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Uwe Jannsen

Mixed Motives and Algebraic K-Theory

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Preface

This is an almost unchanged version of my 1988 Habilitationsschrift at Regensburg. My original plan was to completely rewrite it for publication; in particular I wanted to make it more readable for the non–expert. Finally I chose to rather publish it like it is than turn it into a long range project. So I have only made some minor corrections and added three appendices. The first one reproduces a letter from S. Bloch to me and the second one consists of an example by C. Schoen. I thank both for the permission to publish this material, and the latter for the effort of rewriting the example, which also figured in a letter to me. The third appendix contains some remarks and complements written in 1989.

Uwe Jannsen

Bonn, November 1989

Introduction

This text consists of three parts. In part I we define a category of mixed motives in the setting of absolute Hodge cycles. In part II we investigate, as general as possible, relations between algebraic cycles, algebraic K-theory, and mixed structures in the cohomology of arbitrary varieties. In part III we present some conjectures on Chern characters from K-theory into ℓ-adic cohomology for varieties over finite fields or global fields, and prove these in some (very) specific cases.

Background The concept of motives [Ma] ,[Kl] , [SR] was introduced by Grothendieck to explain phenomena in different cohomology theories of algebraic varieties in a coherent way, in particular those related to algebraic cycles and weights. For example in both the ℓ-adic and the Hodge theory the cohomology $H^i(X)$ of a smooth projective variety is pure of weight i, the class of an algebraic cycle of codimension j can be interpreted as a morphism from the trivial structure into $H^{2j}(X)(j)$, and the parallel formulation of the conjectures of Hodge and of Tate is that the functor sending a motive to its cohomological realization is fully faithful.

All this only concerns cycles modulo homological equivalence and does not cover singular or non-compact varieties, which often arise in algebraic geometry. Concerning these, Deligne shows in [D5] §10 that cycles homologous to zero give rise to non-trivial extensions of pure structures of different weights - this is called a mixed structure - and in his treatments of Hodge theory and ℓ-adic cohomology [D5] , [D9] shows that the cohomology of arbitrary varieties gives rise to mixed structures, too. Indeed, both facts are directly related, and one expects a description of the whole Chow group and a satisfactory treatment of arbitrary varieties in the setting of a category of mixed motives [Bei 4] , [D10] . Finally, work of Beilinson suggests that mixed motives are related to higher algebraic K-theory, like cycles are related to K_o [Bei 1] , [Bei 2].

Grothendieck's definition of motives is quite simple, but only gives a satisfactory theory together with the so-called standard conjectures. Deligne has given a "working definition" of motives for absolute Hodge cycles (the latter ones replacing the algebraic cycles in Grothendieck's definition), which often suffices for the applications [DMOS] . An algebraic definition of mixed motives is problematic, since Grothendieck's methods (algebraic correspondences and idempotents) neither apply nor extend in an obvious way.

<u>Part I</u> In §1 we start with the simple but crucial observation that - in the language introduced later - a subrealization of the realization of a motive for absolute Hodge cycles (AH-motive) is a direct factor and hence a submotive. As a corollary we show that there are natural AH-motives associated to modular forms, having as ℓ-adic realizations the representations constructed by Deligne [D1] (Recently, Scholl [Sch 1] constructed these motives algebraically). Another application is the construction of direct factors in the ℓ-adic cohomology.

In §2 we make a precise definition of a category $\underline{R}_k$ in which the realizations of AH-motives over a field k live, by defining a bigger category $\underline{MR}_k$ of mixed realizations, in which also mixed structures are allowed. These obviously are Tannakian categories, and we study some of their formal properties.

In §3 we prove that for a smooth variety U over a field k of characteristic zero its ℓ-adic, deRham and Betti cohomologies define an object H(U) in $\underline{MR}_k$. The techniques applied here are all taken from papers of Deligne, the main point consisting in showing that one has a weight filtration in each theory which is compatible with the comparison isomorphisms, and that the pure quotients are AH-motives.

