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Gordon Slade

The Lace Expansion and its Applications

Ecole d'Eté de Probabilités de Saint-Flour XXXIV – 2004

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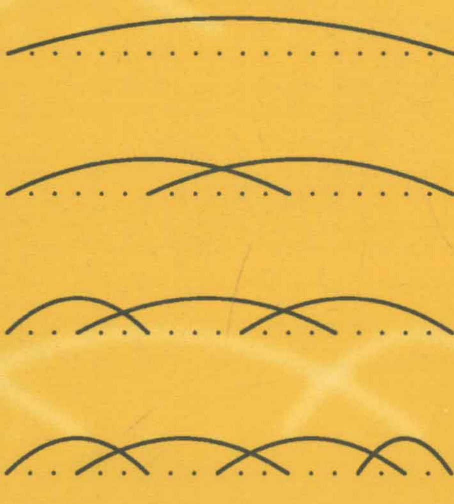

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Foreword

Three series of lectures were given at the 34th Probability Summer School in Saint-Flour (July 6–24, 2004), by the Professors Cerf, Lyons and Slade. We have decided to publish these courses separately. This volume contains the course of Professor Slade. We cordially thank the author for his performance at the summer school, and for the redaction of these notes.

69 participants have attended this school. 35 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

The Saint-Flour Probability Summer School was founded in 1971. Here are the references of Springer volumes which have been published prior to this one. All numbers refer to the *Lecture Notes in Mathematics* series, except S-50 which refers to volume 50 of the *Lecture Notes in Statistics* series.

1971: vol 307	1980: vol 929	1990: vol 1527	1998: vol 1738
1973: vol 390	1981: vol 976	1991: vol 1541	1999: vol 1781
1974: vol 480	1982: vol 1097	1992: vol 1581	2000: vol 1816
1975: vol 539	1983: vol 1117	1993: vol 1608	2001: vol 1837 & 1851
1976: vol 598	1984: vol 1180	1994: vol 1648	2002: vol 1840
1977: vol 678	1985/86/87: vol 1362 & S-50	1995: vol 1690	2003: vol 1869
1978: vol 774	1988: vol 1427	1996: vol 1665	2004: vol. 1878 & 1879
1979: vol 876	1989: vol 1464	1997: vol 1717	

Further details can be found on the summer school web site
`http://math.univ-bpclermont.fr/stflour/`

Jean Picard, Université Blaise Pascal
Chairman of the summer school

Preface

Several superficially simple mathematical models, such as the self-avoiding walk and percolation, are paradigms for the study of critical phenomena in statistical mechanics. Although these models have been studied by mathematicians for about half a century, exciting new developments continue to occur and the subject is flourishing. Much progress has been made, but it remains a major challenge for mathematical physics and probability theory to obtain a complete and mathematically rigorous understanding of the scaling theory of these models at criticality.

These lecture notes concern the lace expansion, which is a powerful tool for the analysis of the critical scaling of several models above their upper critical dimensions, namely:

- the self-avoiding walk on $\mathbb{Z}^d$ for $d > 4$,
- lattice trees and lattice animals on $\mathbb{Z}^d$ for $d > 8$,
- percolation on $\mathbb{Z}^d$ for $d > 6$,
- oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ and the contact process on $\mathbb{Z}^d$ for $d > 4$.

Results include proofs of existence of critical exponents, with mean-field values, and construction of scaling limits. Often, the scaling limit is described in terms of super-Brownian motion.

There are two distinct goals for these notes. The first goal is to provide a written accompaniment to my lectures at the XXXIV Saint–Flour International Probability School, in July 2004, and at the Pacific Institute for the Mathematical Sciences – University of British Columbia Summer School on Probability, in June 2005. The notes contain an introduction to the lace expansion and several of its applications, with sufficient background and depth to prepare a newcomer to do research using the lace expansion. Basic graduate level probability theory will be used, but no previous knowledge of the lace expansion or super-Brownian motion is assumed. The second goal is to provide a survey of the field, so that an interested reader can follow up by consulting the original literature. In pursuit of the second goal, these notes include more material than can be covered during a summer school course.

Following a brief initial chapter concerning random walk, the notes can be divided into four parts, whose contents are summarized as follows.

