Lecture Notes in Mathematics

Michael Bildhauer

Convex Variational Problems

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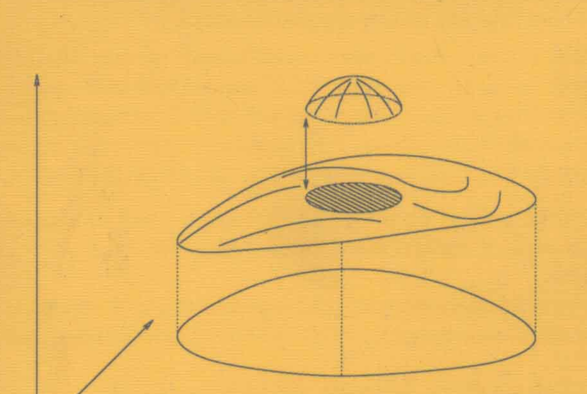
Linear, Nearly Linear and Anisotropic Growth Conditions



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Linear, Nearly Linear and Anisotropic Growth Conditions

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Dedicated to *Christina*

Preface

In recent years, two (at first glance) quite different fields of mathematical interest have attracted my attention.

- Elliptic variational problems with linear growth conditions. Here the notion of a "solution" is not obvious and, in fact, the point of view has to be changed several times in order to get some deeper insight.
- The study of the smoothness properties of solutions to convex anisotropic variational problems with superlinear growth.

It took some time to realize that, in spite of the fundamental differences and with the help of some suitable theorems on the existence and uniqueness of solutions in the case of linear growth conditions, a non-uniform ellipticity condition serves as the main tool towards a unified view of the regularity theory for both kinds of problems.

This is roughly speaking the background of my habilitations thesis at the Saarland University which is the basis for this presentation.

Of course there is a long list of people who have contributed to this monograph in one or the other way and I express my thanks to each of them. Without trying to list them all, I really want to mention:

Prof. G. Mingione is one of the authors of the joint paper [BFM]. The valuable discussions on variational problems with non-standard growth conditions go much beyond this publication.

Prof. G. Seregin took this part in the case of variational problems with linear growth.

Large parts of the presented material are joint work with Prof. M. Fuchs: this, in the best possible sense, requires no further comment. Moreover, I am deeply grateful for the numerous discussions and the helpful suggestions.

Saarbrücken, April 2003 *Michael Bildhauer*

Contents

1

Introduction

One of the most fundamental problems arising in the calculus of variations is to minimize strictly convex energy functionals with respect to prescribed Dirichlet boundary data. Numerous applications for this type of variational problems are found, for instance, in mathematical physics or geometry.

Here we do not want to give an introduction to this topic – we just refer to the monograph of Giaquinta and Hildebrandt ([GH]), where the reader will find in addition an intensive discussion of historical facts, examples and references.

Let us start with a more precise formulation of the problem under consideration: given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a variational integrand $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN})$ we consider the autonomous minimization problem

$$J[w] := \int_\Omega f(\nabla w)\, dx \longrightarrow \min \qquad (\mathcal{P})$$

among mappings $w\colon \Omega \to \mathbb{R}^N$, $N \geq 1$, with prescribed Dirichlet boundary data u_0. Depending on f, the comparison functions are additionally assumed to be elements of a suitable energy class $\mathbb{K}$. In the following, the variational integrand is always assumed to be strictly convex (in the sense of definition), thus we do not touch the quasiconvex case (compare, for instance, [Ev], [FH], [EG1], [AF1], [AF2], [CFM]).

The purpose of our studies is to establish regularity results for (maybe generalized and not necessarily unique) minimizers of the problem $(\mathcal{P})$ under linear, nearly linear and/or anisotropic growth conditions on f together with some appropriate notion of ellipticity: if u denotes a suitable (weak) solution of $(\mathcal{P})$, then three different kinds of results are expected to be true.

