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Géométrie et théorie des groupes

Les groupes hyperboliques de Gromov

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Avant-propos

Ces notes sont issues d'une série d'exposés faits de septembre 1988 à novembre 1988 (et complétés en juin et septembre 1989) au séminaire GT_3 de l'Université Louis Pasteur de Strasbourg. Le thème en était les travaux de Michael Gromov sur l'hyperbolicité. La référence principale est l'article *Hyperbolic Groups* [Gro] de Gromov.

A de rares exceptions près, tous les énoncés se trouvant dans le texte présent sont dûs à M. Gromov. Cependant les preuves de [Gro] sont assez succintes, quand elle ne sont pas laissées entièrement au lecteur, et le fait d'en expliciter le détail nous a paru beaucoup plus qu'un simple exercice.

Nous aimerions que le lecteur partage notre éblouissement devant une théorie qui, presque entièrement basée sur l'inégalité triangulaire dans un espace métrique, permet d'englober un nombre important de résultats de la théorie combinatoire des groupes.

Nous ne sommes pas les seuls commentateurs de Gromov ! Signalons en particulier le recueil [G-H] édité par Etienne Ghys et Pierre de la Harpe, issu d'un séminaire à Berne, les notes [Sho] éditées par Hamish Short, d'un séminaire à Berkeley, consacrées à ce même sujet, et un preprint de B. Bowditch [Bo] écrit dans un esprit analogue. Avec tout cela, que le lecteur se rassure, l'article original de Gromov, d'une inépuisable richesse, laisse encore de multiples faces cachées.

C'est avec plaisir que nous remercions plusieurs personnes qui ont soutenu ce travail, en particulier Daniel Bennequin, Martin Lustig, Bernard Morin (responsable du séminaire GT_3) et Hamish Short qui se sont intéressés de près à ce séminaire et nous ont encouragés à rédiger les notes. Nous remercions aussi toutes les personnes qui ont assisté régulièrement aux exposés, ou qui nous ont aidés, soit par leurs questions, ou bien par leurs remarques ou commentaires : François Apéry, Marie-Claude Arnaud, Michèle Audin, Julien Duval, Francis Bonahon, Henri Gaudier (qui nous a expliqué la théorie des automates), Steve Gersten, Pierre Julg, Marie-Paule Muller, Frédéric Paulin et Marc Troyanov.

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Abstract

This book grew out of a series of lectures made by the three authors at the University of Strasbourg. The purpose of the lectures was to introduce Gromov's theory of hyperbolic groups. Basically, we have tried to write a self-contained text, with complete proofs, for part of Gromov's theory which, sometimes, is only outlined in [Gro].

In **Chapter 1** , we begin by introducing some basic definitions of Gromov's : if X is a metric space, with x_0 a basepoint of X , the *Gromov product* of two points x and y in X is defined as $(x.y) = 1/2(\ \mathrm{dist}(x,x_0) + \mathrm{dist}(y,x_0) - \mathrm{dist}(x,y))$, and the space X is said to be δ-hyperbolic if for every choice of a basepoint, we have the following inequality for any three points x , y and z in the space : $(x.y) \geq \min(\ (x.z)\ ,\ (y.z)\ ) - \delta$. We shall henceforth call δ-hyperbolicity "hyperbolicity in the Gromov sense", or, simply, "hyperbolicity", when there is no ambiguity. In the case where the metric space is geodesic (i. e. the distance between any two points is given by the length of a path joining them), we provide several equivalent definitions of hyperbolicity in terms of uniform geometric properties of the geodesic triangles in that space. One definition, which is easy to state, is the following : a geodesic metric-space X is hyperbolic if there exists a constant $\delta \geq 0$ such that for any geodesic triangle Δ in X, each side of Δ is contained in the δ–neighborhood of the union of the two other sides.

