INTERVAL ORDERS AND INTERVAL GRAPHS

A Study of Partially Ordered Sets

Peter C. Fishburn

Wiley-Interscience Series in Discrete Mathematics

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Preface

Interval orders and interval graphs have emerged during the past few decades as important research subjects in the theory of partially ordered sets and graph theory. The name *interval order* was first used in the late 1960s (Fishburn, 1970a), but the same concept was discussed much earlier as "relations of complete sequence" by Norbert Wiener (1914). The more recent name reflects the fact that a partially ordered set $(X, \prec)$ is an interval order precisely when its points $x, y, \ldots$ can be mapped into intervals in a linearly ordered set, such as $(\mathbb{R}, \prec)$, such that, for all x and y in X, $x \prec y$ if and only if the interval assigned to x completely precedes the interval assigned to y.

The companion term *interval graph* appeared earlier (Gilmore and Hoffman, 1962), having been previously discussed without this designation by Hajös (1957) and Benzer (1959). It refers to a graph $(X, \sim)$ whose points can be mapped into intervals in a linearly ordered set such that, for all distinct x and y, $x \sim y$ if and only if the intervals assigned to x and y have a nonempty intersection.

The close relationship between interval orders and interval graphs is suggested by two observations. First, if $(X, \prec)$ is an interval order, and if $\sim$ is the symmetric complement of $\prec$, then $(X, \sim)$ is an interval graph. Second, if $(X, \sim)$ is an interval graph, and if $\prec$ is defined from an interval representation of $(X, \sim)$ by $x \prec y$ if the interval for x completely precedes the interval for y, then $(X, \prec)$ is an interval order.

The present book is an outgrowth of my research on ordered sets. It attempts to provide a unified treatment of interval orders, interval graphs, and related concepts such as semiorders, comparability graphs, and indifference graphs (named, respectively, by R. D. Luce, A. J. Hoffman, and F. S. Roberts). The presentation is self-contained and is designed to be readily accessible to mathematicians, upper-level students of mathematics, and people in economics, statistics, psychology, computer science, and other fields who are interested in ordered sets and graphs.

My organization of the book around the theme of interval orders reflects its author's particular orientation, as does the choice of subject matter. However, important topics developed by others are included for broader and more balanced coverage. At the same time, the book does not try to treat exhaustively all of the topics mentioned, and some extensions and generalizations of interval orders and interval graphs are not discussed. A great deal of additional material will be found, for example, in the graph-theory books by Berge (1973) and Golumbic (1980), and in the recent treatise on linear orders by Rosenstein (1982).

A few remarks on personal style are in order. Over the years I have come to prefer to work with irreflexive as opposed to reflexive orders and will follow this preference here. Thus partial, interval, linear, and other types of orders symbolized by $\prec$, $<_1$, and so forth, are always irreflexive (it is never true that $x \prec x$) and asymmetric [if $x \prec y$, then not $(y \prec x)$]. Reflexive orders, when used, carry an undersymbol, as in $\precsim$ and $\leq$. The symbol $\sim$, by itself or with scripts, always denotes a symmetric binary relation; it may or may not be reflexive. In most cases, $(X, \sim)$ is a reflexive graph, so every point has a "loop" $(x \sim x)$; the main exception involves comparability graphs.

Theorems within each chapter are numbered consecutively without chapter prefix, but interchapter references add the chapter number (Theorem 5.3 is Theorem 3 in Chapter 5). The end of each proof is marked by $\square$, $|X|$ is the cardinality of set X, $A \setminus B$ is the set of points in A but not B, $\subset$ denotes proper inclusion, $\mathbb{R}$ is the set of real numbers, $\lfloor x \rfloor$ is the integer part of x, and $\lceil x \rceil$ is the smallest integer not less than x.