THEOREM 1 (Regularity in the scalar case)
Assume that $N = 1$ and that f satisfies some appropriate growth and ellipticity conditions. Then u is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

According to an example of DeGiorgi (see [DG3], compare also [GiuM2], [Ne] and the recent example [SY]), there is no hope to prove an analogous result of this strength in the vectorial setting. Here we can only hope for

THEOREM 2 (PARTIAL REGULARITY IN THE VECTOR-VALUED CASE)
Assume that $N > 1$ and that f satisfies some appropriate growth and ellipticity conditions. Then there is an open set $\Omega_0 \subset \Omega$ of full Lebesgue measure, i.e. $|\Omega - \Omega_0| = 0$, such that $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$, $0 < \alpha < 1$.

Finally, an additional structure condition might improve Theorem 2 to full regularity (see [Uh], earlier ideas are due to [Ur]):

THEOREM 3 (FULL REGULARITY IN THE VECTOR-VALUED CASE WITH SOME ADDITIONAL STRUCTURE)
Suppose that in the vectorial setting the integrand f satisfies in addition $f(Z) = g(|Z|^2)$ for some function $g\colon [0,\infty) \to [0,\infty)$ of class C^2 (plus some Hölder condition for the second derivatives). Then u is of class $C^{1,\alpha}(\Omega; \mathbb{R}^N)$, $0 < \alpha < 1$.

As the essential assumptions, the growth and the ellipticity conditions on f are involved in the above theorems. Hence, in order to make our discussion more precise and to summarize the various cases for which Theorems 1–3 are known to be true, we first introduce some brief classification of the integrands under consideration with respect to both growth and ellipticity properties. We also remark that in the cases A and B considered below the existence (and the uniqueness) of minimizers in suitable energy spaces is easily established.

Before going through the following list it should be emphasized that we do not claim to give an historical overview which is complete to some extent.

A.1 POWER GROWTH

Having the standard example $f_p(Z) = (1 + |Z|^2)^{p/2}$, $1 < p$, in mind, let us assume that the growth rates from above and below coincide, i.e. for some number $p > 1$ and with constants c_1, c_2, C, λ, $\Lambda > 0$ the integrand f satisfies for all Z, $Y \in \mathbb{R}^{nN}$ (note that the second line of (1) implies the first one)

$$\begin{aligned} c_1 |Z|^p - c_2 \le \quad & f(Z) \quad \le C\left(1 + |Z|^p\right), \\ \lambda\left(1 + |Z|^2\right)^{\frac{p-2}{2}} |Y|^2 \le & D^2 f(Z)(Y,Y) \le \Lambda\left(1 + |Z|^2\right)^{\frac{p-2}{2}} |Y|^2 . \end{aligned} \tag{1}$$

With the pioneering work of DeGiorgi, Moser, Nash as well as of Ladyzhenskaya and Ural'tseva, Theorem 1 is well known in this setting and of course many other authors could be mentioned (see [DG1], [Mos], [Na] and [LU1] for a complete overview and a detailed list of references).

As already noted above, the third theorem in this setting should be mainly connected to the name of Uhlenbeck (see [Uh], where the full strength of (1) is not needed which means that also degenerate ellipticity can be considered).

Without additional structure conditions in the vectorial case, the two-dimensional case $n = 2$ substantially differs from the situation in higher dimensions: a classical result of Morrey ensures full regularity if $n = 2$ (here we like to refer to [Mor1], the first monograph on multiple integrals in the calculus of variations, where again detailed references can be found).

Finally, Theorem 2 is proved in any dimension and in a quite general setting by Anzellotti/Giaquinta ([AG2]), where the whole scale of integrands up to the limit case of linear growth is covered (with some suitable notion of relaxation). In addition, the assumptions on the second derivatives are much weaker than stated above, i.e. their partial regularity result is true whenever $D^2 f(Z) > 0$ holds for any matrix Z.

To keep the historical line, we like to mention the earlier contributions on partial regularity [Mor2], [GiuM1], [Giu1] (compare also [DG2], [Alm], a detailed overview is found in [Gia1]).

