RICHARD A. MOORE

INTRODUCTION TO DIFFERENTIAL EQUATIONS

College Mathematics Series

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PREFACE

This book is essentially a text for the one-semester undergraduate course in differential equations. As a rule, this is the introductory course taken by students of science, engineering, and mathematics who have completed the traditional calculus sequence. Indeed, the author has tested the substance of this book in just such a course.

The aim of this book is to provide more than a catalogue of explicit solution techniques. The hope is that even the beginning student will reach a point where he can say something useful about the solutions of a differential equation whether it is solvable or not. This is in direct response to the needs of scientists and mathematicians who simply cannot prosper on a knowledge that is restricted to solvable cases or even linear cases. In fact, then, this is an introduction to a theorizing technique for the discussion of differential equations. This technique depends in varying degrees on mathematical theory, geometric inferences, argument by analogy with solvable cases and, in applications, the requirements of the physical situation. It is clear from this that such a technique does not lend itself well to systematic exposition. Rather, it is presented here by illustrative examples found in Chapters 3, 6, and 7. Not all of the applications, however, are of this "unsolvable" type.

The remaining chapters form a unified discussion of explicit solutions methods, fundamental theory, and qualitative geometric arguments in preparation for the above aim. The substance of these chapters is that of a somewhat shortened, but standard, introductory course in differential equations. In order to relate the explicit methods more closely to both the theory and the applications, these methods are made to lead directly to solutions of initial-value problems rather than to general solutions. For example, the definite integral is employed to the exclusion of the indefinite integral in solving first-order initial-value problems, and the Laplace transform is consistently used for all linear equations and systems of equations with constant coefficients. The use of the Laplace transform has the additional advantage of leading in a natural way to the idea and form of integral representations of solutions even in the most general linear case. It is important to note, however, that the Laplace transform is given no independent interest here; therefore, the essential results are given without proof.

Problems of varying difficulty are given with each section. Often an alternative to the method given in the text is developed in a short sequence of problems. Two sections (on undetermined coefficients) have no text and are entirely in the form of problems. The reader is to develop this technique himself by working a sequence of problems of increasing difficulty and generality.

R. A. M.

CONTENTS

1
FIRST-ORDER EQUATIONS

▶ 1.1 INTRODUCTION

A first-order differential equation has for us the standard form

$$y' = f(x, y). \tag{1.11}$$

In some instances, an equation may be presented, or studied, in differential form,

$$M(x, y)\, dx + N(x, y)\, dy = 0,$$

or occasionally in implicit form

$$F(x, y, y') = 0,$$

but the latter forms are of less interest here, and normally are equivalent to (1.11). The obvious problem posed by (1.11) is the finding of solutions, that is, functions which satisfy the equation identically. Specifically, a *solution* of (1.11) is a function $y(x)$, defined on an interval $a < x < b$, such that

$$y'(x) \equiv f(x, y(x))$$

holds for $a < x < b$. For example, e^x is a solution of

$$y' = y$$

for $-\infty < x < +\infty$ since, for all x,

$$(e^x)' \equiv e^x.$$

Similarly, the function $\dfrac{1}{1 - x^2}$ is a solution of

$$y' = 2xy^2$$

for $-1 < x < 1$, as is easily verified.

How we obtained these solutions in these examples is a matter of certain formal solution techniques which we shall describe shortly. It is important, however, to place solution techniques of any nature in proper perspective by observing that a function is (or is not) a solution only according as is satisfies (or does not satisfy) the equation at hand. Along with this, we must remember that a function may exist while a formula for the functional values does not. This is the status of solutions of many differential equations—solutions exist, but no manipulative technique exists to display them. It is natural that purely mathematical interests should center on such equations, and it is not surprising that the most interesting and pressing modern applications rest on equations of the same intractable type.

For a truly useful knowledge of the subject, then, the obvious problem of finding solutions in the sense of displayed formulas is too narrow; this must be, at best, an introduction to and a part of the deeper problem of saying something useful about solutions in every case. For example, we are not able to "solve" the equation

$$y' = x^2y^2 + 1. \tag{1.12}$$

Nonetheless, as a consequence of a general theory, which we develop later, solutions do exist. Given any solution $y(x)$ we can see from the equation that $y'(x)$ is positive; thus, $y(x)$ is increasing. Further, we can say that $y'(x)$, that is to say, $x^2y^2(x) + 1$, is also an increasing function of x for $y(x) \geq 0$. With now only a little more information, say, the value of $y(x)$ at one point, we could produce a very plausible sketch of the graph of this solution which is a convex, increasing function. This is what is meant by useful information, and it is by no means all that can be said of (1.12).