We prove that for any $n \geq 2$, the hyperbolic space $\mathbb{H}^n$ (the n-dimensional Riemannian simply-connected space of constant curvature - 1) is hyperbolic in the Gromov sense, and more generally, any simply connected Riemannian manifold whose sectional curvature is bounded above by a strictly negative constant is hyperbolic. Another important class of examples of hyperbolic metric spaces is the class of $\mathbb{R}$-trees.

Chapter 2 is about the boundary of a hyperbolic space. If X is a hyperbolic metric space, Gromov's boundary ∂X is defined as a set of equivalence classes of sequences of points converging to infinity in an appropriate sense. Gromov's scalar product extends to the union $X \cup \partial X$. In the case of a geodesic space which is proper (i. e. in which every closed ball is compact), we introduce Gromov's topology on the union $X \cup \partial X$, with respect to which this space is compact, and such that X is an open dense subset of $X \cup \partial X$. Whenever X is a simply-connected Riemannian manifold of nonpositive curvature which is complete, there are canonical homeomorphisms between ∂X and the unit sphere in the tangent space at each point of X .

It is often useful to equip a noncompact space with a boundary, such that some of the properties of the space are reflected in the properties of the boundary. Isometries between hyperbolic metric spaces induce topological maps between their boundaries. To give an example of

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how useful is the boundary, we shall see in Chapter 6 that when Γ is a hyperbolic group, acting on itself by left translations, the induced action of Γ on $\partial\Gamma$ gives information about the algebraic properties of Γ . In a subsequent chapter (Chapter 11), we shall see that the boundary of a hyperbolic space is equipped with a natural class of metrics.

In **Chapter 3** , we study quasi-geodesics and quasi-isometries of hyperbolic spaces. We prove an important property of hyperbolic spaces, that is, the stability of quasi-geodesics : for every quasi-geodesic path in X , there exists a geodesic path in X which is close to it (in the Hausdorff metric on closed subsets of X). Another property is some kind of a local criterion for a path in X to be quasi-geodesic : to check that some path g is a quasi-geodesic, it is sufficent to examine individually all subpaths of g of a certain bounded length. All the constants that appear in these statements are uniform.

After quasi-geodesics, we study quasi-isometries. We prove that if $f : X_1 \to X_2$ is a quasi-isometry between metric spaces, and X_2 is hyperbolic, then X_1 is also hyperbolic, and f induces an embedding ∂f from ∂X_1 to ∂X_2 . At the end of this chapter, we prove that every 0-hyperbolic space is an $\mathbb{R}$-tree.

In **Chapter 4** , we introduce hyperbolic groups. A finitely generated group Γ equipped with a finite system of generators G is said to be hyperbolic if Γ is hyperbolic with respect to the word metric associated to G , or equivalently, if the associated Cayley graph is hyperbolic. (It is sometimes more convenient to deal with the Cayley graph rather than the groupitself; in fact, some properties are easier to state, the reason being that this space is geodesic). We show that hyperbolicity of a group is independent of the choice of the generating system. In particular, we have a well-defined notion of boundary for a hyperbolic group. We show that a group Γ is hyperbolic if and only if there exists a proper geodesic hyperbolic metric space X on which Γ acts properly discontinuously, with X/Γ compact. The fundamental group of a compact Riemannian manifolds of negative sectional curvature is hyperbolic, and the boundary of the group is homeomorphic to a sphere. If Γ' is a subgroup of finite index of a group Γ, then Γ' is hyperbolic if and only if Γ is hyperbolic. Finally, we give a solution to the word problem and to the conjugacy problem for hyperbolic groups.

In **Chapter 5** , we construct, for a given hyperbolic group, a contractible locally finite and finite-dimensional simplicial complex P on which Γ acts simplicially, in such a way that this action is free and transitive on the vertices, and such that the quotient space P/Γ is compact. In particular, this proves that a hyperbolic group is finitely presented. From this action, we deduce the following algebraic corollaries : If Γ is a torsion-free hyperbolic group, there exists a finite simplicial complex which is a $K(\Gamma,1)$. In particular, Γ has finite cohomological dimension. Without the hypothesis that Γ is torsion-free, we find that cohomology with rational coefficients if finite-dimensional. In the last section of this chapter, we prove that the simplicial complex referred to above verifies a linear isoperimetric inequality of a combinatorial type.