A.2 ANISOTROPIC POWER GROWTH

The study of anisotropic variational problems was pushed by Marcellini ([Ma2]–[Ma7]) and is a natural extension of (1). To give some motivation we consider the case $n = 2$, $2 \leq p \leq q$ and replace f_p by

$$f_{p,q}(Z) = \left(1 + |Z|^2\right)^{\frac{p}{2}} + \left(1 + |Z_2|^2\right)^{\frac{q}{2}}, \quad Z = (Z_1, Z_2) \in \mathbb{R}^{2N},$$

hence f is allowed to have different growth rates from above and from below. The natural generalization of the structure condition (1) is the requirement that f satisfies (again the growth conditions on the second derivatives imply the corresponding growth rates of f)

$$\begin{aligned} c_1 |Z|^p - c_2 \leq \quad & f(Z) \quad \leq C\left(1 + |Z|^q\right), \\ \lambda\left(1 + |Z|^2\right)^{\frac{p-2}{2}} |Y|^2 \leq & D^2 f(Z)(Y, Y) \leq \Lambda\left(1 + |Z|^2\right)^{\frac{q-2}{2}} |Y|^2 \end{aligned} \tag{2}$$

for all Z, $Y \in \mathbb{R}^{nN}$, where, as usual, c_1, c_2, C, λ, Λ denote some positive constants and $1 < p \leq q$.

If p and q differ too much, then it turns out that even in the scalar case singularities may occur (to mention only one famous example we refer to [Gia2]). However, following the work of Marcellini, suitable assumptions on p and q yield regular solutions (compare Section 3.5 for a discussion of these conditions). Note that [Ma5] also covers the case $N > 1$ with some additional structure condition.

In the general vectorial setting only a few contributions are available, we like to refer to the papers of Acerbi/Fusco ([AF4]) and Passarelli Di Napoli/Siepe ([PS]), where partial regularity results are obtained under quite restrictive assumptions on p and q excluding any subquadratic growth (again see Section 3.5).

If an additional boundedness condition is imposed, then the above results are improved by Esposito/Leonetti/Mingione ([ELM2]) and Choe ([Ch]). In [ELM2] higher integrability (up to a certain extent) is established ($N \geq 1$, $2 \leq p$) under a quite weak relation between p and q. A theorem of the third type is found in [Ch].

B.1 Growth conditions involving N-functions

Studying the monograph of Fuchs and Seregin ([FuS2]) it is obvious that many problems in mathematical physics are not within the reach of power growth models – the theories of Prandtl-Eyring fluids and of plastic materials with logarithmic hardening serve as typical examples. The variational integrands under consideration are now of nearly linear growth, for example we have to study the logarithmic integrand

$$f(Z) = |Z| \ln(1 + |Z|)$$

which satisfies none of the conditions (1) or (2).

The main results on integrands with logarithmic structure are proved by Frehse/Seregin ([FrS]: full regularity if $n = 2$), Fuchs/Seregin ([FuS1]: partial regularity if $n \leq 4$), Esposito/Mingione ([EM2]: partial regularity in any dimension) and finally by Mingione/Siepe ([MS]: full regularity in any dimension).

B.2 The first extension of the logarithm

As a first natural extension one may think of integrands which are bounded from above and below by the same quantity $A(|Z|)$, where A: $[0, \infty) \to [0, \infty)$ denotes some arbitrary N-function satisfying a Δ_2-condition (see [Ad] for the precise definitions). Although this does not imply some natural bounds (in terms of A) on the second derivatives, (1) and (2) suggest the following model: given a N-function A as above, positive constants c, C, λ and Λ, we assume that our integrand f satisfies

$$\begin{gathered} c\,A(|Z|) \leq \quad f(Z) \quad \leq C\,A(|Z|)\,, \\ \lambda\left(1+|Z|^2\right)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y,Y) \leq \Lambda\left(1+|Z|^2\right)^{\frac{q-2}{2}} |Y|^2 \end{gathered} \tag{3}$$