In §4 the category $\underline{MM}_k$ of mixed motives over k is defined as the Tannakian subcategory of $\underline{MR}_k$ generated by the H(U) . We prove that Deligne's category $\underline{M}_k$ can be identified with the Tannakian subcategory generated by the realizations of smooth, projective varieties, and can be identified with the full subcategory of pure objects in $\underline{MM}_k$. This gives a simpler definition of $\underline{M}_k$ than the original one, avoiding the processes of taking the pseudo-abelian hull, inverting the Lefschetz object and changing the commutation constraints. If G and MG are the associated "Galois groups" of the neutral Tannakian categories $\underline{M}_k$ and $\underline{MM}_k$ (for some fibre functor given by Betti cohomology), then the embedding $\underline{M}_k \hookrightarrow \underline{MM}_k$ defines a homomorphism MG → G , and the above is reflected in an exact sequence of pro-algebraic groups

$$1 \to U \to MG \to G \to 1 ,$$

with connected, pro-unipotent U, identifying G with the maximal pro-reductive quotient of MG .

Part II §5 is, except for theorems 5.13 and 5.15 (comparing $\mathcal{O}(X)^{\times}$ with Deligne cohomology $H^1_{\mathcal{D}}(X,\mathbb{Z}(1))$ or étale cohomology $H^1_{ét}(X,\mathbb{Z}_\ell(1))$), mainly motivational. The conjectures stated here for the smooth case are contained in those formulated later for arbitrary varieties.

In §6 a very important tool appears, the notion, due to Bloch and Ogus [BO], of a twisted Poincaré duality theory, axiomatizing the aspects of a cohomology theory and an associated homology theory.

In this setting the "Poincaré duality" is an isomorphism

(0.1) $\quad H^i(X,j) \underset{\sim}{\rightarrow} H_{2d-i}(X,d-j) \quad , \quad d = \dim X$,

between cohomology and homology for smooth X . We define a version with values in a tensor category, also introducing the concept of weights modeled after the situation for mixed Hodge structures or mixed ℓ-adic sheaves. After discussing ℓ-adic, deRham and Betti-cohomology we prove - extending the results in part I - that there is a Poincaré duality theory with values in $\underline{MR}_k$.

In §7 we propose how to extend the conjectures of Hodge and Tate to arbitrary varieties. The basic observation is that the right setting is the homology, the classical formulations being reobtained by (0.1). We show that this Hodge conjecture is true if and only if the classical Hodge conjecture is, and that the same is basically true for the Tate conjectures.

In §8 we recall some properties of Chern characters and Riemann-Roch transformations assuring that the maps

(0.2) $\quad H_a^{\mathcal{M}}(X,\mathbb{Q}(b))\otimes\mathbb{Q}_\ell \rightarrow H_a^{\acute{e}t}(X\times_k\bar{k},\mathbb{Q}_\ell(b))^{G_k} \quad , \quad \text{char } k \neq \ell$,

(0.3) $\quad H_a^{\mathcal{M}}(X,\mathbb{Q}(b)) \rightarrow \Gamma_{\mathcal{H}} H_a(X(\mathbb{C}),\mathbb{Q}(b)) \quad , \quad k = \mathbb{C}$,

(where $H^{\mathcal{M}}$ is the motivic homology defined by Beilinson via $K_*'(X)$ and $\Gamma_{\mathcal{H}}$ denotes the group of Hodge cycles), satisfy all functorialities of morphisms of Poincaré duality theories. We state conjectures on the surjectivity of (0.2) and (0.3) and extend theorems 5.13 and 5.15 to arbitrary varieties, thus proving the conjectures for curves.

In §9 we discuss relations between extensions of realizations and algebraic cycles homologous to zero. As a consequence we show why a naive extension of the conjectures of Hodge and Tate to the surjectivity of (0.2) and (0.3) for arbitrary $a,b \in \mathbb{Z}$ is false. In particular, this disproves a Hodge-theoretic conjecture by Beilinson [Bei 2] . We deduce the counterexample from examples of Mumford on the non-injectivity of the Abel-Jacobi map

$CH^j(X)_o \rightarrow H^{2j-1}(X,\mathbb{C})/H^{2j-1}(X,\mathbb{Z}(j)) + F^j$.