For the record, and to avoid unnecessary references to my own work throughout the text, I note here publications used as source material for parts of the book. Fxy signifies Fishburn (19xy) in the references at the end of the book; numbers in parentheses are the numbers of theorems in the book that first appeared in Fxy. F69 (7.3); F70a (1.4, 2.2, 2.3, 2.8, 3.1); F70b (1.1, 1.2, 1.4, 2.2, 2.3, 2.8); F70c (1.1(c)); F71 (3.1, 4.1, 4.3, 4.5, 4.6); F73a (7.5 through 7.12); F73b (1.2(f), 2.10); F81a (9.1, 9.3); F81b (6.1 through 6.4); F82 (6.2, 6.5, 6.6); F83a (8.2, 8.4, 8.5); F83b (5.18, 7.2); F83c (10.1 through 10.6); F84a (3.12); F84b (8.1, 8.3, 9.2, 9.4); F84c (9.6, 9.7, 9.8).

I am indebted to many people for guidance and encouragement over the years in my work on ordered sets. Special thanks go to Fred Roberts, Duncan Luce, Peter Hammer, Ronald Graham, and Thomas Trotter. I am also indebted to the institutions that have supported my research in this area—The Research Analysis Corporation, The Institute for Advanced Study, The Pennsylvania State University, AT & T Bell Laboratories—and to Bell Labs for making the book possible. The entire manuscript was superbly typed by Marie Wenslau, and I thank her for invaluable assistance.

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1

Introduction

This chapter has two purposes. The first is to introduce basic terminology and facts about relations, orders, and graphs that are used throughout the book. The second is to outline topics discussed later. Our study of interval orders proper begins in the next chapter.

1.1 BINARY RELATIONS

An *n-ary relation* R on a set X is a subset of X^n. Although ternary ($n = 3$) and quaternary ($n = 4$) relations will be used later, most of the relations we shall deal with are binary relations, with $R \subseteq X \times X$. Notationally, xRy means the same thing as $(x, y) \in R$, and not(xRy) or $x\not{R}y$ signifies $(x, y) \notin R$.

We shall say that a binary relation R on X is

reflexive if xRx for every x in X.
irreflexive if not(xRx) for every x in X.
symmetric if $xRy \Rightarrow yRx$ for all x and y in X.
asymmetric if $xRy \Rightarrow$ not(yRx) for all x and y in X.
transitive if $(xRy, yRz) \Rightarrow xRz$ for all x, y, and z in X.
negatively transitive if $xRy \Rightarrow (xRz$ or $zRy)$ for all x, y, and z in X.
complete if $x \neq y \Rightarrow (xRy$ or $yRx)$ for all x and y in X.

Other properties will be introduced as they are needed.

An *equivalence relation* on X is a reflexive, symmetric, and transitive binary relation on X. Given an equivalence relation E on X, X/E denotes the *set of equivalence classes* determined by E. Each class in X/E is a subset of X of the form $\{y\colon yEx\}$, and X/E is a partition of X. Conversely, a partition of X into nonempty subsets determines an equivalence relation E by defining xEy if x and y are in the same element of the partition.

The *composition* RS of binary relations R and S on X is defined by

$$RS = \{(x, y)\colon xRz \text{ and } zSy \text{ for some } z \text{ in } X\}.$$

When $S = R$, we write RS as R^2. For $n > 2$, $R^n = R(R^{n-1})$. Transitivity says that $R^2 \subseteq R$.

For any binary relation R on X, let $a(R)$, $s(R)$, $c(R)$, $d(R)$, and $t(R)$ be respectively the *asymmetric part* of R, the *symmetric part* of R, the *complement* of R, the *dual* of R, and the *transitive closure* of R:

$$a(R) = \{(x, y)\colon xRy \text{ and } \text{not}(yRx)\}.$$
$$s(R) = \{(x, y)\colon xRy \text{ and } yRx\}.$$
$$c(R) = \{(x, y)\colon \text{not}(xRy)\}.$$
$$d(R) = \{(x, y)\colon (y, x) \in R\}.$$
$$t(R) = R \cup R^2 \cup R^3 \dots .$$

These basic operations combine to form compound operations, such as $cd(R)$, the complement of the dual of R, $sc(R)$, the *symmetric complement* of R, and $ta(R)$, the transitive closure of the asymmetric part of R. Fishburn (1978) proves that, in addition to the empty relation $\varnothing$ and the universal relation $X \times X$, at most 110 different relations can be generated from a given relation by sequential applications of the five basic operations, and that 110 is the least upper bound. Examples of duplications for different sequences are $cd(R) = dc(R)$, $ac(R) = ad(R)$, $atcat(R) = atct(R)$, and $stct(R) = tsct(R)$.

