当代数学讲座丛书

Lectures in Contemporary Mathematics **3**

Bennett Chow Peng Lu & Lei Ni

Hamilton's Ricci Flow

(哈密顿Ricci流)

SCIENCE PRESS
Beijing

AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

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Preface
of the Lecture
in Contemporary Mathematics

During the last twenty years, Chinese mathematics has experienced very impressive developments, with significant increases in international academic communications. Different levels of modern mathematical lecture series and summer schools (for example, the Special Mathematics Lecture Series in Beijing University since 1998) were held in many universities and research institutes. Prominent native and overseas mathematicians gave lectures on basic knowledge and recent developments in different areas of mathematics. This has provided very good opportunities for Chinese mathematics researchers and graduate students to get in touch with basic knowledge as well as ongoing research projects in mathematics. In particular, this has substantially promoted the development of young mathematicians in China.

The formulation of the Lecture in Contemporary Mathematics is based on these activities and series lectures. It serves as high level, specialized textbooks for senior undergraduates, graduate students, and young mathematics researchers in mathematics and applied sciences. By publishing lecture notes of top quality, notes from elite courses in summer schools, and other forms of notes, we wish that students and young researchers can harvest a deep understanding of new developments, and grasp basic knowledge and important ongoing projects in different areas of mathematics in a short period of time.

It is the collective wish of chinese mathematicians that China will become an international power in mathematics in the near future. For this purpose, we need many more native young mathematicians with international recognition. We believe that the publication of this lecture note series will contribute to realizing the goal.

Gang Tian
October 16, 2005

Preface

About the book. Ricci flow is a geometric and analytic evolution equation which we believe is related to physical reality. One of the underlying principles of science is unity. In Ricci flow we see the unity of geometry and analysis. It is also expected to exhibit unity with low-dimensional topology. In this book we emphasize the more geometric and analytic aspects of Ricci flow rather than the topological aspects. We also attempt to convey some of the relations and formal similarities between Ricci flow and other geometric flows such as mean curvature flow. The interaction of techniques and ideas between Ricci flow and other geometric flows is a two-way street. So we hope the reader with a more general interest in geometric flows will benefit from the usefulness of applying ideas originating in Ricci flow to the study of other geometric flows. We have not aimed at completeness, even in the realm of the limited material that we cover. A more extensive coverage of the subject of Ricci flow is planned in the book by Dan Knopf and one of the authors [**163**] and its multi-authored sequel [**153**].

At places we follow the informal style of lecture notes and have attempted to cover some of the basic material in a relatively direct and efficient way. At the same time we take the opportunity to expose the reader to techniques, some of which lie outside of the subject of Ricci flow *per se*, which he or she may find useful in pursuing research in Ricci flow. So, metaphorically speaking, this book is a hybrid between rushing to work in the morning on a cold and blistery winter day and a casual stroll through the park on a warm and sunny midsummer afternoon. As much as possible, we have attempted to construct the book so that the chapters, and in some cases, individual sections, are relatively independent. In this way we hope that the book may be used as both an introduction and as a reference. Exercises appear at various places in the book and solutions to selected exercises are collected at the end of the book. We have endeavored to include some open problems which are aimed at conveying to the reader what the limits of our current knowledge are and to point to some interesting directions. We have also attempted to give the appropriate references so that the reader may further pursue the statements and proofs of the various results. To use real estate jargon, we hope that the references are reliable, but they are not guaranteed; in particular, sometimes the references given may not be the first place where a particular result is proven.

The purpose of this book is to give an introduction to the Ricci flow for graduate students and mathematicians interested in working in the subject. We have especially targeted the audience of readers who are novices to the Ricci flow. In order to make the book suitable for beginners, we have included in the first chapter basic material on Riemannian geometry. Many exercises and open problems can also be found scattered throughout this book. Some new progress of Perelman will appear in [**153**]. For example, a detailed coverage of entropy and the reduced distance will be included there. Moreover, comparisons are made between the Ricci flow and the linear heat equation, mean curvature flow, and other geometric evolution equations whenever possible. We have found that the analogies between the Ricci flow and other related heat-type equations are important and suggestive. Some new material on Hamilton's singularity analysis (unpublished recent work of Hamilton) has also been incorporated into this book; more generally, we have attempted to make the exposition of Hamilton's singularity analysis more complete. However, this book does not intend to cover all, or even most, classical aspects of the Ricci flow and the choice of topics reflects the authors' limited knowledge in the subject and the authors' preferences. The references at the end of the book and the comments at the end of the chapters are by no means intended to be exhaustive or complete. We only list the references we are aware of and which are most related to the material treated in the book. Due to the limited knowledge of the authors, it is inevitable that we have missed some important related works; we apologize for any omissions.

