Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Finite Rank Torsion Free Abelian Groups and Rings

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INTRODUCTION

These notes contain a largely expository introduction to the theory of finite rank torsion free abelian groups developed since the publication of "Infinite Abelian Groups," Vol. II, L. Fuchs, in 1973. As reflected in Chapter XIII of that text, the subject consists of a satisfactory theory for direct sums of rank 1 groups due to R. Baer in 1937; a uniqueness of quasi-direct sum decompositions up to quasi-isomorphism due to B. Jónsson in 1959; a realization of subrings of finite dimensional Q-algebras as endomorphism rings due to A.L.S. Corner in 1963; a variety of pathological direct sum decompositions; and some apparently miscellaneous results largely relegated to the exercises.

Substantial progress has been made in the subject since 1973. Most notable are the stable range conditions proved by R.B. Warfield, near isomorphism as introduced by E.L. Lady, and the application of properties of subrings of finite dimensional Q-algebras to finite rank torsion free abelian groups via a Morita-like duality developed by E.L. Lady and the author. Consequently, some older results of R. Beaumont, R. Pierce, and J. Reid (c. 1960) involving subrings of finite dimensional Q-algebras gain new importance. Thus a systematic introduction to the theory of finite rank torsion free abelian groups and subrings of finite dimensional Q-algebras seems timely.

The theory of direct sums of rank-1 torsion free abelian groups has been combined with the theory of totally projective groups to characterize a class of mixed abelian groups (Warfield [7] and Hunter-Richman [1]). The category Walk, as discussed in Warfield [7], has been used to investigate mixed abelian groups. A secondary goal of these notes is to survey the known results for finite rank torsion free abelian groups with an eye towards eventual application to mixed groups of finite torsion free rank via the category Walk. Some progress along these lines is reported by Warfield [7]. Other potential applications include the study of mixed abelian groups of finite torsion free rank via the category Warf, as discussed in Arnold - Hunter - Richman [1], and valuated finite direct sums of torsion free cyclic groups, as discussed in a series of papers by E. Walker, F. Richman, R. Hunter, and the author. In particular, Rotman [1] shows that finite rank torsion free groups are characterized in terms of valuated finite direct sums of torsion free cyclic groups.

These notes were developed for a graduate course taught by the author as part of the Year of Algebra at the University of Connecticut during academic year 1978-1979. The students were assumed to have had a graduate course in algebra (fundamental concepts and classical theory of artinian rings are given in Section 0 and the exercises) but little or no exposure to finite rank torsion free abelian groups or subrings of finite dimensional Q-algebras.

Except for portions of Sections 0, 1, and 2 there is little overlap with the results proved in Fuchs [7], Vol. II. There are exercises at the end of each section, some of which are contributed by others as noted, devoted to an extension and elaboration of the results presented or of the requisite background material. No attempt has been made to state or prove results in maximum generality, but in most cases references are given for more general theorems.

Sections 1-4 include a classical introduction to the subject of finite rank torsion free abelian groups as well as some generalizations of type and applications (Richman [1] and Warfield [1]) in Section 1; properties of rank-2 groups in terms of their typeset (Beaumont-Pierce [2]) in Section 3; and characterizations of pure subgroups of finite rank completely decomposable groups (Butler [1]) in Section 4.

Generalizations of such topics as finite rank completely decomposable groups and Baer's Lemma are developed in Sections 5-6 as derived by Arnold-Lady [1] and Arnold-Hunter-Richman [1].

Section 7 includes a proof of the Krull-Schmidt Theorem in additive categories with Jónsson's quasi-decomposition theorem and some essential properties of near isomorphism, due to Lady [1], as corollaries.

Stable range conditions are considered in Section 8 (Warfield [5]) as well as cancellation and substitution properties (Warfield [5], Fuchs-Loonstra [2], and Arnold-Lady [1]), exchange properties (Warfield [5], Monk [1], Crawley-Jónsson [1]) and self-cancellation (Arnold [7]).