Chapter 6 is about linear isoperimetric inequalities in hyperbolic spaces. We have adopted Gromov's analytic approach, and we suppose in this chapter that our space X is a Riemannian manifold. We prove that hyperbolicity is equivalent to the existence of linear isoperimetric inequality satisfied by the space. To prove this equivalence, the main step is an analytic lemma which gives some bound on the geometry of equilateral triangles of the 2-dimensional plane equipped with a metric which is conformal to the Euclidean metric.

As an application, we give a characterization of hyperbolic groups in terms of linear isoperimetric inequalities. Given a finitely generated group Γ, we can realize it as the fundamental group of some riemannian manifold V. Then Γ is hyperbolic if and only if V satisfies a linear isoperimetric inequality.

In **Chapter 7**, we show how to use the isoperimetric characterization of hyperbolic groups to construct examples of such groups. We prove the following theorem : Let M_1 be a compact Riemannian manifold with negative sectional curvature (in particular, its fundamental group is hyperbolic), and M_2 a manifold whose fundamental group is hyperbolic. Suppose also that M_1 and M_2 are oriented and have the same dimension, which is ≥ 4. Then, the amalgamated product of their fundamental groups over an infinite cyclic group Z is hyperbolic, provided that the image of Z in $\pi_1(M_1)$ is maximal (i. e. generated by a primitive element in $\pi_1(M_1)$), and that Z injects in $\pi_1(M_2)$. In the proof, we use Morrey's theorem on minimal surfaces.

As another application of the analytical techniques we use, we provide another proof of the fact (which we already know from Chapter 1) that if V is a complete simply connected Riemannian manifold whose sectional curvature is bounded above by a strictly negative constant, then V is hyperbolic.

In **Chapter 8**, we prove a useful "approximation lemma" for points in a hyperbolic space with basepoint, by points on a tree. The lemma says that if X is any finite set of points in a δ–hyperbolic space Y which include the basepoint, (more generally, elements of X can be points in $Y \cup \partial Y$ or geodesic segments containing the basepoint of Y), there exists a simplicial tree, $Tr(X)$, and a continuous map f from X into $Tr(X)$ which is distance non-increasing, which preserves distances to the basepoints (the basepoint in $Tr(X)$ being the image by f of the basepoint of Y), and which cannot decrease distances between points by more than $2k\delta$, where k is of the order of the logarithm of the number of elements in X.

This approximation lemma is easy to prove, and will be used several times in the subsequent chapters. The general pattern in which the lemma is applied is the following : Consider any metric property of a tree, such that the definition of the property involves a bounded number n of points, for instance, convexity of the distance function, whose definition involves taking three points in the space. With the help of the approximation lemma, we can prove that in an arbitrary δ–hyperbolic space, the same property holds up to an additive constant (which depends only on δ and on n). We therefore get a "quasi-convexity" property in a hyperbolic space.

In **Chapter 9**, we show that, in analogy with the isometries of the classical hyperbolic space $\mathbb{H}^n$, isometries of a Gromov-hyperbolic space are classified into three types : hyperbolic, parabolic and elliptic. A *hyperbolic* isometry is characterized by the fact that orbits of points define quasi-isometries from the integers into the space X (in chapter 10, we give more information about the dynamics of such an isometry; in particular, we prove that its action on the boundary of the space has, as in the classical case, two fixed points, one attractive and one repulsive). An isometry is *elliptic* if the orbit of every point stays in a compact region of space. Finally, an isometry is *parabolic* if every orbit has exactly one accumulation point on ∂X. While we establish this classification, we prove several useful lemmas which give sufficient conditions for the type of an isometry in terms of the the relative distances of the first three iterates of an arbitrary point.