for all Z, $Y \in \mathbb{R}^{nN}$ and for some real numbers $1 \leq \mu$, $1 < q \leq 2$, this choice being adapted to the logarithmic integrand which satisfies (3) with $\mu = 1$ and $q = 1 + \varepsilon$ for any $\varepsilon > 0$. Note that the correspondence to (1) and (2) is only of formal nature: since we require $\mu \geq 1$, the μ-ellipticity condition, i.e. the first inequality in the second line of (3), does not give any information on the lower growth rate of f in terms of a power function with exponent $p > 1$.

A first investigation of variational problems with the structure (3) under some additional balancing conditions is due to Fuchs and Osmolovskii ([FO]), where Theorem 2 is shown in the case that $\mu < 4/n$.

Theorems of type 1 and 3 are established by Fuchs and Mingione (see [FuM]) – their assumptions on μ and q are discussed in Section 3.5.

C LINEAR GROWTH

It remains to discuss the case of variational problems with linear growth. On account of the lack of compactness in the non-reflexive Sobolev space $W_1^1(\Omega;\mathbb{R}^N)$, the problem $(\mathcal{P})$ in general fails to have solutions. Thus one either has to introduce a suitable notion of generalized minimizers (possibility *i*)) or one must pass to the dual variational problem (possibility *ii*)).

ad *i*). Since the integrand f under consideration is of linear growth, any J-minimizing sequence $\{u_m\}$, $u_m \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$, is uniformly bounded in the space $BV(\Omega;\mathbb{R}^N)$. This ensures the existence of a subsequence (not relabeled) and a function u in $BV(\Omega;\mathbb{R}^N)$ such that $u_m \to u$ in $L^1(\Omega;\mathbb{R}^N)$. Thus, one suitable definition of a generalized minimizer u is to require $u \in \mathcal{M}$, where the set $\mathcal{M}$ is given by

$$\mathcal{M} = \Big\{u \in BV\big(\Omega;\mathbb{R}^N\big) : u \text{ is the } L^1\text{-limit of a } J\text{-minimizing sequence} \\ \text{from } u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)\Big\} .$$

Another point of view is to define a relaxed functional $\hat{J}$ on the space $BV(\Omega;\mathbb{R}^N)$ (a precise notion of relaxation is given in Appendix A). Then generalized solutions of the problem $(\mathcal{P})$ are introduced as minimizers of a relaxed problem $(\hat{\mathcal{P}})$.

Remark 1.1. *We already like to mention that these formally different points of view in fact lead to the same set of functions. Moreover, the third approach to the definition of generalized minimizers given in [Se1], [ST] also leads to the same class of minimizing objects.*

ad *ii*). Following [ET] we write

$$J[w] = \sup_{\tau \in L^\infty(\Omega;\mathbb{R}^{nN})} l(w,\tau) , \quad w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N) ,$$

where $l(w,\tau)$ denotes some natural Lagrangian (see Section 2.1.1). If we let

$$R:\ L^\infty\big(\Omega;\mathbb{R}^{nN}\big) \to \overline{\mathbb{R}} ,$$

$$R(\tau) := \inf_{u \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} l(u,\tau) = \begin{cases} -\infty , & \text{if } \operatorname{div}\tau \neq 0 , \\ l(u_0,\tau) , & \text{if } \operatorname{div}\tau = 0 , \end{cases}$$

then the dual problem reads as

$$\text{to maximize } R \text{ among all functions in } L^\infty\left(\Omega;\mathbb{R}^{nN}\right)\,, \qquad (\mathcal{P}^*)$$

where the existence of solutions easily is established.

In any of the above definitions the set of generalized minimizers of the problem $(\mathcal{P})$ may be very "large". In contrast to this fact, the solution of the dual problem is unique (see the discussion of Section 2.2). Moreover, the dual solution σ admits a clear physical or geometrical interpretation, for instance as a stress tensor or the normal to a surface. Hence, in the linear growth situation we wish to complete the above theorems by analogous regularity results for σ.