Sections 9-11 include an extensive introduction to the subject of subrings of finite dimensional Q-algebras, including a proof of the Jordan-Zassenhaus Theorem for Z-orders, derived in part from Reiner [1] and Swan-Evans [1]. The fact that the additive groups of such rings are finite rank torsion free is exploited to avoid completions in the derivation of the theory. Moreover, localization at primes of Z is consistently used instead of localization at prime ideals of more general domains.

The relationship between near isomorphism and genus class of lattices over orders is examined in Section 12. Classical properties of genus classes of lattices over orders are derived and used to develop properties of near isomorphism of finite rank torsion free abelian groups.

The structure of Grothendieck groups of finite rank torsion free abelian groups is considered in Section 13, as developed by Lady [2] and Rotman [2].

Section 14 includes characterizations of additive groups of subrings of finite dimensional Q-algebras, due to Beaumont-Pierce [1] and [3], including a proof of the Wedderburn principal theorem and a simplified proof of the analog for subrings of finite dimensional Q-algebras.

Several classes of groups are given in Section 15, providing an appropriate setting for the development of Murley groups (Murley [1]) and strongly homogeneous groups (Arnold [6]).

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§0. Notation and Preliminaries

It is assumed that rings have identities and that ring homomorphisms preserve identities. In particular, if R is a subring of S then $1_R = 1_S$.

Suppose that R is a ring, A is a right R-module and B is a left R-module. Then $A \otimes_R B$ is defined to be F/N where F is the free abelian group with elements of $A \times B$ as a basis and N is the subgroup of F generated by $\{(a_1 + a_2, b) - (a_1, b) - (a_2, b), (a, rb) - (ar, b), (a, b_1 + b_2) - (a, b_1) - (a, b_2) | a, a_1, a_2 \in A; b, b_1, b_2 \in A$ and $r \in R\}$. Write $a \otimes b$ for $(a, b) + N$ so that if $x \in A \otimes_R B$ then $x = \Sigma\, a_i \otimes b_i$ for some $a_i \in A$, $b_i \in B$. Then $\psi : A \times B \to A \otimes_R B$, defined by $\psi(a,b) = a \otimes b$ is an R bi-linear map. Furthermore, if G is an abelian group and $g : A \times B \to G$ is an R bi-linear map then there is a unique homomorphism $\psi : A \otimes_R B \to G$ with $\phi\psi = g$.

With the above properties, one can prove that

(i) $A \otimes_R (\oplus B_i) \simeq \oplus (A \otimes_R B_i)$;

(ii) If $f \in \mathrm{Hom}_R(A, A')$ and $g \in \mathrm{Hom}_R(B, B')$ then $f \otimes g : A \otimes_R B \to A' \otimes_R B'$, induced by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$, is a homomorphism of abelian groups;

(iii) $A \otimes_R *$ and $* \otimes_R B$ are functors, e.g., if $f \in \mathrm{Hom}_R(B_1, B_2)$ and $g \in \mathrm{Hom}_R(B_2, B_3)$ then $(1_A \otimes g)(1_A \otimes f) = 1_A \otimes gf : A \otimes_R B_1 \to A \otimes_R B_3$ and $1 \otimes 1 : A \otimes_R A \to A \otimes_R A$ is the identity homomorphism;

(iv) If $0 \to B_1 \to B_2 \to B_3 \to 0$ is an exact sequence of left R-modules then $A \otimes_R B_1 \to A \otimes_R B_2 \to A \otimes_R B_3 \to 0$ is an exact sequence of abelian groups and if $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of right R-module then $A_1 \otimes_R B \to A_2 \otimes_R B \to A_3 \otimes_R B \to 0$ is an exact sequence of abelian groups;

and (v) Torsion free abelian groups are flat, i.e., if $0 \to B_1 \to B_2 \to B_3 \to 0$ is an exact sequence of abelian groups and if A is a torsion free abelian group then $0 \to A \otimes_Z B_1 \to A \otimes_Z B_2 \to A \otimes_Z B_3 \to 0$ is exact, where Z is the ring of integers.