Then we extend everything to the ℓ-adic Abel-Jacobi maps

$$CH^j(X)_o \to H^1_{cont}(G_k, H^{2j-1}(X\times_k\bar{k}, \mathbb{Z}_\ell(j))) ,$$

by using results of Bloch [Bl 1] .

In §10 we extend Bloch's results to higher-dimensional varieties and show that Abel-Jacobi maps are non-injective quite principally, for any reasonable Poincaré duality theory - provided the base field contains too many parameters. The main theme of our conjectures, and of several conjectures of Bloch and Beilinson, is that the situation is different for finite fields, global function fields, and number fields.

In §11 we recall some ideas of Beilinson on mixed motives [Bei 4]. We stress the fact that his philosophy of mixed motivic sheaves would imply some quite explicit conjectures - extending earlier ones by Bloch - on the structure of Chow groups of smooth projective varieties over arbitrary fields. I think these should be regarded as an extension of Grothendieck's standard conjectures to the whole Chow group. We remark that they would follow from the injectivity of some cycle map.

Part III Our basic conjecture for varieties over finite fields is that here (0.2) is an isomorphism. In §12 we prove it in some cases and show that it would follow from several "classical" conjectures on smooth, projective varieties, at least if we assume a weak form of resolution of singularities. The conjecture would imply a description of motivic homology of arbitrary varieties X over arbitrary fields of positive characteristic, by writing $X = \varprojlim_\alpha \mathcal{X}_\alpha$, with varieties $\mathcal{X}_\alpha$ over $\mathbb{F}_p$ and flat transition maps, since $H^{\mathcal{M}}_a(X,\mathbb{Q}(b)) = \varinjlim_\alpha H^{\mathcal{M}}_{a'}(\mathcal{X}_\alpha,\mathbb{Q}(b'))$. We explain this in more detail for the case of a global function field k . Note that we need non-proper $\mathcal{X}_\alpha$ even for a smooth, projective X , and observe the similarities and the differences to the approach of Artin and Tate in [D.E.] .

We don't have a similarly general conjecture for number fields, but in §13 we discuss a conjecture on the bijectivity of

(0.4) $\quad H^M_a(X,\mathbb{Q}(b))\otimes\mathbb{Q}_\ell \to \tilde{H}^{et}_a(X,\mathbb{Q}_\ell(b))$,

(where $\tilde{H}^{et}_*$ is a certain modified étale homology) in the "stable range" $a > \dim X + b$. This is related to certain Galois cohomological investigations in [J3] .

The extreme counterpart of pure structures are mixed structures whose pure pieces are as simple as possible, i.e., Tate objects, so that only mixed phenomena remain. In §14 we define a class of varieties (containing those stratified by linear spaces, like Grassmannians or flag varieties) with this property, and prove most of our conjectures for these varieties.

Final remarks and acknowledgements

I learnt about motives for absolute Hodge cycles in inspiring lectures by G. Anderson (Harvard 1983/84), and my own investigations were started by a question of N. Schappacher whether the realizations for modular forms come from such motives (see §1). A. Scholl brought my attention to the paper by Bloch and Ogus, and communicated to me some ideas on K-homology and extension classes (cf. §6). It is a pleasure to thank them for this inspiration and the latter two for further discussions.

The first four chapters exist in this form since end of 1985 and were communicated to a few mathematicians. It should be noted that a construction similar to our category $\underline{MR}_k$ also appears in a recent paper by Deligne. It will be clear to the reader how much parts II and III are influenced by work and ideas of Bloch and Beilinson, but I would also like to stress the influence of Deligne's work on 1-motives [D5] and his reinterpretation of Beilinson's ideas in [D10] .

I would like to thank J. Neukirch heartily for his constant enthusiasm and encouragement, and all friends in Regensburg for their interest and support. Also I thank the Max-Planck-Institut at Bonn, where this program was started and where the final part was written. Special thanks go to K. Deutler, M. Grau and H. Wolf-Gazo from the MPI, and in particular to M. Hertl from Regensburg for a phantastic typing under big pressure of time.