It is easily seen that $a(R) = R \cap cd(R)$ and $s(R) = R \cap d(R)$. In addition, we have the following expressions for properties defined earlier:

symmetry: $d(R) = R$
asymmetry: $R \cap d(R) = \varnothing$
negative transitivity: $c(R)^2 \subseteq c(R)$.

When A and B are nonempty subsets of X, and R is a binary relation on X, we shall write

$$ARB \quad \text{if } aRb \quad \text{for all } (a, b) \in A \times B.$$

Similarly, when $a, b \in X$, aRB means $\{a\}RB$, and ARb means $AR\{b\}$.

1.2 RELATED SETS

The simple relational system (X, R) in which R is a binary relation on X will be referred to as a *related set*. More specifically, when E is an equivalence relation on X, (X, E) is an equivalence set; when $\prec$ is a partial order on X, $(X, \prec)$ is a partially ordered set; and so forth.

We shall say that (X, R) is reflexive (irreflexive, ..., complete) if R is reflexive (irreflexive, ..., complete). In addition, the operations on binary relations defined in the preceding section apply to related sets by the definitions

$$\gamma(X, R) = (X, \gamma(R)) \qquad \gamma \in \{a, s, c, d, t\}.$$

Thus the asymmetric part of (X, R) is $(X, a(R))$, the complement of (X, R) is $(X, c(R))$, and so forth.

Two related sets (X, R) and (Y, S) are *isomorphic* if there is a one-to-one mapping f from X onto Y such that, for all $x, y \in X$,

$$xRy \Leftrightarrow f(x)Sf(y).$$

Isomorphism between (X, R) and (Y, S) is expressed by $(X, R) \cong (Y, S)$. It is easily seen that isomorphism is an equivalence relation on the class of related sets.

The *restriction* of a binary relation R on X to a subset Y of X is $R \cap (Y \times Y)$, and the restriction of the related set (X, R) to $Y \subseteq X$ is $(Y, R \cap (Y \times Y))$. For convenience, we shall sometimes abbreviate $(Y, R \cap (Y \times Y))$ as (Y, R), or just Y. A subset $Z \subseteq X$ in the context of (X, R) is said to be isomorphic to (Y, S) if (Z, R) is isomorphic to (Y, S).

1.3 ORDERED SETS

A binary relation R on X is a:

partial order if R is irreflexive and transitive.
weak order if R is asymmetric and negatively transitive.
linear order if R is a complete weak order.

We refer to (X, R) respectively as a *poset* (partially ordered set), a weakly ordered set, and a linearly ordered set. I leave it to the reader to show that a partial order is asymmetric, a weak order is transitive, and a complete partial order is a linear order.

Linearly ordered sets will also be called *chains*. A *linearly ordered subset* or chain in a related set (X, R) is a $Y \subseteq X$ such that $(Y, R \cap (Y \times Y))$ is a linearly ordered set. A chain in (X, R) is *maximal* if it is not properly included in another chain in (X, R), and it is *maximum* if no chain has greater cardinality.

The distinctions among partial, weak, and linear orders are illustrated by the Hasse diagrams in Fig. 1.1. In such a diagram, either xRy if x is above y and there is a downward path from x to y, or (exclusionary) xRy if x is below y and there is an upward path from x to y. The latter orientation will be used when we write R as $\prec$.