Let Q be the field of rational numbers and let A be an abelian group. Then $Q\otimes_Z A$ is a Q-vector space. If A is a torsion group then $Q\otimes_Z A = 0$.

If A is a torsion free group define <u>rank(A)</u> $= \dim_Q(Q\otimes_Z A)$. Note that rank(A) = cardinality of a maximal Z-independent subset of A since if $\{a_i\}$ is a maximal Z-independent subset of A then $F = \oplus Za_i$ is a subgroup of A with A/F torsion. Thus, $Q\otimes_Z F \simeq Q\otimes_Z A$ is a Q-vector space with dimension = cardinality of $\{a_i\}$. If rank(A) = n and F is a free subgroup of A of rank n then there is an exact sequence $0 \to F \to Q\otimes_Z A \to T \to 0$ where T is the direct sum of n copies of Q/Z since $0 \to Z \to Q \to Q/Z \to 0$ exact implies that $0 \to Z\otimes_Z F \to Q\otimes_Z F \to (Q/Z)\otimes_Z F \to 0$ is exact. If $0 \to B_1 \to B_2 \to B_3 \to 0$ is an exact sequence of finite rank torsion free abelian groups then $\text{rank}(B_1) + \text{rank}(B_3) = \text{rank}(B_2)$ as a consequence of (v).

An abelian group A is torsion free iff A is isomorphic to a subgroup of $Q\otimes_Z A$ via $a \to 1\otimes a$ since $Z \subseteq Q$ implies that $A \simeq Z\otimes_Z A$ is isomorphic to a subgroup of $Q\otimes_Z A$ whenever A is torsion free by (v). Thus, we may assume that $A \subseteq Q\otimes_Z A$, $(Q\otimes_Z A)/A$ is torsion, and every element of $Q\otimes_Z A$ is of the form qa for $q \in Z$, $a \in A$. Consequently, if B is a subgroup of A with A/B torsion then A/B is isomorphic to a subgroup of the direct sum of rank(A) copies of Q/Z.

If A and B are torsion free of finite rank then $A\otimes_Z B$ is torsion free with $\text{rank}(A\otimes_Z B) = \text{rank}(A)\,\text{rank}(B)$ since a monomorphism $A \to Q\otimes_Z A$ induces a monomorphism $A\otimes_Z B \to (Q\otimes_Z A)\otimes_Z B$ and $(Q\otimes_Z A)\otimes_Z B \simeq (Q\otimes_Z A)\otimes_Z (Q\otimes_Z B)$ is a vector space of dimension = $\dim(Q\otimes_Z A)\,\dim(Q\otimes_Z B)$.

If A and B are torsion free of finite rank and $f \in \text{Hom}_Z(A,B)$ then f extends uniquely to $\bar{f} \in \text{Hom}_Q(Q\otimes_Z A,\ Q\otimes_Z B)$. Thus $\text{Hom}_Z(A,B)$ is a finite rank torsion free group with rank $\leq \text{rank}(A)\,\text{rank}(B) = \dim_Q(\text{Hom}_Q(Q\otimes_Z A,\ Q\otimes_Z B))$.

A subgroup B of a torsion free group A is <u>pure in A</u> if $B \cap nA = nB$ for all $n \in Z$. Note that B is pure in A iff A/B is torsion free. If S is a subset of A let $\langle S \rangle$ be the subgroup of A generated by S and $\langle S \rangle_* = \{a \in A \mid na \in \langle S \rangle \text{ for some } 0 \neq n \in Z\}$, the <u>pure subgroup of A generated by S,</u> noting that, in fact, $\langle S \rangle_*$ is a pure subgroup of A and the smallest pure subgroup of A containing S. If $S = \{x\}$ then $\langle S \rangle$ is denoted by Zx and $\langle S \rangle_*$ by $\langle x \rangle_*$.

If B is a pure subgroup of the torsion free group A and if C is an abelian group then $0 \to C \otimes_Z B \to C \otimes_Z A$ is exact. Moreover, if B_i is a pure subgroup of the torsion free group A_i for $i = 1, 2$ then $B_1 \otimes_Z B_2$ is isomorphic to a pure subgroup of $A_1 \otimes_Z A_2$, the isomorphism being given by $b_1 \otimes b_2 \to b_1 \otimes b_2$.