Part I, which concerns the self-avoiding walk, consists of Chaps. 2–6. A complete and self-contained proof is given of the convergence of the lace expansion for the nearest-neighbour model in dimensions $d \gg 4$, and for the spread-out model of self-avoiding walks which take steps of length at most L, with $L \gg 1$, in dimensions $d > 4$. The convergence proof presented here seems simpler than all previous lace expansion convergence proofs. As a consequence of convergence, it is shown that the critical exponent γ for the generating function of the number of n-step self-avoiding walks exists and is equal to 1. A survey is then given of the many extensions of this result that have been obtained using the lace expansion.

Part II, which concerns lattice trees and lattice animals, consists of Chaps. 7–8. It is shown how a minor modification of the expansion for the self-avoiding walk can be applied to give expansions for lattice trees and lattice animals, and an indication is given of the diagrammatic estimates that are necessary for proving convergence of the expansion. The relevance of the square condition is indicated, and results concerning existence of critical exponents in dimensions $d > 8$ are surveyed.

Part III, which concerns percolation, oriented percolation, and the contact process, consists of Chaps. 9–14. Detailed discussions are given of expansions for each of these models. Differential inequalities involving the triangle condition are stated (and usually proved) and are shown to imply mean-field behaviour of various critical exponents. Results concerning existence of critical exponents in dimensions $d > 6$ (for percolation) and $d > 4$ (for oriented percolation and the contact process) are surveyed.

Part IV, which concerns super-Brownian scaling limits, consists of Chaps. 15–17. Critical branching random walk with Poisson offspring distribution is analyzed in detail and used to give a self-contained construction of integrated super-Brownian excursion (ISE). The role of ISE as the scaling limit of lattice trees and of critical percolation clusters, above the upper critical dimensions, is discussed. The canonical measure of super-Brownian motion is also described, as is its role as scaling limit of critical oriented percolation clusters and the critical contact process in dimensions $d > 4$, and of lattice trees in dimensions $d > 8$.

Mathematics is not a spectator sport, and true understanding requires active participation in working out the ideas. To help facilitate this, a number of exercises for the reader appear throughout the notes. Some can be solved in a few lines, and others require more effort. I am grateful to Jeremy Flowers, Jesse Goodman, Jeffrey Hood, Sandra Kliem, Richard Liang, and Terry Soo, who collectively wrote solutions to all the exercises during the PIMS–UBC summer school.

It would not be possible to include detailed proofs of all the results discussed in these lecture notes without substantially increasing their length, and a number of important topics are only alluded to. These include: the inductive approach to the lace expansion, which is in many respects the most powerful method to prove convergence of the expansion; the "double" expansions that have been used to analyze r-point functions for $r \geq 3$; and the lace expansion on a tree, which is a method that can sometimes be used to replace a double expansion. (Two of these topics—the inductive method and double expansions—are discussed in recent lecture notes by Remco van der Hofstad [110].) Also, a complete proof of the convergence of the expansion is given only for the self-avoiding walk. This is the simplest setting for proving convergence, and convergence for the other models can be based on the ideas used in this setting. Finally, in an important new development about which it is too early to provide details, Sakai [181] has shown how to apply the lace expansion to analyze the Ising model in dimensions $d > 4$.

This work was supported in part by NSERC of Canada. Versions of the lectures were given at the University of British Columbia in Spring 2003, at EURANDOM in Fall 2003, at Saint-Flour in Summer 2004, and at PIMS/UBC in Summer 2005. The lecture notes were written primarily while I was travelling during 2003-04. I thank EURANDOM and the Thomas Stieltjes Institute, the University of Melbourne, Microsoft Research, and my hosts at these institutions, for their hospitality during visits to Eindhoven, Melbourne and Redmond.

I am grateful to the friends and colleagues with whom I have had the good fortune to work on topics related to these lecture notes. I thank Markus Heydenreich, Remco van der Hofstad, Mark Holmes, Sandra Kliem, Ed Perkins and Akira Sakai for suggesting improvements and for comments on earlier drafts of these notes. Many others have also made helpful comments of one form or another. Most of the illustrations (and all of the best ones) were produced by Bill Casselman, my colleague at the University of British Columbia and Graphics Editor of *Notices of the American Mathematical Society.*

I extend special thanks to David Brydges, whose patient teaching brought me into the subject, and to Takashi Hara and Remco van der Hofstad, who have played profound roles in the development of the ideas presented in these notes.

Vancouver, Gordon Slade
August 9, 2005

Contents

1

Simple Random Walk

The point of departure for the lace expansion is simple (ordinary) random walk, and it is helpful first to recall some elementary facts about random walk on $\mathbb{Z}^d$. This will also set some notation for later use.