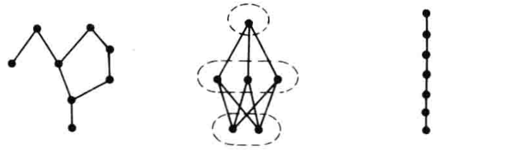

Figure 1.1 Posets.

When $(X, \prec)$ is a poset, we shall often use $\sim$ to denote $sc(\prec)$, in which case

$$x \sim y \quad \text{if not}(x \prec y) \text{ and not}(y \prec x).$$

We now present two theorems that record useful facts about partial orders and foreshadow some of the results proved later for interval orders. These theorems describe qualitative properties of $\prec$ with the use of $\sim = sc(\prec)$ and two other binary relations on X that are based on $\prec$. A numerical representation theorem is proved in the next section.

The first new relation is defined succinctly by

$$\approx = c[(\sim)c(\sim) \cup c(\sim)(\sim)],$$

where $(\sim)c(\sim)$ is the composition of $\sim$ and $c(\sim)$, and $c(\sim)(\sim)$ is the composition of $c(\sim)$ and $\sim$. In more detail, $x \approx y$ if it is false that there is a z for which $x \sim z$ and not$(z \sim y)$, and it is false that there is a z for which not$(x \sim z)$ and $z \sim y$, that is,

$$x \approx y \quad \text{if } \{z\colon z \sim x\} = \{z\colon z \sim y\}.$$

When $(X, \prec)$ is a poset, Theorem 1 notes that $x \approx y$ if x and y have the same upper sets ($\{z\colon x \prec z\} = \{z\colon y \prec z\}$) and the same lower sets ($\{z\colon z \prec x\} = \{z\colon z \prec y\}$).

The second new relation, called the *sequel* of $\prec$, is defined by

$$S_0(\prec) = [(\sim)(\prec) \cup (\prec)(\sim)] \cap cd[(\sim)(\prec) \cup (\prec)(\sim)],$$

where $(\ )(\ )$ denotes composition. Thus, $xS_0(\prec)y$ if either $x(\sim)(\prec)y$ or $x(\prec)(\sim)y$, and neither $y(\sim)(\prec)x$ nor $y(\prec)(\sim)x$. When $(X, \prec)$ is a poset, so is $(X, S_0(\prec))$; moreover, $\prec$ is included in $S_0(\prec)$.

Here and later X is assumed to be nonempty. Unless it is noted otherwise, there is no restriction on the cardinality of X apart from $|X| > 0$.

Theorem 1. *Suppose* $(X, \prec)$ *is a poset. Let* $\sim\, = sc(\prec)$, *with* $\approx$ *and* $S_0(\prec)$ *as defined above. Then*

(a) $\approx$ *is an equivalence relation.*

(b) $\prec$, $(\prec)(\approx)$, *and* $(\approx)(\prec)$ *are identical.*

(c) $x \approx y \Leftrightarrow [\{z\colon x \prec z\} = \{z\colon y \prec z\}, \{z\colon z \prec x\} = \{z\colon z \prec y\}]$.

(d) $(X/\approx, \prec)$ *is a poset.*

(e) $S_0(\prec)$ *is a partial order that includes* $\prec$.

Proof. (a) Since $\prec$ is irreflexive, its symmetric complement $\sim$ is reflexive and symmetric, and $\approx$ is clearly reflexive and symmetric. Suppose $x \approx y$ and $y \approx z$. Then $\{w\colon w \sim x\} = \{w\colon w \sim y\} = \{w\colon w \sim z\}$, hence $\{w\colon w \sim x\} = \{w\colon w \sim z\}$, hence $x \approx z$, so $\approx$ is transitive.

Proofs of (b), (c), and (d) are left to the reader.