In **Chapter 10**, we study quasi-convexity and its applications in hyperbolic spaces. A subspace Y of a metric space X is said to be *ε–quasi-convex* (or quasi-convex) if every geodesic segment whose endpoints are in Y is contained in the ε–neighborhood of Y. For example, the image of a quasi-geodesic path in a hyperbolic space is a quasiconvex set. In the same way, a real-valued function is said to be quasi-convex if it is convex up to a bounded additive constant. Quasi-convexity in Gromov's hyperbolic spaces replaces convexity in the classical setting, and is just as useful. It is invariant by quasi-isometries between hyperbolic spaces (the image of a quasi-convex set is quasi-convex). We study projections into convex sets, and we prove a contraction property for these projections, which will be useful in the sequel. As for the local quasi-geodesicity criterion, there is a local quasi-convexity criterion : to see that a subspace Y of a hyperbolic space is quasi-convex, it is sufficient to examine all geodesic segments whose endpoints are in Y, and which are of some uniformly bounded length.

We then introduce quasi-convex subgroups of hyperbolic groups. This notion is independent of the choice of a generating system. If Γ' is a quasi-convex subgroup of a hyperbolic group Γ, then Γ' is hyperbolic and there is a canonical topological embedding from $\partial\Gamma'$ into $\partial\Gamma$.

Using the properties of the projection over a quasi-convex set, we study the displacement function of an isometry of a hyperbolic space. As another application of the notion of quasi-convexity, we prove a few algebraic properties of hyperbolic groups. For instance, in such a group, the centralizer of a non-torsion element g is a finite extension of the infinite cyclic group generated by g. We deduce that a hyperbolic group cannot not contain $\mathbb{Z}\oplus\mathbb{Z}$ as a subgroup.

Chapter 11 is about the boundary of a hyperbolic space (which is supposed to be proper and geodesic). We develop metric properties of this boundary.

Following Gromov, we begin by defining a class of "conformal" metrics on the hyperbolic space itself, each of which is obtained by multiplying the initial metric of the space by a function μ

which has some special properties. This process can be thought of as multiplying the length element of the hyperbolic metric by the function μ , or in other words, instead of calculating lengths of curves as the infimum of Riemann sums of distances between neighboring points on the curve, we calculate lengths of curves by integrating the function μ . To define the new metric, we take the associated path-metric.

By taking μ in a convenient class of functions, we obtain a metric space which is not complete and whose completion is, from the topological point of view, Gromov's compactification of the hyperbolic space which was introduced earlier. This construction is comparable with the construction of the open Euclidean unit disk, from the hyperbolic disk, by scaling the metric by the right real-valued function. The metric completion of the Euclidean unit disk is exactly the union of the hyperbolic disk with its boundary.

With respect to any of the new metrics, the action of an isometry of X on ∂X is Lipschitz. The action of a quasi-isometry is Hölder. Furthermore, the action of an isometry of hyperbolic type is a contraction near the attractive fixed point; using this fact, we show how to construct free groups of rank 2 in the isometry group of a hyperbolic space, as groups generated by sufficiently large powers of two isometries of X which have disjoint fixed point sets.

In **Chapter 12** , we show that hyperbolic groups are automatic. The idea of studying automatic groups from the geometric point of view seems to have been initiated in the last few years by J. Cannon and W. Thurston, and have been recently developed in a paper by J. Cannon, D. Epstein, D. Holt, M. Paterson and W. Thurston. The class of automatic groups is much larger than that of hyperbolic groups. For instance, finitely generated abelian groups, discrete groups of isometries of $\mathbb{R}^n$, braid groups are all automatic.

This chapter contains an account of some basic facts of the theory of automata (languages, finite state automata, regular words) which are needed to understand the theorem which says that hyperbolic groups are automatic. The proof uses the notion of *cone type* in the Cayley graph of a group, which has been introduced by J. Cannon in his proof that fundamental groups of compact Riemannian manifolds of negative sectional curvature are automatic. The proof we give here is an adaptation of Cannon's proof to the general case of Gromov's hyperbolic groups.

In the last section, we use these ideas to prove that the growth function of a hyperbolic group, relative to any finite system of generators, is rational.