Contents

PART I

MIXED MOTIVES FOR ABSOLUTE HODGE CYCLES

§1. Some remarks on absolute Hodge cycles

Let k be a field of characteristic zero, which is embeddable in $\mathbb{C}$. Fix an algebraic closure $\bar{k}$ of k and let $G_k = \mathrm{Gal}(\bar{k}/k)$. In the following we deal with motives for absolute Hodge cycles as defined by Deligne in [D6], see also [DMOS]II §6, in particular we use similar notations as in these references. Then a motive M over k has realizations

$H_{DR}(M)$ - a k-vector space with a descending filtration F^p

$H_l(M)$ - (for each prime number l) a $\mathbb{Q}_l$-vector space, on which G_k acts continuously,

$H_\sigma(M)$ - (for each embedding $\sigma: k \hookrightarrow \mathbb{C}$) a $\mathbb{Q}$-vector space with a Hodge structure on $H_\sigma(M) \otimes \mathbb{R}$, i.e., a $\mathbb{Q}$-Hodge structure,

all of the same finite dimension. Furthermore, there are comparison isomorphisms

$$I_{\infty,\sigma} : H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{DR}(M) \otimes_{k,\sigma} \mathbb{C}$$

and

$$I_{l,\bar{\sigma}} : H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} H_l(M)$$

for each extension $\bar{\sigma} : \bar{k} \hookrightarrow \mathbb{C}$ of σ.

If X is a smooth projective variety over k and $n \geq 0$ an integer, the motive $M = h^n(X)$ is given by the realizations

$$H_{DR}(M) = H^n_{DR}(X) = H^n_{DR}(X/k) \qquad \text{(de Rham cohomology)}$$
$$H_l(M) = H^n_l(X) = H^n_{et}(X \times_k \bar{k}, \mathbb{Q}_l) \qquad \text{(l-adic cohomology)}$$
$$H_\sigma(M) = H^n_\sigma(X) = H^n(X \times_{k,\sigma} \mathbb{C}, \mathbb{Q}) \qquad \text{(singular cohomology)} .$$

The comparison isomorphisms are obtained from the canonical ones between the cohomology theories of the variety $\sigma X = X \times_{k,\sigma} \mathbb{C}$ over $\mathbb{C}$. Namely $I_{1,\bar\sigma}$ is given by

$$H^n(X \times_{k,\sigma} \mathbb{C}, \mathbb{Q}_1) \xrightarrow[\sim]{can} H^n_{et}(X \times_{k,\sigma} \mathbb{C}, \mathbb{Q}_1) \xrightarrow[\sim]{\bar\sigma^*} H^n_{et}(X \times_k \bar{k}, \mathbb{Q}_1)$$

and $I_{\infty,\sigma}$ is induced by

$$H^n(\sigma X, \mathbb{C}) \xrightarrow[\sim]{can} H^n_{DR}(\sigma X/\mathbb{C}) .$$

If we let $h(X) = \bigoplus_{n=0}^{\dim X} h^n(X)$, any motive M is a direct summand of $h(X)(m)$, the m-fold Tate-twist of $h(X)$, for some smooth projective X and some $m \in \mathbb{Z}$.

The following lemma, which describes the possible summands, is rather easy but very important for the following.

1.1. Lemma Let M be a motive over k. Suppose given a k-subspace $U_{DR} \subseteq H_{DR}(M)$, for each l a $\mathbb{Q}_1$-subspace $U_1 \subseteq H_1(M)$, which is a G_k-submodule, and for each $\sigma: k \hookrightarrow \mathbb{C}$ a $\mathbb{Q}$-subspace $U_\sigma \subseteq H_\sigma(M)$, which is a sub-$\mathbb{Q}$-Hodge structure, such that these subspaces correspond under the comparison isomorphisms. Then there is a decomposition $M = M_1 \oplus M_2$ in motives such that $U_\alpha = H_\alpha(M_1) \subseteq H_\alpha(M)$ where α runs through the indices DR, 1 and σ.