1.1 Asymptotic Behaviour

Fix a finite set $\Omega \subset \mathbb{Z}^d$ that is invariant under the symmetry group of $\mathbb{Z}^d$, i.e., under permutation of coordinates or replacement of any coordinate x_i by $-x_i$. Our two basic examples are the *nearest-neighbour model*

$$\Omega = \{x \in \mathbb{Z}^d : \|x\|_1 = 1\} \tag{1.1}$$

and the *spread-out model*

$$\Omega = \{x \in \mathbb{Z}^d : 0 < \|x\|_\infty \leq L\}, \tag{1.2}$$

where L is a fixed (usually large) constant. The norms are defined, for $x = (x_1, \ldots, x_d)$, by $\|x\|_1 = \sum_{j=1}^d |x_j|$ and $\|x\|_\infty = \max_{1 \leq j \leq d} |x_j|$.

For $n \geq 1$, an n-step walk taking steps in Ω is defined to be a sequence $(\omega(0), \omega(1), \ldots, \omega(n))$ of vertices in $\mathbb{Z}^d$ such that $\omega(i) - \omega(i-1) \in \Omega$ for $i = 1, \ldots, n$. Let $\mathcal{W}_n(x,y)$ be the set of n-step walks with $\omega(0) = x$ and $\omega(n) = y$, and let $\mathcal{W}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{W}_n(0,x)$ denote the set of all n-step walks starting from the origin. Let $c_n^{(0)}(x)$ denote the cardinality of $\mathcal{W}_n(0,x)$. The superscript (0) is there to indicate that we are working with the random walk with no interaction. We allow for the degenerate case $n = 0$ by defining $\mathcal{W}_0(x,y)$ to consist of the zero-step walk (x) if $x = y$, and to be empty otherwise. Then $c_0^{(0)}(x,y) = \delta_{x,y}$. Taking into account the translation invariance, we will use the abbreviations $\mathcal{W}_n(y-x) = \mathcal{W}_n(x,y)$ and $c_n^{(0)}(y-x) = c_n^{(0)}(x,y)$.

For $n \geq 1$, by considering the possible values $y \in \Omega$ of the walk's first step, we have

$$c_n^{(0)}(x) = \sum_{y \in \Omega} c_{n-1}^{(0)}(x-y) = \sum_{y \in \mathbb{Z}^d} c_1^{(0)}(y) c_{n-1}^{(0)}(x-y). \tag{1.3}$$

Denoting the convolution of functions f and g by

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x-y), \tag{1.4}$$

(1.3) can be written as

$$c_n^{(0)}(x) = \left(c_1^{(0)} * c_{n-1}^{(0)}\right)(x). \tag{1.5}$$

The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) \mathrm{e}^{\mathrm{i}k \cdot x} \qquad (k \in [-\pi, \pi]^d), \tag{1.6}$$

where $k \cdot x = \sum_{j=1}^d k_j x_j$, with inverse

$$f(x) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{f}(k) \mathrm{e}^{-\mathrm{i}k \cdot x}. \tag{1.7}$$

The fact stated in part (a) of the following exercise makes the use of Fourier transforms very convenient.

Exercise 1.1. (a) Show that the Fourier transform of $f * g$ is $\hat{f}\hat{g}$.
(b) A closely related statement is the following. Denote the generating functions of the sequences f_n and g_n by $F(z) = \sum_{n=0}^\infty f_n z^n$ and $G(z) = \sum_{n=0}^\infty g_n z^n$, and assume these series both have positive radius of convergence. Show that the generating function $H(z)$ of the sequence $h_n = \sum_{m=0}^n f_m g_{n-m}$ is $H(z) = F(z)G(z)$.