If p is a prime of Z and A is an abelian group then $A/pA \simeq (Z/pZ) \otimes_Z A$. Define <u>p-rank(A)</u> $= \dim_{Z/pZ}(A/pA)$. If $0 \to A \to B \to C \to 0$ is an exact sequence of torsion free abelian groups, each having finite p-rank, then p-rank(A) + p-rank(C) = p-rank(B) since $0 \to A/pA \to B/pB \to C/pC \to 0$ is exact.

<u>Theorem 0.1</u>. If A is a torsion free abelian group of finite rank and if $0 \neq n \in Z$ then A/nA is finite. Moreover, p-rank(A) $\leq$ rank(A) for each prime p of Z.

<u>Proof</u>. If $n = p_1^{e_1} \cdots p_h^{e_h}$ is a product of powers of distinct primes of Z then $A/nA \simeq A/p_1^{e_1}A \oplus \cdots \oplus A/p_h^{e_h}A$ since $Z/nZ \simeq Z/p_1^{e_1}Z \oplus \cdots \oplus Z/p_h^{e_h}Z$ and $A/nA \simeq (Z/nZ) \otimes_Z A$. If A/p_iA is finite for each p_i then $A/p_i^{e_i}A$ is finite by induction on e_i.

It is now sufficient to assume that $n = p$ is a prime and to prove that p-rank(A) $= \dim_{Z/pZ}(A/pA) \leq$ rank(A). Let $a_1 + pA, \ldots, a_n + pA$ be independent in A/pA. Then $\{a_1, \ldots, a_n\}$ is a Z-independent subset of A for if $m_1a_1 + \cdots + m_na_n = 0$ with $m_i \in Z$ and g.c.d. $(m_1, \ldots, m_n) = 1$ then $m_ia_i \in pA$ for each i, whence $m_i/p \in Z$ for each i, a contradiction. ///

If T is a torsion abelian group and p is a prime then define $T[p] = \{x \in T \mid px = 0\}$, a Z/pZ-vector space.

<u>Theorem 0.2</u>. Suppose that $0 \to A \to B \to T \to 0$ is an exact sequence of abelian groups where A and B are finite rank torsion free and T is torsion. Then $\dim(T[p])$ + p-rank(B) = p-rank(A) + p-rank(T). In particular, if T is finite then p-rank(A) = p-rank(B).

<u>Proof</u>. There is an exact sequence $0 \to K \to A/pA \to B/pB \to T/pT \to 0$ where $K = (A \cap pB)/pA$. Now $T[p] \simeq C/A$, where $C = \{x \in B \mid px \in A\}$. Define

$\theta : C/A \to K$ by $\theta(c + A) = pc + pA$, a well-defined isomorphism. Thus $\dim(K) + \dim(B/pB) = \dim(A/pA) + \dim(T/pT)$ as desired. Finally, if T is finite then $\dim(T[p]) = \dim(T/pT)$ so that $p\text{-rank}(A) = p\text{-rank}(B)$. ///

A torsion free group A is divisible if A is a Q-vector space, i.e., $nA = A$ for all $0 \neq n \in Z$. If A is a torsion free group then there is a unique maximal divisible subgroup $d(A)$ of A with $d(A/d(A)) = 0$. Moreover, $A = d(A) \oplus B$ for some B and B is reduced (i.e., $d(B) = 0$). If A is reduced then the endomorphism ring of A is reduced as a group.

Let p be a prime and define $Z_p = \{m/n \in Q \mid \text{g.c.d.}(n,p) = 1\}$, the localization of Z at p. If A is a group then let $A_p = Z_p \otimes_Z A$, a Z_p-module. If A is finite rank torsion free then $p\text{-rank}(A_p) = p\text{-rank}(A)$ since $A/pA \simeq A_p/pA_p$. Moreover, $A \subseteq A_p \subseteq Q \otimes_Z A$ for each prime p and $A = \cap_p A_p$. If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups then $0 \to A_p \to B_p \to C_p \to 0$ is an exact sequence of Z_p-modules. Consequently, if B is torsion free and A is pure in B then B_p is torsion free and A_p is pure in B_p.