Proof As the subspaces U_α are compatible with the weight gradings (this is implicit in the statement that the U_σ are sub-$\mathbb{Q}$-Hodge structures), we may assume M pure of weight r, say. Then there exists a morphism of motives

$$\Psi : M \to \check{M}(-r) \qquad (\check{M} = \text{dual of } M)$$

giving rise to non-degenerate pairings for $\alpha \in \{DR, 1, \sigma\}$

$$\psi_\alpha : H_\alpha(M) \otimes H_\alpha(M) \to H_\alpha(1(-r)) = \begin{cases} k & \alpha = DR \\ \mathbb{Q}_1(-r) & \alpha = 1 \\ \mathbb{Q}(-r) & \alpha = \sigma \end{cases}$$

which are compatible with the various structures like G_k-action for $\alpha = 1$ and Hodge structure for $\alpha = \sigma$ etc., and correspond under the comparison isomorphisms. Moreover, the ψ_σ induce polarizations of real Hodge structures.

$$H_\sigma(M)\otimes\mathbb{R} \otimes H_\sigma(M)\otimes\mathbb{R} \to \mathbb{R}(-r)$$

In fact, to fix ideas we may assume - by twisting with powers of the Tate motive and adding other motives - that M is $h^r(X)$ for a smooth projective variety X of dimension d over k. Then by using a very ample divisor and the hard Lefschetz theorem one constructs an absolute Hodge cycle in $C_{AH}^{2d-r}(X\times X)$ giving a homomorphism

$$\Phi : h^r(X) \to h^{2d-r}(X)(d-r) \quad ,$$

the motivic version of the "*-operator" in Hodge theory, see [DMOS] II 6.2. The pairings ψ_α above are then obtained by combining with the Poincaré pairings

$$H_\alpha^r(X) \otimes H_\alpha^{2d-r}(X) \to H_\alpha^{2d}(X) \underset{\sim}{\overset{tr}{\to}} H_\alpha(1(-d))$$

and twists by $d-r$. Or: the Poincaré pairings give an isomorphism $h^{2d-r}(X)(d-r) \underset{\sim}{\to} h^r(X)^\vee(-r)$, whose composition with Φ is Ψ.

Let V_{DR}, V_1 and V_σ be the orthogonal complements of U_{DR}, U_1 and U_σ, respectively, with respect to the pairings ψ_{DR}, ψ_1 and ψ_σ. By the compatibility of the ψ_α these spaces then correspond under the comparison isomorphisms. Also the V_α are substructures of the $H_\alpha(M)$ like the U_α : the G_k-invariance of V_1 follows from the G_k-invariance of U_1 and ψ_1, and V_σ is a sub-$\mathbb{Q}$-Hodge structure, as ψ_σ is a polarization of $\mathbb{Q}$-Hodge structures. This also shows that $U_\sigma \cap V_\sigma = 0$ (compare Deligne's argument [D4] p. 44, that any sub-structure of a polarized $\mathbb{Q}$-Hodge structure is a direct factor): one has $(2\pi i)^r\psi_\sigma(x,Cx) > 0$ for all $0 \neq x \in H_\sigma(M) \otimes \mathbb{R}$, where C is the Weil operator: $C = i \in S(\mathbb{R}) = \mathbb{C}^{\times}$ acting on every $\mathbb{R}$-Hodge structure, see [D4] (2.1.14). As C respects the sub-Hodge structure $U_\sigma \otimes \mathbb{R}$ we conclude $U_\sigma \otimes \mathbb{R} \cap (U_\sigma \otimes \mathbb{R})^{\perp} = 0$ as claimed. By the comparison isomorphisms we also get $U_1 \cap V_1 = 0$ and $U_{DR} \cap V_{DR} = 0$. The decompositions $H_\alpha(M) = U_\alpha \oplus V_\alpha$ then induce endomorphisms

$$p_\alpha : H_\alpha(M) \xrightarrow{\text{projection}} U_\alpha \to H_\alpha(M)$$