Prerequisites. We have tried to make this book as self-contained as possible, given the subject matter. The reader is assumed to have a basic knowledge of differentiable manifolds. Chapter 1 is an introduction to Riemannian geometry, and further basic topics in Riemannian geometry and geometric analysis are given in Appendix A. The reader familiar with Riemannian geometry may skip Chapter 1. Throughout the book there is a reliance on tensor calculus. Geometric (comparison geometry) methods also play an important role in this book. This is especially evident in the study of singularities of the Ricci flow. For these reasons we have included a number of exercises intended to develop the reader's proficiency in carrying out local coordinate calculations involving tensors and also applying comparison geometry techniques.

Course suggestions. A one semester course might consist of Chapter 1, Sections 2–5, Chapter 2, Sections 2–3 and 5–7, Chapter 3, Sections 1–6, Chapter 4, Sections 1–4, Chapter 5, Sections 1 and 4–6, with the rest of the sections in Chapters 1 through 5 and Appendix A optional. The second semester topics might include Chapter 6, Sections 1–3 and 5–7, Chapter 7, Sections 1–3, Chapter 8, Sections 1–4, Chapter 9, Sections 1–3 and 5–6, and Chapter 10, Sections 1–5, with the rest of the sections in Chapters 6 through 11 and Appendix B optional. The order of Chapters 6 and 7 is essentially interchangeable. See also the 'overall structure of the book' and 'suggested course outline' flowcharts at the end of the guide for the reader.

A word from our sponsor. Like its cousin, *The Ricci Flow: An Introduction* by Dan Knopf and one of the authors [**163**], which we will, for the moment, nickname "g_{ij}" after the metric, the present book, which we will nickname "Γ_{ijk}" after the connection, comprises an introduction to the subject of Ricci flow. As the notation suggests, Γ_{ijk} is derived from g_{ij}. In the following we compare and contrast the "metric" and its "connection".

1. Substance

In g_{ij} we begin by emphasizing the special geometries and their role in the development of Ricci flow. Then we give detailed and comprehensive treatments of short time existence, maximum principles, Ricci flow on surfaces, and Hamilton's '3-manifolds with positive Ricci curvature theorem'. The remainder of g_{ij} is primarily devoted to proving some classical results on Type I singularities and background on such topics as the strong maximum principle, derivative estimates, the statement of the compactness theorem, and elementary singularity analysis including how to dilate about singularities. In the appendices tensor calculus and basic comparison geometry are presented.

In Γ_{ijk} we begin by giving a review of those aspects of Riemannian geometry which are needed for the study of the Ricci flow. We then discuss the same basic topics of short time existence, maximum principle, and 3-manifolds with positive Ricci curvature. We present the topics of Sobolev inequalities, isoperimetric estimates, Perelman's no local collapsing, Ricci solitons, higher-dimensional Ricci flow, and the Ricci flow on noncompact manifolds. After this we give a comprehensive introduction to singularity analysis, mixing classical results with the application of the no local collapsing theorem. Ancient solutions are presented in detail including the classical theory and Perelman's classification of 3-dimensional noncompact shrinking solitons. Finally we develop differential Harnack inequalities and the space-time approach to Ricci flow which leads to Perelman's reduced distance function. In the appendices geometric analysis related to Ricci flow and topics in geometric evolution equations are presented.

2. Style We would like to imagine g_{ij} as an attempt to write in the style of *jazz* and Γ_{ijk} as an attempt in the style of *rock 'n' roll* (with apologies to rhythm and blues, gospel, country and western, hip hop, etc.). In g_{ij} we dive right into Ricci flow and then proceed at a metric pace, taking the time to appreciate the intricacies and nuances of the melody and structure of the mathematical music. In Γ_{ijk}, after starting from more basic material, as a connection to Ricci flow, the tempo is slightly more upbeat. The recital is defined on a longer page interval, and consequently more ground is covered, with the intention of leading up to the forefront of mathematical research. In g_{ij} calculations are carried out in detail in the main body of the book whereas in Γ_{ijk} the details appear either in the main body or at the end of the book in the solutions to the exercises. With the addition of basic core material on Riemannian geometry and a substantial number of solved exercises, Γ_{ijk} is accessible to graduate students and suitable for use in a semester or year-long graduate course.