What we have to say in the first six sections concerns solvable cases. Since here solutions can be displayed, we do not need a theorem which asserts that each of a large class of differential equations has solutions. Since for a solvable equation we can show directly that there is only one solution which satisfies some side condition, a general uniqueness theorem is not necessary. The theory is a necessity for our deeper aims, and it will be stated and proved in later sections. The ideas of the theory, however, can be simply and concisely stated; moreover, they form a useful guide and principle even in routine solvable cases. It is not unnatural to give these ideas a physical meaning and, indeed, to derive them from physics. Typically, a physical experiment may be represented mathematically as a problem in differential equations; e.g., the first-order equation

$$\frac{dv}{dt} = -32,$$

together with the initial condition $v(t_0) = 50$, describes (ignoring friction among other things) the experiment of throwing a ball from the surface of the earth at time t_0 with velocity 50 ft/sec. Now, a well-posed experiment has three obvious properties:

(*i*) Something happens.

(*ii*) Only one thing happens; that is, the experiment can be repeated under the same conditions with the same results.

(*iii*) Small variations in initial state, physical components, or any other physical parameters produce only small variations in results.

These hardly need comment, but from them we infer that a mathematical problem which purports to describe a well-posed experiment must be well posed in the mathematical sense; namely,

(*i*) A solution exists.

(*ii*) The solution is unique.

(*iii*) The solution is continuous in all parameters of the problem.

It is now a simple matter to describe concisely the fundamental theory of differential equations. It is a collection of theorems which asserts that a wide class of standard problems are well posed. It is not, of course, restricted to first-order equations. Specifically, the fundamental theory takes the form of an *existence* theorem, a *uniqueness* theorem, and various theorems on *continuity* in parameters, each of substantial generality.

We see, for example, that the initial value problem

$$\frac{dv}{dt} = -32$$

$$v(t_0) = 50$$

is indeed well posed, for, by well-known results of elementary calculus, all solutions of the differential equation are given by

$$v(t) = -32t + c$$

where c is any constant. Only one of these functions satisfies the initial condition, and c is determined by

$$50 = v(t_0) = -32t_0 + c.$$

The unique solution is thus given by

$$v(t) = -32(t - t_0) + 50.$$

PROBLEMS

1. Show that xe^x is a solution of

$$y' = y + e^x.$$

2. For what constant a is e^{ax} a solution of

$$y' = 3y.$$

3. Show that

$$x^2 - xy - y^2 = c$$

defines solutions of

$$y' = \frac{2x - y}{x + 2y}.$$

4. For what n is x^n a solution of

$$y' = \frac{2y}{x}.$$

5. Exhibit a solution of

$$y' = y^2 - 1.$$

6. Exhibit a solution of

$$y' = y^2 - x^4 + 2x.$$

7. Show that $e^{x^2} \int_0^x e^{-t^2}\, dt$ is a solution of

$$y' = 2xy + 1.$$

8. Show that

$$y(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & x > 0 \end{cases}$$

is a solution of

$$y' = 2\sqrt{y}$$

for all x.

9. Find the unique solution of the initial value problem

$$\frac{dv}{dt} = -g$$

$$v(t_0) = v_0.$$

Verify that this solution is continuous in all the parameters t_0, v_0, and g.

▶ 1.2 SEPARABLE EQUATIONS

The equation

$$y' = f(x)g(y) \tag{1.21}$$

is said to be *separable*, where, unless otherwise specified, the functions $f(x)$ and $g(y)$ are assumed to be continuous in their domains. For example,

$$y' = 2x + 1 \tag{1.22}$$

is separable. All solutions of (1.22) are given by the formula

$$y(x) = x^2 + x + c \tag{1.23}$$

because $x^2 + x$ is one primitive of (1.22), and any two primitives differ by a constant; that is, (1.22) is solved by integration.