for $\alpha \in \{DR, l, \sigma\}$, which are compatible with the various structures and the comparison isomorphisms, as this is the case for the U- and V-spaces. Therefore the family of the p_α gives an element $p \in \mathrm{End}(M)$ (see [DMOS]II 6.7 (g) or 6.1 for $M = h(X)$, note that p_{DR} respects the Hodge filtration as it is compatible with p_σ and p_σ is a homomorphism of Hodge structures), which is a projector and gives the wanted decomposition by taking $M_1 = \mathrm{Im}\, p$ and $M_2 = \mathrm{Im}(1-p)$; for $M = h(X)$ we have $M_1 = (h(X), p)$ in the notation of [DMOS].

<u>1.2. Corollary</u> If X, Y are smooth varieties over k with X projective, then for any morphism $f: Y \to X$ and $g: X \to Y$ the kernel of

$$f_\alpha^*: H_\alpha^r(X) \to H_\alpha^r(Y) \qquad \alpha \in \{DR, l, \sigma\}$$

is represented by a motive $\mathrm{Ker}\, f^* \subseteq h^r(X)$ and the image of

$$g_\alpha^* : H_\alpha^r(Y) \to H_\alpha^r(X) \qquad \alpha \in \{DR, l, \sigma\}$$

is represented by a motive $\mathrm{Im}\, g^* \subseteq h^r(X)$, and these are direct factors of $h^r(X)$.

<u>Proof</u> The cohomology groups $H_\sigma^r(Y)$ have mixed $\mathbb{Q}$-Hodge structures, and f_σ^* and g_σ^* are morphisms of mixed $\mathbb{Q}$-Hodge structures [D4] . So $\mathrm{Ker}\, f_\sigma^*$ and $\mathrm{Im}\, g_\sigma^*$ are (pure) sub-$\mathbb{Q}$-Hodge structures

of the pure, polarized $\mathbb{Q}$-Hodge structures $H^r_\sigma(X)$. Ker f^*_α and Im g^*_α in the other realizations correspond to Ker f^*_σ and Im g^*_σ under the comparison isomorphisms, as these are functorial and also exist for Y , and of course in the l-adic realizations one gets G_k-invariant subspaces. So we can apply the lemma (with U_α = Ker f^*_α or Im g^*_α) .

In particular we get a result which should be true more generally by a conjecture of Grothendieck-Serre on the semi-simplicity of the action of G_k on the l-adic cohomology.

<u>1.3. Corollary</u> In the situation above, the kernel of

$$f^*_l : H^r_l(X) \to H^r_l(Y)$$

and the image of

$$g^*_l : H^r_l(Y) \to H^r_l(X)$$

are direct factors of $H^r_l(X)$ as G_k-modules .

Of course, similar considerations apply to other natural maps like Gysin maps or the canonical map

$$H^r_c(U) \to H^r(X)$$

of the cohomology with compact support of an open subvariety U of X into the cohomology of a smooth projective variety X . This is needed in the proof of the next corollary.

<u>1.4. Corollary</u> The realizations attached to an elliptic modular form f by Deligne ([D6] §7) belong to a motive $M(f)$.

<u>Proof</u> Let f be a new form of weight $k+2$ $(k \geq 0)$, conductor N and character ε for

$$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \} .$$

There is a smooth projective curve $X_1(N)$ over $\mathbb{Q}$ and an open subvariety

$$j: Y_1(N) \hookrightarrow X_1(N)$$

such that the $\mathbb{C}$-valued points can be identified with

$$\Gamma_1(N)\backslash\mathfrak{h} \hookrightarrow \overline{\Gamma_1(N)\backslash\mathfrak{h}} = \text{compactification by adding the cusps} ,$$

where $\mathfrak{h}$ is the Poincaré upper halfplane.