By Exercise 1.1(a), (1.5) implies that

$$\hat{c}_n^{(0)}(k) = \hat{c}_1^{(0)}(k) \hat{c}_{n-1}^{(0)}(k). \tag{1.8}$$

Since $\hat{c}_0^{(0)}(k) = 1$, solving (1.8) by iteration gives

$$\hat{c}_n^{(0)}(k) = \hat{c}_1^{(0)}(k)^n \qquad (n \geq 0). \tag{1.9}$$

If we define the transition probability

$$D(x) = \frac{1}{|\Omega|} I[x \in \Omega] = \frac{1}{|\Omega|} c_1^{(0)}(x), \tag{1.10}$$

where $|\Omega|$ denotes the cardinality of the set Ω and I denotes the indicator function, then (1.9) can be rewritten as

$$\hat{c}_n^{(0)}(k) = |\Omega|^n \hat{D}(k)^n \qquad (n \geq 0). \tag{1.11}$$

Exercise 1.2. (a) Show that for the nearest-neighbour model,

$$\hat{D}(k) = \frac{1}{d}\sum_{j=1}^{d} \cos k_j, \tag{1.12}$$

and for the spread-out model

$$\hat{D}(k) = \frac{1}{|\Omega|}\left[\prod_{j=1}^{d} M(k_j) - 1\right], \tag{1.13}$$

where

$$M(t) = \frac{\sin[(2L+1)t/2]}{\sin(t/2)} \tag{1.14}$$

is the Dirichlet kernel.
(b) Denote the variance of D by $\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x)$. Show that $\sigma = 1$ for the nearest-neighbour model and that σ is asymptotic to a multiple of L as $L \to \infty$ for the spread-out model.

The number of n-step walks starting from a given vertex is of course $|\Omega|^n$, because each step can be chosen in $|\Omega|$ different ways. This fact is contained in (1.11), since the number of n-step walks starting from the origin is $\sum_{x\in\mathbb{Z}^d} c_n^{(0)}(x) = \hat{c}_n^{(0)}(0) = |\Omega|^n$, using $\hat{D}(0) = 1$.

By symmetry, $\sigma^2 = -\nabla^2 \hat{D}|_{k=0}$, where $\nabla^2 = \sum_{j=1}^{d} \nabla_j^2$ is the Laplacian, with ∇_j denoting partial differentiation with respect to the component k_j of k. Then, by (1.11) and by the symmetry of Ω, the central limit theorem

$$\lim_{n\to\infty} \frac{\hat{c}_n^{(0)}(k/\sigma\sqrt{n})}{\hat{c}_n^{(0)}(0)} = \mathrm{e}^{-|k|^2/2d} \tag{1.15}$$

follows, as does the fact that the mean-square displacement is given by

$$\frac{\sum_{x\in\mathbb{Z}^d} |x|^2 c_n^{(0)}(x)}{\sum_{x\in\mathbb{Z}^d} c_n^{(0)}(x)} = - \left.\nabla^2 \hat{D}^n\right|_{k=0} = n\sigma^2. \tag{1.16}$$

Exercise 1.3. Prove (1.15) and (1.16).

The *two-point function* is defined by

$$C_z(x,y) = \sum_{n=0}^{\infty} \sum_{\omega\in\mathcal{W}_n(x,y)} z^n = \sum_{n=0}^{\infty} c_n^{(0)}(x,y) z^n. \tag{1.17}$$

The two-point function is finite for $z \in [0, 1/|\Omega|)$. For $d > 2$, it is also known to be finite for $z = 1/|\Omega|$, and for this value of z it is called the Green function. By translation invariance, we may regard the two-point function as a function of a single variable, writing $C_z(x,y) = C_z(y-x)$. By (1.11) and (1.17), its Fourier transform is

$$\hat{C}_z(k) = \sum_{n=0}^{\infty} \hat{c}_n^{(0)}(k) z^n = \frac{1}{1 - z|\Omega|\hat{D}(k)}. \tag{1.18}$$

The *susceptibility* is defined by

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} C_z(0,x) = \hat{C}_z(0) = \frac{1}{1 - z|\Omega|}. \tag{1.19}$$

The *critical point* is the singularity $z_c = 1/|\Omega|$ of the susceptibility.

The inverse Fourier transform of (1.18) is

$$C_z(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1 - z|\Omega|\hat{D}(k)}. \tag{1.20}$$

For $d > 2$,

$$C_{z_c}(x) \sim \text{const} \frac{1}{|x|^{d-2}} \tag{1.21}$$

as $|x| \to \infty$, where the constant depends on d and on Ω (see [149,195], or [203] for a more general statement of this fact). The notation

$$f(x) \sim g(x) \qquad \text{denotes} \qquad \lim_{x\to\infty} f(x)/g(x) = 1, \tag{1.22}$$

and this notation will used in general for asymptotic formulas.