(e) Suppose $x \prec y$. Then $x \sim x \prec y$, and either $y \sim z \prec x$ or $y \prec z \sim x$ yields a contradiction by transitivity, so $xS_0(\prec)y$. By its definition, $S_0(\prec)$ is irreflexive. To verify that it is transitive, suppose $xS_0(\prec)y$ and $yS_0(\prec)z$. Contrary to transitivity, suppose not($xS_0(\prec)z$). Then either

(i) $z \sim a$ and $a \prec x$ for some $a \in X$, or

(ii) $z \prec a$ and $a \sim x$ for some $a \in X$, or

(iii) there is no $b \in A$ such that either $(x \sim b,\ b \prec z)$ or $(x \prec b,\ b \sim z)$.

If (i) holds, then $a \prec y$ by $xS_0(\prec)y$, but $a \prec y$ and $z \sim a$ contradict $yS_0(\prec)z$. A similar contradiction follows from (ii). Suppose then that (iii) holds. Assume first that $xS_0(\prec)y$ is realized in part by $(x \sim c,\ c \prec y)$. Then (iii) implies not($c \prec z$), so either $z \prec c$ or $z \sim c$. However, since $c \prec y$, each of $z \prec c$ and $z \sim c$ contradicts $yS_0(\prec)z$. A similar contradiction obtains if $(x \prec c,\ c \sim y)$, so (iii) leads to a contradiction in any event, and we conclude that $xS_0(\prec)z$. □

Theorem 2. *Suppose the hypotheses and definitions of Theorem* 1 *hold. Then the following are mutually equivalent:*

(a) $(X, \prec)$ *is a weakly ordered set.*

(b) $\sim$ *is transitive.*

(c) $\sim\, = \,\approx$.

(d) $\prec\, = (\prec)(\sim) = (\sim)(\prec)$.

(e) $(X/\approx, \prec)$ *is a linearly ordered set.*

Moreover, if X is finite, then each of (a)–(e) *is equivalent to*

(f) $S_0(\prec) = \,\prec$.

In addition, $(X, \prec)$ *is a linearly ordered set if and only if* $\sim$ *is the identity relation* $\{(x, x)\colon x \in X\}$.

Remark 1. We show why (f) can fail to imply weak order when X is not finite. Suppose $(X, \prec)$ consists of the linearly ordered set of integers $(\mathbb{Z}, <)$

plus an additional point ω such that $\omega \sim n$ for all $n \in \mathbb{Z}$. Then $n \prec n+1 \sim \omega \Rightarrow \text{not}(\omega S_0(\prec) n)$, and $\omega \sim (n-1) \prec n \Rightarrow \text{not}(n S_0(\prec)\omega)$. Theorem 1(e) then implies $S_0(\prec) = \prec$, but $\prec$ is not a weak order since $\sim$ is not transitive.

Proof. The proof of the final sentence of Theorem 2 is left to the reader. The proofs of the other implications follow.

(a) $\Rightarrow$ (b). Since negative transitivity says that $x \prec z \Rightarrow (x \prec y \text{ or } y \prec z)$ and $z \prec x \Rightarrow (z \prec y \text{ or } y \prec x)$, $x \sim y$ and $y \sim z$ imply $x \sim z$, so $\sim$ is transitive.

(b) $\Rightarrow$ (c). Clearly $x \approx y \Rightarrow x \sim y$, and $x \sim y \Rightarrow (z \sim x \Leftrightarrow z \sim y)$.

(c) $\Rightarrow$ (d). See Theorem 1(b).

(d) $\Rightarrow$ (b). Obvious.

(d) $\Rightarrow$ (e). By Theorem 1(d), $(X/\approx, \prec)$ is a poset. By (d) $\Rightarrow$ (b) $\Rightarrow$ (c), $\approx$ equals $\sim$. To show that $\prec$ on $X/\approx$ is complete, suppose $A, B \in X/\sim$ and $A \neq B$. Then $A \cap B = \varnothing$ since $\sim$ is an equivalence relation by Theorem 1(a). Hence $a \prec b$ or $b \prec a$ for some $(a, b) \in A \times B$. If $a \prec b$ then (d) $\Rightarrow A \prec B$, and if $b \prec a$ then (d) $\Rightarrow B \prec A$.