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Chapitre 1

Espaces métriques hyperboliques

§1._ Définition de l'hyperbolicité

Le "produit scalaire" (x.y).

Soit (X,x_o) un espace métrique pointé. On adopte les notations suivantes :

$|x - y| =$ distance entre les points x et y.

$|x| = |x|_{x_o} = |x - x_o|$.

$(x.y) = (x.y)_{x_o} = 1/2\ (|x| + |y| - |x - y|)$.

On a immédiatement :

$(x.y) = (y.x)$; $(x.x) = |x|$; $(x.x_o) = 0$,

et d'après l'inégalité triangulaire :

$0 \leq (x.y) \leq \min(|x|, |y|)$.

On peut calculer $(x.y)$ de la manière suivante. Considérons dans le plan euclidien un triangle $[x'_o, x', y']$ qui *réalise* l'espace métrique $\{x_o, x, y\}$, c'est-à-dire tel que :

$$|x'_o - x'| = |x_o - x|, \quad |x'_o - y'| = |x_o - y| \quad \text{et} \quad |x' - y'| = |x - y|.$$

Soit c' le point de contact du cercle inscrit dans le triangle $[x'_o, x', y']$ avec l'un des côtés issus de x'_o (figure 1). Alors $(x.y)$ est la distance entre les points x'_o et c'. (Remarquons que, dans cette construction, le plan euclidien peut être remplacé par le plan hyperbolique $\mathbb{H}^2$ ou encore par une sphère de rayon assez grand.)

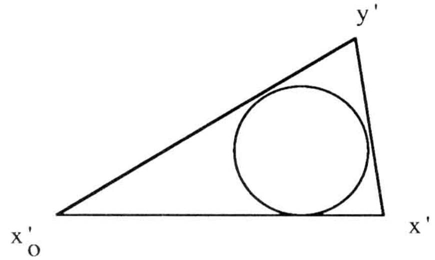

Figure 1

Définition d'un espace hyperbolique.

Définition 1.1 *Soit $\delta \geq 0$. On dit que l'espace métrique pointé (X,x_o) est δ-hyperbolique si :*

$$(x.y) \geq \min\left((x.z), (y.z) \right) - \delta$$

pour tous les $x, y, z \in X$.

Etudions l'effet d'un changement de point base. L'inégalité triangulaire donne

$$|(x.y)_{x_o} - (x.y)_{x_1}| \leq |x_o - x_1|.$$

Par conséquent, si (X,x_o) est δ-hyperbolique alors (X,x_1) sera δ'-hyperbolique avec $\delta' = \delta + 2\,|x_o - x_1|$. On a d'autre part :

Proposition 1.2 *Si (X,x_o) est δ-hyperbolique, alors (X,x_1) est 2δ-hyperbolique pour tout point $x_1 \in X$.*

Pour montrer la proposition, on utilise le

Lemme 1.3 *Si (X,x_o) est δ-hyperbolique, alors*

(1) $$(x.y) + (z.t) \geq \min\left((x.z) + (y.t) , (x.t) + (y.z) \right) -2\delta ,$$

pour tous les $x, y, z \in X$.

Démonstration du lemme 1.3 Des nombres $(x.z)$, $(x.t)$ et $(y.z)$ on peut supposer (quitte à échanger x et y , ou z et t) que $(x.z)$ est le plus grand. On a alors, d'après la δ-hyperbolicité :

$$(x.y) \geq \min\big((x.z), (y.z) \big) - \delta = (y.z) - \delta$$

et
$$(z.t) \geq \min\big((z.x), (t.x) \big) - \delta = (t.x) - \delta.$$

En ajoutant ces inégalités membre à membre, on obtient (1).