Exercise 1.4. Some care is needed with (1.20) when $z = z_c$, since $C_{z_c}(x)$ is not summable by (1.21) and thus its Fourier transform is problematic. Using the symmetry of Ω, prove that (1.20) does hold when $z = z_c$ for $d > 2$, and that the integral is infinite when $z = z_c$ for $d \leq 2$.

Exercise 1.5. Let $f : \mathbb{Z}^d \to \mathbb{C}$. For $y \in \Omega$, define forward and backward discrete partial derivatives by $\partial_y^+ f(x) = f(x+y) - f(x)$ and $\partial_y^- f(x) = f(x) - f(x-y)$. Define the discrete Laplacian by

$$\Delta f(x) = \frac{1}{2}\frac{1}{|\Omega|}\sum_{y\in\Omega} \partial_y^- \partial_y^+ f(x) = \frac{1}{|\Omega|}\sum_{y\in\Omega} f(x+y) - f(x), \tag{1.23}$$

and let $\delta_{x,y}$ denote the Kronecker delta which takes the value 1 if $x = y$ and 0 if $x \neq y$. Show that $-\Delta C_{1/|\Omega|}(x) = \delta_{0,x}$. Thus $C_{1/|\Omega|}(x)$ is the Green function for $-\Delta$.

Exercise 1.6. Consider a simple random walk started at the origin.

(a) Let u denote the probability that the walk ever returns to the origin. The walk is *recurrent* if $u = 1$ and *transient* if $u < 1$. Let N denote the (random) number of visits to the origin, including the initial visit at time 0, and let

$m = \mathbb{E}N$. Show that $m = \frac{1}{1-u}$, so the walk is recurrent if and only if $m = \infty$.
(b) Show that

$$m = \sum_{n=0}^{\infty} \mathbb{P}(\omega(n) = 0) = \int_{[-\pi,\pi]^d} \frac{1}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d}. \tag{1.24}$$

Thus transience is characterized by the integrability of $\hat{C}_{1/|\Omega|}(k)$.
(c) For simplicity, consider the nearest-neighbour model, with Ω given by (1.1). Show that the walk is recurrent in dimensions $d \leq 2$ and transient in dimensions $d > 2$.

Exercise 1.7. Let $\omega^{(1)}$ and $\omega^{(2)}$ denote two independent simple random walks started at the origin, and let

$$X = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} I[\omega^{(1)}(i) = \omega^{(2)}(j)] \tag{1.25}$$

denote the number of intersections of the two walks. Here I denotes an indicator function. Show that

$$\mathbb{E}X = \int_{[-\pi,\pi]} \frac{1}{[1 - \hat{D}(k)]^2} \frac{d^d k}{(2\pi)^d}. \tag{1.26}$$

Thus $\mathbb{E}X$ is finite if and only if $\hat{C}_{1/|\Omega|}(k)$ is square integrable. Conclude, for simplicity for the nearest-neighbour model, that the expected number of intersections is finite if $d > 4$ and infinite if $d \leq 4$.

The integral $(2\pi)^{-d} \int_{[-\pi,\pi]^d} \hat{C}_{z_c}(k)^2 d^d k$ of (1.26) is equal, by the Parseval relation, to $\sum_{x \in \mathbb{Z}^d} C_{z_c}(x)^2$. The relevance of the condition $d > 4$ for the latter is evident from the asymptotic behaviour (1.21). However, the k-space analysis is more elementary, as it relies on the easy formulas given in (1.12) and (1.18) rather than the deeper statement (1.21). It is often much easier to use estimates in k-space than to work directly in x-space.

It is a consequence of Donsker's Theorem [24] that the scaling limit of simple random walk is Brownian motion, in all dimensions. This means that if we define a random continuous function X_n from the interval $[0, 1]$ into $\mathbb{R}^d$ by setting $X_n(j/n) = \sigma^{-1} n^{-1/2} \omega(j)$ for integers $j \in [0, n]$, and interpolating linearly between consecutive vertices, then the distribution of X_n converges weakly to the Wiener measure. See Fig. 1.1.