(e) $\Rightarrow$ (a). Since $(X, \prec)$ is a poset, $\prec$ is asymmetric. To prove that $\prec$ is negatively transitive, let $[x]$ denote the equivalence class in $X/\approx$ that contains x. Suppose $x \prec z$. Then $[x] \cap [z] = \varnothing$. Given any $y \in X$, either $[y] \cap [x] = \varnothing$ or $[y] \cap [z] = \varnothing$. Suppose for definiteness that $[y] \cap [x] = \varnothing$. Then (e) implies either $[x] \prec [y]$ or $[y] \prec [x]$. If $[x] \prec [y]$ then $x \prec y$, and if $[y] \prec [x]$ then $y \prec x$, hence $y \prec z$ by transitivity. Therefore $x \prec z \Rightarrow (x \prec y \text{ or } y \prec z)$, so $\prec$ is negatively transitive.

(a) $\Rightarrow$ (f). Since (a) $\Rightarrow$ (d), $S_0(\prec) = \prec \cap cd(\prec)$. Since $\prec$ is asymmetric (see preceding paragraph), $x \prec y \Leftrightarrow y d(\prec) x \Rightarrow \text{not}(x d(\prec) y) \Leftrightarrow x cd(\prec) y$, and therefore $\prec \subseteq cd(\prec)$. Hence $S_0(\prec) = \prec$.

[(f) and $|X| < \infty$] $\Rightarrow$ (a). The conclusion is obvious if $\prec$ is empty. Assume henceforth that $S_0(\prec) = \prec \neq \varnothing$. Let Y be a maximum-cardinality linearly ordered subset of $(X, \prec)$ and let $|Y| = m$. The existence of Y is guaranteed by the finiteness of X. Let

$$A_k = \{x\text{: the maximum-cardinality chain in which } x \text{ is the final element has } k \text{ elements}\}.$$

Clearly, $x \sim y$ when $x, y \in A_k$. This and the transitivity of $\prec$ show that $(x \in A_j, y \in A_k, j < k) \Rightarrow \text{not}(y \prec x)$. Hence to prove that $\prec$ is a weak order, we need only show that $A_1 \prec A_2 \prec \cdots \prec A_m$. Suppose this is false. Let k be the smallest integer for which $\text{not}(A_{k-1} \prec A_k)$. Then $x_{k-1} \sim x_j$ for some $x_{k-1} \in A_{k-1}$ and $x_j \in A_j$ with $j \geq k$. Choose x_j for this so that j is as large as possible. Then $x_{k-1}(\sim)(\prec)x_j$ since, by the definition of A_j, $y_{k-1} \prec x_j$ for some $y_{k-1} \in A_{k-1}$. If $z \prec x_{k-1}$ then, by the choice of k, $z \prec y_{k-1}$, hence $z \prec x_j$, hence $\text{not}(x_j \sim z \prec x_{k-1})$, so that $(x_{k-1}, x_j) \in cd((\sim)(\prec))$. Moreover, $(x_{k-1}, x_j) \in cd((\prec)(\sim))$ since if $x_j(\prec)(\sim)x_{k-1}$ then $x_j \prec v \sim x_{k-1}$ for some v, which contradicts the choice of j. Since $cd((\sim)(\prec)) \cap cd((\prec)(\sim)) =$

$cd((\sim)(\prec) \cup (\prec)(\sim))$, $x_{k-1}S_0(\prec)x_j$, and therefore $x_{k-1} \prec x_j$ by (f). But this contradicts our hypothesis that $x_{k-1} \sim x_j$, and we conclude that $A_1 \prec A_2 \prec \cdots \prec A_m$. □

1.4 LINEAR EXTENSIONS

A *linear extension* of a poset $(X, \prec)$ is a linearly ordered set $(X, \prec^*)$ for which $\prec \subseteq \prec^*$. This section first proves the extension theorem of Szpilrajn (1930), then uses his theorem in the proof of a numerical representation theorem for posets that have countably many $\approx$ classes.