Ricci flow and geometrization. The subject of Hamilton's Ricci flow lies in the more general field of geometric flows, which in turn lies in the even more general field of geometric analysis. Ricci flow deforms Riemannian metrics on manifolds by their Ricci tensor, an equation which turns out to exhibit many similarities with the heat equation. Other geometric flows, such as the mean curvature flow of submanifolds, demonstrate similar smoothing properties. The aim for many geometric flows is to produce canonical geometric structures by deforming rather general initial data to these structures. Depending on the initial data, the solutions to geometric flows may encounter singularities where at some time the solution can no longer be defined smoothly. For various reasons, in Ricci flow, the study of the qualitative aspects of solutions, especially ones which form singularities, is at present more amenable in dimension 3. This is precisely the dimension in which the Poincaré conjecture was originally stated; the higher-dimensional generalizations have been solved by Smale in dimensions at least 5 and by Freedman in dimension 4. Remaining in dimension 3, a vast generalization of this conjecture was proposed by Thurston, called the geometrization conjecture, which, roughly speaking, says that each closed

3-manifold admits a geometric decomposition, i.e., can be decomposed into pieces which admit complete locally homogeneous metrics.

Hamilton's program is to use Ricci flow to approach this conjecture. Perelman's work aims at completing this program. Through their works one hopes/expects that the Ricci flow may be used to *infer* the existence of a geometric decomposition by taking any initial Riemannian metric on any closed 3-manifold and proving enough analytic, geometric and topological results about the corresponding solution of the Ricci flow with surgery. Note on the other hand that, in this regard, one does *not* expect to need to prove the convergence of the solution to the Ricci flow with surgery to a (possibly disconnected) homogeneous Riemannian manifold. The reason for this is Cheeger and Gromov's theory of collapsing manifolds and its extension to the case where the curvature is only bounded from below. Furthermore, if one is only interested in approaching the Poincaré conjecture, then one does not expect to need the theory of collapsing manifolds. For these geometric and topological reasons, the study of the Ricci flow as an approach to the Poincaré and geometrization conjectures is reduced to proving certain analytic and geometric results. In many respects the Ricci flow appears to be a very natural equation and we feel that the study of its analytic and geometric properties is of interest in its own right. Independent of the resolution of the above conjectures, there remain a number of interesting open problems concerning the Ricci flow in dimension 3. In higher dimensions, the situation is perhaps even more interesting in that, in general, much less is known.

The year 1982 marked the beginning of Ricci flow with the appearance of Hamilton's paper on 3-manifolds with positive Ricci curvature. Since then, the development of Hamilton's program is primarily scattered throughout several of his papers (see **[81]** for a selection of Ricci flow papers edited by Cao, Chu, Yau and one of the authors). In Hamilton's papers (sometimes implicitly and sometimes by analogy) a well-developed theory of Ricci flow is created as an approach toward the geometrization conjecture. We encourage the reader to go back to these original papers which contain a wealth of information and ideas. Hamilton's program especially takes shape in the three papers [**287**], [**290**] and [**291**]. The first two papers discuss (among other important topics) singularity formation, the classification of singularities, applications of estimates and singularity analysis to the Ricci flow with surgery. The third paper discusses applications of the compactness theorem, minimal surface theory and Mostow rigidity to obtain geometric decompositions of 3-manifolds via Ricci flow under certain assumptions.

The recent spectacular developments due to Grisha Perelman aimed at completing Hamilton's program appear in [**452**] and [**453**]. Again the reader is encouraged to go directly to these sources which contain a plethora of ideas. Perelman's work centers on the further development of singularity and surgery theory. Of primary importance in this regard is Perelman's reduced distance function which has its precursors in the work of Li and Yau on gradient estimates for the heat equation and of Hamilton on matrix differential Harnack inequalities for the Ricci flow. A main theme in Perelman's work is the use of comparison geometry, including the understanding of space-time distance, geodesics, and volume, to obtain estimates. These estimates build upon in an ingenious way the earlier gradient estimates of Li and Yau and of Hamilton and the volume comparison theorem of Bishop and Gromov. In a sense, Perelman has further strengthened the bridge between the partial differential equation and comparison approaches to differential geometry in the setting of Ricci flow.