EXERCISES

0.1: Prove any statement in Section 0 that you have not previously proved. (Properties of tensor products are standard, e.g., Hungerford [1], and the remaining unproved statements may be found in Fuchs [7].)

§1. Types and rank - 1 groups

A height sequence, $\alpha = (\alpha_p)$, is a sequence of non-negative integers, together with ∞, indexed by the elements of Π, the set of primes of Z. Given a torsion free group A, an element a of A and a prime p of Z, define the p-height of a in A, $h_p^A(a)$, to be n if there is a non-negative integer n with $a \in p^n A \setminus p^{n+1} A$ and ∞ if no such n exists. The height sequence of a in A, $h^A(a) = (h_p^A(a))$, is a height sequence.

If $\alpha = (\alpha_p)$ is a height sequence and m is a positive integer define $m\alpha = (\beta_p)$, where $\beta_p = h_p^Z(m) + \alpha_p$ for each $p \in \Pi$ (agreeing that $\infty + k = \infty$ for each non-negative k). Two height sequences $\alpha = (\alpha_p)$ and $\beta = (\beta_p)$ are equivalent if there are positive integers m and n with $m\alpha = n\beta$, i.e., $\alpha_p = \beta_p$ for all but a finite number of p and $\alpha_p = \beta_p$ if either $\alpha_p = \infty$ or $\beta_p = \infty$. This relation is easily seen to be an equivalence relation. An equivalence class τ of height sequences is called a type, written $\tau = [\alpha]$ for some height sequence α.

Define the type of a in A, $\text{type}_A(a)$, to be $[h^A(a)]$. The group A is homogeneous if any two non-zero elements of A have the same type, the common value being denoted by type(A).

A rank-1 torsion free group A is homogeneous since if a and b are non-zero elements of A then $ma = nb$ for some non-zero integers m and n. Thus, $|m|h^A(a) = h^A(ma) = h^A(nb) = |n|h^A(b)$ so that $\text{type}_A(a) = [h^A(a)] = [h^A(b)] = \text{type}_A(b)$.

For example, $\text{type}(Z) = [(\alpha_p)]$, where $\alpha_p = 0$ for each p and $\text{type}(Q) = [(\beta_p)]$, where $\beta_p = \infty$ for each p. The set of types is a complete set of invariants for torsion free groups of rank 1:

Theorem 1.1.

(a) Suppose A and B are rank-1 torsion free groups. Then A and B are isomorphic iff type(A) = type(B).

(b) If τ is a type then there is a torsion free group A of rank 1 with $\text{type}(A) = \tau$.

<u>Proof.</u>

(a) $(\rightarrow)$ is a consequence of the observation that if $f : A \rightarrow B$ is an isomorphism then $h^A(a) = h^B(f(a))$ for each $a \in A$.

$(\leftarrow)$ If $0 \neq a \in A$ and $0 \neq b \in B$ then $[h^A(a)] = [h^B(b)]$. Choose positive integers m and n with $h^A(ma) = mh^A(a) = nh^B(b) = h^B(nb)$. The correspondence $ma \rightarrow nb$ lifts to an isomorphism $f : A \rightarrow B$ since for integers k and ℓ the equation $kx = \ell ma$ has a solution x in A iff $ky = \ell nb$ has a solution y in B. Moreover, the solution of either equation is unique since A and B are torsion free.