The proof of Szpilrajn's theorem uses

Kuratowski's Lemma. *Each chain in a poset is included in a maximal chain of the poset.*

Kelley (1955, p. 33) notes that this is equivalent to several other set-theoretic axioms, including the Axiom of Choice—if $\mathscr{F}$ is a set of nonempty sets, then there is a function g on $\mathscr{F}$ such that $g(A) \in A$ for every $A \in \mathscr{F}$.

Theorem 3 (Szpilrajn). *Every poset has a linear extension.*

Proof. Let $(X, \prec)$ be a poset. Assume that $\prec$ is not complete since otherwise $(X, \prec)$ is a chain. Let $\sim = sc(\prec)$, take $x \sim y$ for $x \neq y$, and define $\prec'$ on X by

$$\prec' = \prec \cup [\{a: a \prec x \text{ or } a = x\} \times \{b: y \prec b \text{ or } y = b\}].$$

It is routine to verify that $(X, \prec')$ is a poset. Moreover, $\prec \subset \prec'$ since $x \prec' y$.

Let $\mathscr{P}$ be the set of all partial orders P on X such that $\prec \subseteq P$, and order $\mathscr{P}$ by inclusion so that $(\mathscr{P}, \subset)$ is a poset. Let $(\mathscr{P}', \subset)$ be any nonempty chain in $(\mathscr{P}, \subset)$. Then, by Kuratowski's lemma, there is a maximal chain $(\mathscr{P}^*, \subset)$ in $(\mathscr{P}, \subset)$ that includes $(\mathscr{P}', \subset)$. Given such a $(\mathscr{P}^*, \subset)$, let

$$\prec^* = \bigcup\{P: P \in \mathscr{P}^*\}.$$

Since $\mathscr{P}^*$ is linearly ordered by $\subset$, it is easily seen that $\prec^*$ is a partial order. Suppose $\prec^*$ is not complete. Then, by the preceding paragraph, there is a partial order $\prec^{*\prime}$ on X that properly includes $\prec^*$. But then $(\mathscr{P}^*, \subset)$ is not maximal since $P \subset \prec^{*\prime}$ for every $P \in \mathscr{P}^*$, and we obtain a contradiction. Therefore $\prec^*$ is complete, so it gives a linear extension of $\prec$. □

Since x and y can be interchanged in the first paragraph of the preceding proof, it follows that for every $(x, y) \in sc(\prec)$ for which $x \neq y$, there are linear extensions of $(X, \prec)$ in which x precedes y, and others in which y precedes x. Consequently, the intersection of all linear extensions of a poset equals the poset. The minimal cardinality over the sets of linear extensions whose intersections equal the poset is called the *dimension* of the poset. We shall return to this in Chapter 5.

We now use Szpilrajn's theorem to prove

Theorem 4. *Suppose $(X, \prec)$ is a poset for which $X/\approx$ is countable when $\approx$ is defined as in the preceding section. Then there exists $f\colon X \to \mathbb{R}$ such that, for all $x, y \in X$,*

$$x \approx y \Leftrightarrow f(x) = f(y),$$

$$x \prec y \Rightarrow f(x) < f(y).$$

Remark 2. Theorems 2 and 4 imply that if $(X, \prec)$ is a weakly ordered set such that $X/sc(\prec)$ is countable, then there exists $f\colon X \to \mathbb{R}$ such that, for all $x, y \in X$,

$$x \prec y \Leftrightarrow f(x) < f(y).$$

This is true also if $(X, \prec)$ is a countable chain, but may be false when X is uncountable. The latter case is discussed further in Chapter 7.