For finite extinction time of the Ricci flow with surgery on 3-manifolds, which is aimed at proving the Poincaré conjecture using neither the theory of collapse nor most of Hamilton's 'nonsingular' techniques, see [**454**] and Colding and Minicozzi [**176**]. The relevant results on collapsing manifolds with only lower curvature bounds were announced in Perelman [**452**] (with earlier related work in [**451**]) and appear in Shioya and Yamaguchi [**504**]. For expositions of Perelman's work, see Kleiner and Lott [**342**], Sesum, Tian, and Wang [**495**] and Morgan [**419**] (there is also a discussion of [**453**] in Ding [**193**]). We also encourage the reader to consult these excellent expository sources which clarify and fill in the details for much of Perelman's work.[1]

The authors
December 6, 2005

[1]We also understand that there are forthcoming works of Cao and Zhu [**86**], Morgan and Tian [**421**], and Topping [**536**] on Ricci flow devoted primarily to Perelman's work. We encourage the reader to consult these works.

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This book originated as notes by the authors for lectures on Hamilton's Ricci flow given in China during the summer of 2004. These lecture notes were written during the authors' stays at Fudan, East China Normal, Beijing, and Zhejiang Universities. We soon decided to expand these notes into a book which covers many more topics, including some of Perelman's recent results. The work on this book was completed when the authors were at their respective institutions, the University of California at San Diego and the University of Oregon.

The first author gave lectures at the Beijing University Summer School and the Summer School at the Zhejiang University Center of Mathematical Sciences and one lecture each at East China Normal University, Fudan University, and the Nankai Institute. The second author gave lectures at the Summer School at the Zhejiang University Center of Mathematical Sciences. The third author gave lectures at Fudan University and a lecture at East China Normal University.

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The first author would especially like to thank Dan Knopf for collaboration over the years on the book "The Ricci Flow: An Introduction" **[163]**, published recently by the American Mathematical Society. It has been a fun ride and we have enjoyed and benefited from this collaboration quite a bit. We have tried to minimize the overlap between these two books; unfortunately, to retain some degree of completeness, a certain amount of overlap has been unavoidable. Parts of this book are derived from notes written in the 1990's. This author would like to thank editors Ed Dunne, Sergei Gelfand, and Ina Lindemann at the AMS for their help with the AMS Ricci flow series of books and especially to Ed Dunne for all of his help in publishing his book "The Ricci flow: An Introduction" **[163]** with Dan Knopf and the future coauthored book "The Ricci flow: Techniques and Applications" **[153]**. This author would especially like to thank his teachers Professors Shing-Tung Yau and Richard Hamilton for their help, guidance and encouragement over many years. The benefit he has received from them encompasses much of his mathematical career.

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The second author dedicates this book to his teacher Professor Kung-Ching Chang on the occasion of his seventieth birthday.

The last author dedicates this book to both his wife, Jing-Jing Wu, and his teacher Professor Peter Li.

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A Detailed Guide for the Reader

Chapter 1. We present some basic results and facts from Riemannian geometry. The results in this chapter, for the most part, either are used or are analogous to results in the latter chapters on Ricci flow and other geometric flows. Although we have included in this chapter what we feel ideally the reader should know, he or she should be reassured that mastering all of the contents of this chapter is not a prerequisite for studying Ricci flow! Indeed, this chapter may be used as a reference to which the reader may refer when necessary.