C.1 GEOMETRIC PROBLEMS OF LINEAR GROWTH

One of the most important (scalar) examples is the minimal surface case $f(Z) = \sqrt{1+|Z|^2}$. A variety of references is available for the study of this variational integrand, let us mention the monographs of Giusti ([Giu2]) and Giaquinta/Modica/Souček ([GMS2]) at this point.

At first sight, ellipticity now is very bad since the inequalities in the second line of (3) just hold for the choices $\mu = 3$ and $q = 1$. On the other hand, this rough estimate is not needed because it is possible to benefit from the geometric structure of the problem (see Remark 4.3). A class of integrands with this structure is studied, for instance, in [GMS1] following the a priori gradient bounds given in [LU2]. It turns out that in the minimal surface case generalized $\hat{J}$-minimizers are of class $C^{1,\alpha}(\Omega)$ and that we have uniqueness up to a constant.

C.2 LINEAR GROWTH PROBLEMS WITHOUT GEOMETRIC STRUCTURE

The theory of perfect plasticity provides another famous variational integrand of linear growth. In this case the assumptions of smoothness and strict convexity imposed on f are no longer satisfied. Nevertheless, the example should be included in our discussion since we will benefit in Chapter 2 from the studies of Seregin ([Se1]–[Se6]) on this topic (compare the recent monograph [FuS2]).

The quantity of physical interest is the stress tensor σ, which is only known to be partially regular (compare [Se4]). Even in the two-dimensional setting $n = 2$ we just have some additional information on the singular set (see [Se6]) and the model of plastic materials with logarithmic hardening (as described in B.1) serves as a regular approximation.

It is already mentioned above that the vector-valued linear growth situation is covered by [AG2], provided that we restrict ourselves to smooth and strictly convex integrands. Anzellotti and Giaquinta prove Theorem 2 for generalized $\hat{J}$-minimizers, hence the same regularity result turns out to be true for any $u \in \mathcal{M}$ (see Section 2.3.1 for details). It remains to study the properties of the dual solution which (as noted above) for linear growth problems is a quantity of particular interest.

Before we summarize this brief overview in the table given below, we like to mention that of course there is a variety of further contributions where the class of admissible energy densities is equipped with some additional structure (see [AF4], [Lie2], [UU] and many others).

SOME KNOWN REGULARITY RESULTS IN THE CONVEX CASE

	$N = 1$	$N > 1$
A.1	(1) DeGiorgi, Moser, Nash, Ladyzhenskaya/Ural'tseva $\leq$ '65	(2) Anzellotti/Giaquinta '88 (3) Uhlenbeck, '77
A.2	(1) $1 < p \leq q < \dots$ Marcellini $\approx$ '90	(2) $2 \leq p \leq q < \dots$ Acerbi/Fusco '94, (3) bounded $\dots$, Choe '92
B.1	see $N > 1$	(3) $n = 2$: Frehse/Seregin '98 (2) $n \leq 4$: Fuchs/Seregin '98 (2) Esposito/Mingione '00 (3) Mingione/Siepe '99
B.2	(1) $\mu < 1 + 2/n$, $q < \dots$ Fuchs/Mingione '00	(2) $\mu \leq 4/n$, "balanced" Fuchs/Osmolovskii '98 (3) [FM] (see $N = 1$)
C.1	$(1)_{\hat{J}}$ Giaquinta/Modica/ Souček '79	—
C.2	—	$(2)_{\hat{J}}$ [AG] (see A.1, $N > 1$) $(\mathrm{P})_{\sigma,\mathrm{pl}}$ Seregin $\approx$ '90

(1), (2), (3): Theorems 1–3, respectively

$(1)_{\hat{J}}$, $(2)_{\hat{J}}$: corresponding results for generalized $\hat{J}$-minimizers

$(\mathrm{P})_{\sigma,\mathrm{pl}}$: partial regularity for the stress tensor in the theory of perfect plasticity

In the following we are going to

- have a close look at linear growth problems;
- unify the results of A and B by the way including new classes of integrands;
- discuss the substantial extensions which follow in cases A, B and C from a natural boundedness condition.