Proof. Given the hypotheses of Theorem 4, Theorems 1(d) and 3 imply that there is a linear order $\prec^*$ on $X/\approx$ that includes $\prec$ on $X/\approx$. Since $X/\approx$ is countable, it can be enumerated as $A_1, A_2, \ldots$. Define $F\colon X/\approx \to \mathbb{R}$ by

$$F(A_k) = \Sigma\{2^{-j}\colon A_j \prec^* A_k\},$$

where the summation is over those j for which $A_j \prec^* A_k$. If $A_i \prec^* A_k$ then $\{j\colon A_j \prec^* A_i\} \subset \{j\colon A_j \prec^* A_k\}$ so that $F(A_i) < F(A_k)$. Since $\prec^*$ is a chain, all F values are distinct. Moreover, if $A, B \in X/\approx$ and $A \prec B$, then $A \prec^* B$, so $F(A) < F(B)$. The conclusion of the theorem follows on defining $f(x)$ as $F(A)$ for all $x \in A$ and all $A \in X/\approx$. □

1.5 TRANSITIVE REORIENTATIONS

A *reorientation* of an asymmetric related set (X, R) is an asymmetric related set (X, S) for which

$$S \cup d(S) = R \cup d(R).$$

Thus $xRy \Rightarrow (xSy \text{ or } ySx)$, and $xSy \Rightarrow (xRy \text{ or } yRx)$. A reorientation (X, S)

of an asymmetric related set (X, R) is *transitive* if S is transitive, that is, if (X, S) is a poset.

In this section we shall prove an intriguing result of Ghouilà-Houri (1962) which says that an asymmetric related set (X, R) has a transitive reorientation if it satisfies the pseudo-transitivity property $R^2 \subseteq R \cup d(R)$, that is, if for all x, y, and z in X,

$$(xRy, yRz) \Rightarrow (xRz \text{ or } zRx).$$

The proof for finite X follows Ghouilà-Houri (1962) and Berge (1973, pp. 365–366). The proof for infinite X is suggested by Wolk (1965) on the basis of Rado's theorem (Rado, 1949) which, as noted by Mirski and Perfect (1966), has relevance for diverse problems.

I state Rado's theorem without proof: see Mirski and Perfect (1966, p. 540) for references to several proofs. A *choice function* g on a set $\mathscr{F}$ of nonempty sets is a function g on $\mathscr{F}$ such that $g(A) \in A$ for every $A \in \mathscr{F}$. (Cf. Axiom of Choice in the preceding section.)

Rado's Theorem. *Suppose J is a nonempty set, $\mathscr{F} = \{X_j\colon j \in J\}$ is a family of nonempty finite sets (one for each j, with or without duplications) and, for every finite $A \subseteq J$, g_A is a choice function on $\{X_j\colon j \in A\}$. Then there is a choice function g on $\mathscr{F}$ such that, for every finite $A \subseteq J$ there is a finite $B \subseteq J$ for which $A \subseteq B$ and $g(X_i) = g_B(X_i)$ for all $i \in A$.*

If J is finite then the theorem is trivial: just take $g = g_J$. Its general form presumes the Axiom of Choice since it presumes the existence of a choice function g on $\mathscr{F}$ regardless of the nature of J.

Theorem 5 (Ghouilà-Houri). *An asymmetric related set (X, R) has a transitive reorientation if $R^2 \subseteq R \cup d(R)$.*

Proof. Assume throughout that (X, R) is asymmetric with $R^2 \subseteq R \cup d(R)$. Call x and y *adjacent* if xRy or yRx, and observe that if x, y, and z form an R-cycle, say $(x, y), (y, z), (z, x) \in R$, then every other point in X is adjacent to 0, 2, or 3 of x, y, and z.

We consider finite X first and proceed by induction on $|X|$, noting that each nonempty restriction of (X, R) inherits the properties assumed for (X, R). The theorem is clearly true when $|X| = 1$. Suppose $|X| = n > 1$ and the theorem holds for sets with fewer than n points. Suppose in addition that R is not transitive, since otherwise there is nothing to prove. Then there exist x_1, x_2, and x_3 in X that cycle in R:

$$x_1 R x_2,\; x_2 R x_3,\; x_3 R x_1.$$

Two cases require analysis.