In Section 2 we give a quick review of metrics, connections, curvature, and covariant differentiation. Of particular note are the Bianchi identities, the Lie derivative, and the covariant derivative commutator formulas in Section 3, which have applications to the formulas we shall derive for solutions of the Ricci flow. In Section 4 we recall the theory of differential forms and discuss the Laplace operator for tensors and Bochner formulas for differential forms. Since integration by parts is a useful technique in geometric analysis and in particular Ricci flow, in Section 5 we recall the divergence theorem and its consequences. We also give a quick review of the de Rham theorem and the Hodge decomposition theorem. In Section 6 we introduce the Weyl tensor and the decomposition of the Riemann curvature tensor into its irreducible components. We consider some basic aspects of locally conformally flat manifolds. In Section 7 we discuss Cartan's method of moving frames since it is a useful technique for computing curvatures, especially in the presence of symmetry. As an application, we give a proof of the Gauss-Bonnet formula for surfaces using moving frames. We also discuss hypersurfaces from the point of view of moving frames. Metric geometry has important implications in Ricci flow, so we discuss (Section 8) the first and second variation of arc length and energy of paths. Applications are Synge's theorem and the Hessian comparison theorem. We include the application of the second variation formula to long geodesics and a variational proof of the fact that Jacobi fields minimize the index form. We also discuss the first and second variation formulas for the areas of hypersurfaces. Such formulas have applications to minimal surface theory. In Section 9 we recall basic facts about the exponential map such as the Gauss lemma, the Hopf-Rinow theorem, Jacobi fields, conjugate and cut points, and injectivity radius estimates for positively curved manifolds. Next (Section 10) we present geodesic spherical coordinates using the exponential map. This is a convenient way of studying (Hessian) comparison theorems for Jacobi fields, and by taking the determinant, volume (Laplacian) comparison theorems. One may think of these calculations as associated to hypersurfaces (the distance spheres from a point) evolving in their normal directions with unit speed. (More generally, one may consider arbitrary speeds, including the mean curvature flow.) Observe that the Laplacian of the distance function is the radial derivative of the logarithm of the Jacobian, which is the mean curvature of the distance spheres. Similarly, the Hessian of the distance function is the radial derivative of the logarithm of the inner product of Jacobi fields, which is the second fundamental form of the distance spheres. We then begin to discuss in more detail the Laplacian and Hessian comparison theorems in Section 11. These results are essentially equivalent to the Bishop-Gromov volume and Rauch comparison theorems and have important analogues in Ricci flow. As an application, we prove the mean value inequality. In Section 12 we give detailed proofs of the Laplacian, volume and Hessian comparison theorems. In the case of the Laplacian comparison theorem, we prove the inequality holds in the sense of distributions. In Section 13 we discuss the Cheeger-Gromoll splitting theorem, which relies on the Busemann functions associated to a line being subharmonic (and hence harmonic), and the mean value inequality. We then discuss the Toponogov comparison theorem. Left-invariant metrics on Lie groups provide nice examples of solutions on Ricci flow which can often be analyzed, so we introduce some background material in Section 14. In the notes and commentary (Section 15) we review some basic facts about the first and second fundamental forms of hypersurfaces in Euclidean space, since we shall later discuss curvature flows of hypersurfaces to compare with Ricci flow.

Chapter 2. Here we begin the study of Ricci flow proper. Before we describe the contents of this chapter, we suggest that the reader may occasionally refer to Chapter 4 for some explicit examples of solutions to the Ricci flow; these examples may guide the reader's intuition when studying the abstract derivations throughout the book. We start in Section 1 with some historical remarks about geometric evolution equations. We also give a brief layman's description of how Ricci flow approaches the Thurston geometrization conjecture. The Ricci flow is like a heat equation for metrics. One quick way of seeing this (Section 2) is to compute the evolution equation for the scalar curvature using a variation formula we derive later. We get a heat equation with a nonnegative term. Because of this, we can apply the maximum principle (Section 3) to show that the minimum of the scalar curvature increases. The variation formula for the scalar curvature yields a short derivation of Einstein's equations as the Euler-Lagrange equation for the total scalar curvature (Section 4). By modifying the total scalar curvature, we are led to Perelman's energy functional. Next (Section 5) we carry out the actual computations of the variation of the connection and curvatures. When the variation is minus twice the Ricci tensor, we obtain the evolution equations for the connection, scalar and Ricci curvatures under the Ricci flow. This is the first place we encounter the Lichnerowicz Laplacian which arises in the variation formula for the Ricci tensor. Since the Ricci flow is a weakly parabolic equation, to prove short time existence, we use DeTurck's trick, which shows that it is equivalent to a strictly parabolic equation (Section 6). Having derived the evolution equations for the Ricci and scalar curvatures, we derive the evolution equation for the full Riemann curvature tensor (Section 7). This takes the form of a heat equation with a quadratic term on the right-hand side. In dimension 3, the form of the quadratic term is especially simple. In the notes and commentary (Section 8) we discuss the symbol of the linearization of the Ricci tensor.

Chapter 3. We give a proof of Hamilton's classification of closed 3-manifolds with positive Ricci curvature using Ricci flow. Hamilton's theorem (Section 1) says that under the normalized (volume-preserving) Ricci flow on a closed 3-manifold with positive Ricci curvature, the metric converges exponentially fast in every C^k-norm to a constant positive sectional curvature metric. The maximum principle for tensors (Section 2) enables us to estimate the Ricci and sectional curvatures. We first show that positive Ricci curvature and Ricci pinching are preserved. To control the curvatures, it is convenient to generalize the maximum principle for symmetric 2-tensors to a maximum principle for the curvature operator, and more generally, to systems of parabolic equations on a vector bundle (Section 3). Using this formalism, we show that the pinching of the curvatures improves and tends to constant curvature at points and times where the curvature tends to infinity. This is the central estimate in the study of 3-manifolds with positive Ricci curvature. Once we have a pointwise estimate for the curvatures, we need a gradient estimate for the curvature in order to compare curvatures at different points at the same time (Section 4). Based on the fact that the