Our main line skips from linear to superlinear growth and vice versa: in spite of the essential differences, these two items are strongly related by a non-uniform ellipticity condition (see Definition 3.4 and Assumption 4.1), by the applied techniques and to a certain extent by the obtained results. In particular, this relationship becomes evident while studying scalar variational problems with

- mixed anisotropic linear/superlinear growth conditions.

As the first center of interest, the discussion starts in Chapter 2 by considering the general linear growth situation. Here no uniqueness results for generalized minimizers can be expected and we concentrate on the dual solution σ which, according to the above remarks, is a reasonable physical point of view. The main contributions are

i) uniqueness of the dual solution under very weak assumptions;
ii) partial $C^{1,\alpha}$-regularity for weak cluster points of J-minimizing sequences and, as a consequence, partial $C^{0,\alpha}$-regularity for σ;
iii) a proof of the duality relation $\sigma = \nabla f(\nabla^a u^*)$ for a class of degenerate variational problems with linear growth. Here $\nabla^a u^*$ denotes the absolutely continuous part of ∇u^* with respect to the Lebesgue measure.

ad *i)*. Standard arguments from convex analysis (compare [ET]) yield the uniqueness of the dual solution by assuming the conjugate function f^* to be strictly convex. We do not want to impose this condition since it is formulated in terms of f^*, hence there might be no easy way to check this assumption. In fact, using more or less elementary arguments, it is proved in Section 2.2 that there is no need to involve the conjugate function in an uniqueness theorem for the dual solution (see [Bi1]).

ad *ii)*. Following the lines of [GMS1], any weak cluster point $u \in \mathcal{M}$ minimizes the relaxed problem $(\hat{\mathcal{P}})$ associated to the original problem (see Appendix A.1). Alternatively (and as outlined in [BF1]), a local approach is preferred in Section 2.3.1 (see Remark 2.16 for a brief comment). In any case, the results of Anzellotti and Giaquinta apply and u is seen to be of class $C^{1,\alpha}$ on the non-degenerate regular set Ω_u (see (23), Section 2.3). As a next step, the duality relation $\sigma = \nabla f(\nabla u^*)$, $x \in \Omega_{u^*}$, is shown for a particular solution u^*, hence σ is of class $C^{0,\alpha}$ on this set.

ad *iii)*. The duality relation is proved using local $C^{1,\alpha}$-results for some u^* as above. As a consequence, information on the behavior of σ is only obtained on the u^*-regular set. In Section 2.4, the almost everywhere identity

$\sigma = \nabla f(\nabla^a u^*)$ is established for a class of degenerate problems which gives intrinsic regularity results in terms of σ (this is due to [Bi2]). Note that the applied technique completely differs from the previous considerations since we cannot rely on regularity results: arguments from measure theory are combined with the construction of local comparison functions (see Appendix B.3).

Chapter 3 deals with the nearly linear and/or anisotropic situation. Here

i) we introduce the notion of integrands with (s, μ, q)-growth;

and give a unified and extended approach to

ii) the results of type (1) and (3) outlined in the above table;
iii) the corresponding theorems (2).