Démonstration de la proposition 1.2 Le lemme donne (point base x_o) :

$$(x.y) + (z.t) \geq \min\big((x.z) + (y.t) , (x.t) + (y.z) \big) - 2\delta.$$

On ajoute aux deux membres de l'inégalité la quantité :
$(|x - t| + |y - t| + |z - t| - |x| - |y| - |z| - |t|)/2$,
ce qui donne $(x.y)_t \geq \min\big((x.z)_t , (y.z)_t \big) - 2\delta$; c'est-à-dire la 2δ-hyperbolicité en t.

Définition 1.4 *L'espace métrique X est dit δ-hyperbolique si (X,x_o) est δ-hyperbolique pour tout $x_o \in X$. On dit que X est hyperbolique s'il existe un $\delta \geq 0$ tel que X soit δ–hyperbolique.*

La proposition 1.2 montre que s'il existe un point $x_o \in X$ tel que (X,x_o) soit δ-hyperbolique alors X est 2δ-hyperbolique.

Soit X un espace métrique et $Y \subset X$. On dit que Y est *coborné* si la fonction dist $(., Y)$ est bornée sur X. On a facilement la

Proposition 1.5 *Soit X un espace métrique et $Y \subset X$. Si X est δ-hyperbolique, alors Y est δ–hyperbolique. Inversement, si Y est δ-hyperbolique et coborné alors X est δ'-hyperbolique pour $\delta' = \delta + 6\eta$, où $\eta = \sup_{x \in X} \operatorname{dist}(x,Y)$.*

Nous pouvons maintenant énoncer une nouvelle formulation de l'hyperbolicité :

Proposition 1.6 *L'espace métrique X est δ-hyperbolique si et seulement si*
(2) $\quad |x - y| + |z - t| \leq \max\big(|x - z| + |y - t| , |x - t| + |y - z|\big) + 2\delta,$
pour tous les $x, y, z, t \in X$.

Démonstration En retranchant $|x - t| + |y - t| + |z - t|$ aux deux membres de (2), on obtient

$$(x.y)_t \geq \min\big((x.z)_t , (y.z)_t \big) - \delta.$$

§2._ Premiers exemples d'espaces hyperboliques

Espaces bornés : Tout espace métrique borné X est δ-hyperbolique, pour $\delta = \text{diam } X$, puisque $0 \leq (x.y) \leq d$.

La droite réelle : La droite réelle est 0-hyperbolique. En effet, on remarque que dans $\mathbb{R}$, $(x.y)$ est la distance du point x_o au segment $[x,y]$. L'inégalité $(x.y) \geq \min ((x.z) , (y.z))$ est alors immédiate.

Par contre, l'espace euclidien $\mathbb{R}^n$ n'est pas hyperbolique pour $n \geq 2$. En effet, considérons un vrai triangle $[x, y, z]$ dans $\mathbb{R}^n$, et prenons un point base x_o à l'intérieur du segment $[x, y]$. On a $(x.y) = 0$ et $\min ((x.z) , (y.z)) = r > 0$. Or, la δ-hyperbolicité appliquée aux images de x, y, z par une homothétie de centre x_o et de rapport λ donnerait $0 \geq \lambda r - \delta$; une contradiction pour λ assez grand.

Arbres réels : La notion d'arbre réel a été introduite par J. Morgan et P. Shalen [Mo-Sh]; elle généralise celle d'arbre (1-complexe simplicial contractile) muni d'une métrique simpliciale.

Définition 2.1 *On appelle arbre réel un espace métrique T qui vérifie les propriétés suivantes:*
(i) étant donnés deux points quelconques de T, il existe un segment topologique et un seul qui relie ces deux points
(ii) tous les segments topologiques de T sont des segments géodésiques.

(un *segment topologique* est un espace topologique homéomorphe à un segment de $\mathbb{R}$. Rappelons aussi qu'une application $f : X \to Y$ entre espaces métriques est dite *isométrique* si $|f(x_1) - f(x_2)| = |x_1 - x_2|$ pour tous les points x_1 et x_2 de X. Un *segment géodésique* est l'image par une application isométrique d'un segment $[a, b] \subset \mathbb{R}$ dans X. (Notons que tout segment géodésique est un segment topologique.)

Exemples.