A standard problem for (1.22) and, indeed, for any first-order equation, is that of finding the solution which satisfies an initial condition

$$y(x_0) = y_0.$$

We may solve this by specializing (1.23); thus

$$y(x_0) = x_0^2 + x_0 + c,$$

from which the unique solution is

$$y(x) = x^2 + x + y_0 - x_0^2 - x_0. \tag{1.24}$$

Formula (1.23) is said to be the *general solution* of (1.22), while (1.24) defines a single *particular* solution, but if we allow x_0 and y_0 to take on all values, these two are virtually the same. The difference lies only in point of view, for (1.23) gives all possible solutions while (1.24) gives each solution possible. A general solution is usually the result of indefinite integration. It may or may not contain all solutions. It is certainly useful, but in the long run the idea of a differential equation is best revealed by going directly after the solution(s) of each possible initial value problem.

This would be

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \tag{1.25}$$

in the general case and

$$\begin{aligned} y' &= f(x)g(y) \\ y(x_0) &= y_0 \end{aligned} \tag{1.26}$$

in the separable case.

The definite integral is an apt device for these purposes. Thus, our view of the problem

$$\begin{aligned} y' &= f(x) \\ y(x_0) &= y_0 \end{aligned} \tag{1.27}$$

is as follows. If (1.27) has a solution, then

$$\int_{x_0}^{x} y'(t)\,dt \equiv \int_{x_0}^{x} f(t)\,dt,^*$$

hence

$$y(x) \equiv y_0 + \int_{x_0}^{x} f(t)\,dt \tag{1.28}$$

in view of the initial condition $y(x_0) = y_0$. That is, if there is a solution, it is uniquely given by (1.28). On the other hand, this is a solution since

$$y'(x) \equiv \left(y_0 + \int_{x_0}^{x} f(t)\,dt\right)' \equiv f(x)$$

and

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0$$

hold.

To deal with more complicated problems we combine these simple ideas with permissible arithmetic operations. For example, if $y(x)$ is a solution of

$$\begin{aligned} y' &= xy \\ y(x_0) &= y_0, \end{aligned} \tag{1.29}$$

then

$$\int_{x_0}^{x} \frac{y'(t)}{y(t)}\,dt = \int_{x_0}^{x} t\,dt$$

* Note that

$$\int_{x_0}^{x} f(x)\,dx \equiv \int_{x_0}^{x} f(t)\,dt \equiv \int_{x_0}^{x} f(\theta)\,d\theta,$$

but, of these, the first equation is obviously confusing and will be avoided.

provided $y(x)$ is not zero on some interval about x_0. This will be true, for example, if $y(x_0) = y_0 \neq 0$, and assuming this we have

$$\ln [\pm y(x)] - \ln (\pm y_0) = \frac{x^2}{2} - \frac{x_0^2}{2},$$

where + or − apply according as $y_0 > 0$ or $y_0 < 0$. In each case the result is the same,

$$y(x) = y_0 e^{x^2/2 - x_0^2/2} \tag{1.210}$$

is the only possible solution, and the above arguments are easily reversed to show that this is a solution. Moreover, this is a solution defined for all x; the technical restriction to some interval about x_0 is, in this instance, not necessary. Solutions which are not zero at any one point x_0 are evidently never zero since the exponential function is never zero.

We must now consider the missing case, $y_0 = 0$. We cannot integrate, but once our attention is drawn to it we see a solution,

$$y(x) \equiv 0.$$

That this is the only solution is implied by the content of the last sentence of the preceding paragraph.

The need for careful procedure is illustrated by the following example,

$$\begin{aligned} y' &= \sqrt{y} \\ y(x_0) &= y_0. \end{aligned} \tag{1.211}$$

The arithmetic operations needed to separate this equation draw our attention to two cases: $y_0 = 0$ and $y_0 > 0$ ($y_0 < 0$ is, of course, impossible). If $y_0 = 0$, we cannot integrate; however, there is a solution,

$$y(x) \equiv 0,$$

by inspection. We must see later whether this is the only solution.

If $y_0 > 0$, we integrate:

$$\int_{x_0}^{x} \frac{y'(t)}{\sqrt{y(t)}}\, dt = \int_{x_0}^{x} dt.$$

From this, the solution is uniquely given by

$$2\sqrt{y(x)} \equiv (x - x_0 + 2\sqrt{y_0})$$

as long as $y(x) > 0$; i.e., as long as $x > x_0 - 2\sqrt{y_0}$. Here, the restriction of x to some interval about x_0 in order to permit integration is, in fact, a real one, and evidently a solution is defined only on the interval

$$x_0 - 2\sqrt{y_0} < x < +\infty.$$