Finally, reducing the generality of the previous sections, a theorem on

iv) full $C^{1,\alpha}$-regularity of solutions of two-dimensional vector-valued problems with anisotropic power growth

completes Chapter 3.

ad *i*). The main observation is clarified in Example 3.7. Three free parameters occurring in the structure and growth conditions imposed on the integrand f determine the behavior of solutions, which now uniquely exist in an appropriate energy class: the growth rate s of the integrand f under consideration, and the exponents μ, q of a non-uniform ellipticity condition. This leads to the notion of integrands with (s, μ, q)-growth which includes and extends the list given in A and B in a natural way. Note that related structure conditions for variational integrands with superquadratic growth are introduced in [Ma5]–[Ma7] (see Section 3.5 for a brief discussion).

ad *ii*). Since regular solutions cannot be expected for the whole range of s, μ and q (we already mentioned [Gia2]), we impose the so called (s, μ, q)-condition. Observe that we do not lose information in comparison with the known results (see Section 3.5).

As a next step, uniform a priori L^q_{loc}-estimates for the gradients of a regularizing sequence are proved. This enables us to apply DeGiorgi-type arguments with uniform local a priori gradient bounds as the result. The conclusion then follows in a well known manner (we refer to [BFM] for a discussion of scalar variational problems with (s, μ, q)-growth).

It should be emphasized that the proof covers the whole scale of (s, μ, q)-integrands without distinguishing several cases.

ad *iii*). Here a blow-up procedure (compare [Ev], [CFM]) is used to prove partial regularity in the above setting (compare [BF2]). This generalizes the known results to a large extent (see Section 3.5).

ad iv). With the higher integrability results of the previous sections it is possible (following [BF6]) to refer to a lemma due to Frehse and Seregin.

In Chapter 4 we return to problems with linear growth, where we first benefit from some of the techniques outlined in Chapter 3, i.e.

i) a regular class of μ-elliptic integrands with linear growth is introduced.

Then the results are substantially improved by

ii) studying bounded solutions (in some natural sense);
iii) considering two-dimensional problems.

We finish the study of linear growth problems by proving the

iv) sharpness of the results.

ad i). Example 3.9 also provides a class of μ-elliptic integrands with linear growth in the sense that for all $Z, Y \in \mathbb{R}^{nN}$

$$\lambda \left(1 + |Z|^2\right)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y,Y) \leq \Lambda \left(1 + |Z|^2\right)^{-\frac{1}{2}} |Y|^2 \qquad (4)$$

holds for some $\mu > 1$ and with constants λ, Λ. If $\mu < 1 + 2/n$, then this class is called a regular one since generalized minimizers are unique up to a constant and since Theorems 1 and 3 for functions $u \in \mathcal{M}$ will be established following the arguments of Chapter 3 (see [BF3]). Let us shortly discuss the limitation $\mu < 1 + 2/n$. Given a suitable regularization u_δ, it is shown that

$$\omega_\delta := \left(1 + |\nabla u_\delta|^2\right)^{\frac{2-\mu}{4}}$$

is uniformly bounded in the class $W^1_{2,loc}(\Omega)$. This provides no information at all if the exponent is negative, i.e. if $\mu > 2$. An application of Sobolev's inequality, which needs the bound $\mu < 1 + 2/n$, proves uniform local higher integrability of the gradients. The final DeGiorgi-type arguments will lead to the same limitation on the ellipticity exponent μ.

ad ii). The minimal surface integrand can be interpreted as a μ-elliptic example with limit exponent $\mu = 3$ (recall that in the minimal surface case the regularity of solutions is obtained by using the geometric structure).

Section 4.2 and [Bi4] are devoted to the question, whether the limit $\mu = 3$ is of some relevance if the geometric structure condition is dropped. To this purpose some examples are discussed.

Then, imposing a natural boundedness condition, we prove even in the vector-valued setting (without assuming $f(Z) = g(|Z|^2)$) that a generalized minimizer u^* of class $W^1_1(\Omega;\mathbb{R}^N)$ exists. Moreover, u^* uniquely (up to a constant) determines the solutions of the problem

$$\int_\Omega f(\nabla w)\, dx + \int_{\partial\Omega} f_\infty\big((u_0 - w) \otimes \nu\big)\, d\mathcal{H}^{n-1} \to \min \text{ in } W^1_1\big(\Omega;\mathbb{R}^N\big) . \quad (\mathcal{P}')$$