1) Tout intervalle $\subset \mathbb{R}$.

2) $\mathbb{R}^2$ muni de la distance SNCF ("French railway metric") :

$|x - y| = |x - y|_{eucl}$ si x et y sont alignés avec l'origine,

$|x - y| = |x|_{eucl} + |y|_{eucl}$ sinon. ($|\ |_{eucl}$ désigne la norme euclidienne.)

Proposition 2.2 *Tout arbre réel est 0-hyperbolique.*

Démonstration On remarque tout d'abord que le produit $(x.y)$ est égal à $\mathrm{dist}(x_o, [x, y])$. Il ne reste plus alors qu'à examiner (figure 2) le sous-arbre de T formé de la réunion des segments géodésiques reliant deux points de $\{x_o, x, y, z\}$.

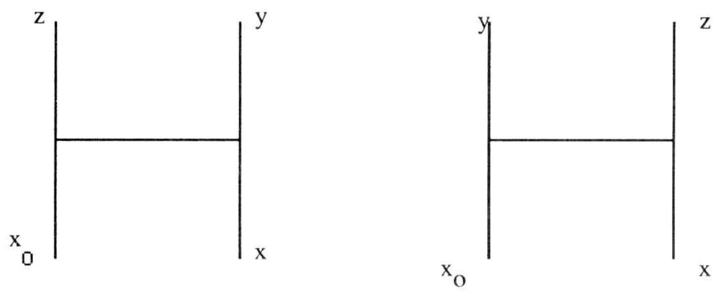

Figure 2

Espaces δ-ultramétriques

Disons qu'un espace métrique X vérifie *l'inégalité δ-ultramétrique* si

$$|x - y| \leq \max(|x - z|, |y - z|) + \delta \text{ pour tous les } x, y, z \in X.$$

Proposition 2.3 *Tout espace métrique qui vérifie l'inégalité δ–ultramétrique est δ-hyperbolique.*

Démonstration En effet, l'inégalité δ-ultramétrique donne (cf. la démonstration du lemme 1.3):

$$|x - y| + |z - t| \leq \max(|x - z| + |y - t|, |x - t| + |y - z|) + 2\delta,$$
ce qui montre la δ-hyperbolicité d'après la proposition 1.6.

Exemples

1) Soit K un corps muni d'une valuation v à valeurs réelles (par exemple le corps $\mathbb{Q}_p$ des nombres p-adiques muni de la valuation p-adique). Alors $|x - y| = e^{-v(x-y)}$ définit une métrique sur K qui vérifie l'inégalité 0-ultramétrique ; $(K, |\ |)$ est donc 0-hyperbolique.

2) Soit $(X, |\ |)$ un espace métrique quelconque. On obtient une nouvelle métrique sur X en posant

$$|x - y|' = \log(1 + |x - y|).$$

Cette métrique vérifie l'inégalité δ-ultramétrique avec $\delta = \log 2$. En effet, on a :

$$|x - y| \le |x - z| + |y - z| \le 2 \max(|x - z|, |y - z|),$$

d'où $|x - y|' \le \max(|x - z|', |y - z|') + \log 2$.

On en déduit que $(X, |\ |')$ est δ-hyperbolique avec $\delta = \log 2$.

Remarque La droite réelle est hyperbolique mais ne vérifie aucune inégalité δ-ultramétrique.

§3._ Hyperbolicité et triangles géodésiques

Espaces géodésiques.

Un espace métrique X est dit *géodésique* si deux points quelconques peuvent être reliés par un segment géodésique.

Remarques et exemples.

1) Un espace géodésique est un espace de longueur au sens de [Gro-1].

2) $\mathbb{R}^2$ privé de l'origine est un espace de longueur qui n'est pas géodésique.

3) Tout arbre réel est géodésique.

4) D'après le théorème de Hopf-Rinow, toute variété riemannienne complète est géodésique.

5) Plus généralement, tout espace de longueur complet et localement compact est géodésique (voir [Gro 1], théorème 1.10).

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