1.2 Universality and Spread-Out Models

In these notes, we study several models that live on the integer lattice, and each has a nearest-neighbour and a spread-out version. In the nearest-neighbour model, specified by (1.1), bonds (also called edges) join pairs of

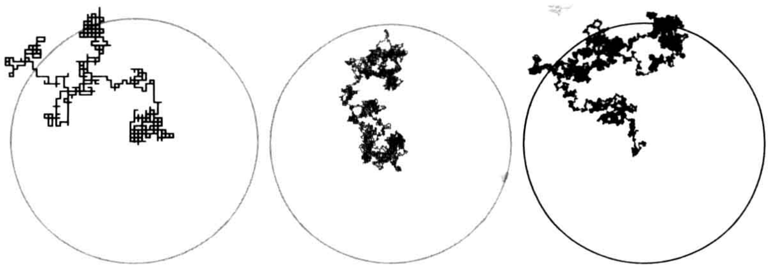

Fig. 1.1. Nearest-neighbour random walks on $\mathbb{Z}^2$ taking $n = 1,000$, 10,000 and 100,000 steps. The circles have radius $\sqrt{n}$, in units of the step size of the random walk.

vertices separated by unit Euclidean distance. In the spread-out model, specified by (1.2), bonds join pairs of vertices separated by distance between 1 and L, where L is a parameter usually taken to be large. According to the deep hypothesis of *universality*, the critical scaling of the models to be studied should be the same for the nearest-neighbour and spread-out models.

We use the spread-out model because proofs of convergence of the lace expansion require *large degree*. The degree is the cardinality of Ω. For the nearest-neighbour model the degree is $2d$, and can be taken large by increasing the dimension. The degree of the spread-out model is of order L^d for large L, and this allows for convergence proofs for the lace expansion without taking the dimension d to be large in an uncontrolled way. In the applications to be discussed, results will typically be obtained: (i) for the nearest-neighbour model for $d \geq d_0$ for some d_0 having no physical meaning, and (ii) for the spread-out model with L larger than some L_0 and d strictly greater than the upper critical dimension (4 for the self-avoiding walk, oriented percolation and the contact process; 6 for percolation; 8 for lattice trees and lattice animals). While it is of interest to prove results of type (i) with d_0 equal to the upper critical dimension plus one, failing this, results of type (ii) seem more important, as they indicate clearly the role of the upper critical dimension. Also, the fact that all large L give rise to the same scaling behaviour provides a partial proof of universality in this context. In fact, much more general spread-out models than (1.2) can be handled using the lace expansion (see, e.g., [94,120]), but we restrict attention in these notes to (1.2) for the sake of simplicity.

2

The Self-Avoiding Walk

The self-avoiding walk is a model of fundamental interest in combinatorics, probability theory, statistical physics and polymer chemistry. It is a model of random walk paths but it cannot be described in terms of transition probabilities and thus is not even a stochastic process. It is certainly non-Markovian. These features makes the subject difficult, and many of the central problems remain unsolved. See [127, 158] for extensive surveys.

The self-avoiding walk is a basic example in the theory of critical phenomena, due to its close links with models of ferromagnetism such as the Ising model. In particular, it can be understood as the $N \to 0$ limit of the N-vector model [79] (see also [158, Sect. 2.3]). In polymer chemistry, self-avoiding walks are used to model a single linear polymer molecule in a good solution [80, 205]. The flexibility of the polymer is modelled by the possible configurations of a self-avoiding walk, while the self-avoidance constraint models the excluded volume effect that causes the polymer to repel itself.

In this chapter, we first give an overview of the self-avoiding walk and its predicted asymptotic behaviour. Then we define the bubble condition and show that it is a sufficient condition for a particular critical exponent (namely γ) to exist and take its mean-field value.

2.1 Asymptotic Behaviour

An n-step self-avoiding walk starting at x and ending at y is an n-step walk $(\omega(0), \omega(1), \ldots, \omega(n))$ with $\omega(0) = x$, $\omega(n) = y$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We will assume for simplicity that the walks take steps in Ω given either by (1.1) or (1.2). Let $\mathcal{S}_n(x, y)$ be the set of n-step self-avoiding walks from x to y, let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(0, x)$ denote the set of all n-step self-avoiding walks starting from the origin, and let $\mathcal{S}(x, y) = \cup_{n=0}^{\infty} \mathcal{S}_n(x, y)$ denote the set of all self-avoiding walks of any length from x to y. Let $c_n(x, y)$ denote the cardinality of $\mathcal{S}_n(x, y)$. In particular, $c_0(x, y) = \delta_{x,y}$. We will use the abbreviations $\mathcal{S}_n(x) = \mathcal{S}_n(0, x)$, $c_n(x) = c_n(0, x)$, and $c_n = \sum_{x \in \mathbb{Z}^d} c_n(x)$.