all multi-indices α, β and integers γ, δ there is a constant $C = C_{\alpha,\beta,\gamma,\delta}$ so that

$$|t^\delta[\frac{\partial}{\partial t}]^\gamma[\frac{\partial}{\partial x}]^\beta[\frac{\partial}{\partial \xi}]^\alpha p(x,t;\xi)| \leq C(1+|\xi|)^{k-|\alpha|+\gamma-\delta}.$$