Let $N \geq 3$; then there is the universal elliptic curve

$$g: E \to Y_1(N) ,$$

and Deligne describes the realizations of $M(f)$ as parts of the "universal cohomology"

$$H^1(X_1(N) , j_*\mathrm{Sym}^k(R^1g_*\mathbb{Q}))$$

(i.e., one has to form the l-adic,de Rham and singular versions of this cohomology), namely as kernel of $T_n - a_n$ for all n prime to N , where the T_n are the Hecke correspondences acting on the cohomology and $f(z) = \sum_{n\geq 1} a_n q^n$, $q = e^{2\pi iz}$. If the a_n are not in $\mathbb{Q}$, one has to take the kernel in the following sense: Let T be the $\mathbb{Q}$-algebra generated by the T_n and $E = \mathbb{Q}(a_1,a_2,\ldots)$, then we have a morphism $T \twoheadrightarrow E$ by $T_n \mapsto a_n$. If $\mathfrak{a}$ is the kernel of this morphism, define the realizations of $M(f)$ as the part annihilated by $\mathfrak{a}$.

By the commutative diagram

$$\begin{array}{ccc} H^1_c(Y_1(N),\mathrm{Sym}^k(R^1g_*\mathbb{Q})) & \twoheadrightarrow & H^1(X_1(N),j_*\mathrm{Sym}^k(R^1g_*\mathbb{Q})) \\ & \searrow^{\varphi} & \downarrow \\ & & H^1(Y_1(N),\mathrm{Sym}^k(R^1g_*\mathbb{Q})) \end{array} ,$$

in which H^1_c denotes cohomology with compact support and the maps are the canonical ones, one can also define the realizations of $M(f)$ to be the kernel of the $T_n - a_n$ in the parabolic cohomology

$$H^1_p(Y_1(N), \mathrm{Sym}^k(R^1 g_* \mathbb{Q})) = \mathrm{Im}(H^1_c(Y_1(N),\ldots) \overset{\varphi}{\to} H^1(Y_1(N),\ldots)).$$

$\mathrm{Sym}^k(R^1 g_* \mathbb{Q})$ is a direct factor of $(R^1 g_* \mathbb{Q})^{\otimes k}$ which in turn is a direct factor of $R^k(g_k)_* \mathbb{Q}$, for

$$g_k : E_k = E \times_{Y_1(N)} \cdots \times_{Y_1(N)} E \to Y_1(N)$$

the k-fold fibre product of g (relative version of the Künneth formula), where by definition $E_o = Y_1(N)$.

Finally the spectral sequence

$$H^p(Y_1(N), R^q(g_k)_* \mathbb{Q}) \Rightarrow H^{p+q}(E_k, \mathbb{Q})$$

degenerates and moreover, as remarked by Lieberman, identifies $H^p(Y_1(N), R^q(g_k)_* \mathbb{Q})$ with the subspace of $H^{p+q}(E_k, \mathbb{Q})$, on which $m \cdot id_{E_k}$ induces the multiplication by m^q , compare [D1] p. 168. The same is true for the cohomology with compact support.

Altogether the realizations of $M(f)$ are direct factors of the cohomology

$$H^{k+1}_p(E_k, \mathbb{Q}) = \mathrm{Im}(H^{k+1}_c(E_k, \mathbb{Q}) \to H^{k+1}(E_k, \mathbb{Q}))$$

which are defined as the kernel of several algebraic correspondences: the T_n are also defined as correspondences of E and so of E_k , see [D1] (3.16), the subquotient of $H^{k+1}_p(E_k, \mathbb{Q})$ which corresponds to

$$H^1_p(Y_1(N), (R^1 g_* \mathbb{Q})^{\otimes k}) \subseteq H^1_p(Y_1(N), R^k(g_k)_* \mathbb{Q})$$

via the spectral sequence can be identified with the subspace of $H^{k+1}_p(E_k, \mathbb{Q})$ where the morphism $m_1 id_E \times \ldots \times m_K id_E$ $(m_i \in \mathbb{Z})$ induces the multiplication by $m_1 \ldots m_k$, and the part corresponding to $\mathrm{Sym}^k(R^1 g_* \mathbb{Q})$ in $(R^1 g_* \mathbb{Q})^{\otimes k}$ can be identified by the action of the symmetric group S_K on E_K .

If one likes - and in particular if one does not like to