(b) Let $\tau = [(\alpha_p)]$ and define A to be the subgroup of Q generated by $\{1/p^i | p \in \Pi, 0 \leq i \leq \alpha_p\}$. Then $\text{type}(a) = \tau$, since $h^A(1) = (\alpha_p)$. ///

The set of height sequences has a partial ordering given by $\alpha = (\alpha_p) \leq \beta = (\beta_p)$ if $\alpha_p \leq \beta_p$ for each $p \in \pi$. The operations <u>$\sup\{\alpha,\beta\}$</u> $= (\max\{\alpha_p, \beta_p\})$ and <u>$\inf\{\alpha, \beta\}$</u> $= (\min\{\alpha_p, \beta_p\})$ induce a lattice structure on the set of height sequences. The set of height sequences is closed under the operation <u>$\alpha + \beta$</u> $= (\alpha_p + \beta_p)$.

If $a, b \in A$, a torsion free group, then $h^A(a+b) \geq \inf\{h^A(a), h^A(b)\}$. Furthermore, if $A = A_1 \oplus A_2$ and $a_i \in A_i$ then $h^A(a_1+a_2) = \inf\{h^A(a_1), h^A(a_2)\}$.

The partial order on the set of height sequences induces a partial order on the set of types where <u>$\tau \leq \sigma$</u> if there is $\alpha \in \tau$ and $\beta \in \sigma$ with $\alpha \leq \beta$. To show, for example, that this relation is anti-symmetric assume that $\tau \leq \sigma$ and $\sigma \leq \tau$. There are $\alpha, \alpha' \in \tau$ and $\beta, \beta' \in \sigma$ with $\alpha \leq \beta$ and $\beta' \leq \alpha'$. Choose positive integers m and n with $m\beta = n\beta'$. Then $m\alpha \leq m\beta = n\beta' \leq n\alpha'$ so that $km\beta = n\alpha'$ for some positive integer k. Thus $\sigma = [\beta] = [\alpha'] = \tau$. For types $\tau = [\alpha]$ and $\sigma = [\beta]$, define <u>$\tau + \sigma$</u> $= [\alpha+\beta]$, a well-defined operation.

The partial order on the set of types corresponds to the existence of non-zero homomorphisms between rank-1 groups:

Proposition 1.2. Let A and B be rank-1 torsion free groups. Then the following are equivalent:

(a) $\mathrm{Hom}(A,B) \neq 0$;

(b) There is a monomorphism $A \to B$;

(c) $\mathrm{type}(A) \leq \mathrm{type}(B)$.

Proof. (a) → (b) Since A and B are rank-1 torsion free every non-zero homomorphism from A to B is a monomorphism.

(b) → (c) If $f : A \to B$ is a monomorphism and $0 \neq a \in A$ then $0 \neq f(a) \in B$ with $h^A(a) \leq h^B(f(a))$.

(c) → (a) Choose $0 \neq a \in A$ and $0 \neq b \in B$ with $h^A(a) \leq h^B(b)$. Then $a \to b$ extends to a non-zero homomorphism $A \to B$. ///

Corollary 1.3. Let A and B be torsion free groups of rank 1. Then the following are equivalent:

(a) A and B are isomorphic;

(b) $\mathrm{Hom}(A,B) \neq 0$ and $\mathrm{Hom}(B,A) \neq 0$;

(c) There is a monomorphism $f : A \to B$ such that $B/f(A)$ is finite.

Proof. (a) → (c) is clear.

(c) → (b) Suppose that $nB \leq f(A) \leq B$ for some $0 \neq n \in Z$. Then $0 \neq f^{-1}n : B \to A$ is a well defined homomorphism.

(b) → (a) As a consequence of Corollary 1.2, $\mathrm{type}(A) = \mathrm{type}(B)$ so that Theorem 1.1 applies. ///

The following theorem gives a description of quotients of rank-1 torsion free groups, as well as an alternative definition for the height of an element. For a prime p and a non-negative integer i, let $Z(p^i)$ denote the cyclic group of order p^i. Define $Z(p^\infty)$ to be the p-torsion subgroup of Q/Z. If C is a subgroup of $Z(p^\infty)$ then $C \simeq Z(p^i)$ for some $0 \leq i < \infty$ and $Z(p^\infty)/C \simeq Z(p^\infty)$, or $C = Z(p^\infty)$.

Theorem 1.4: Assume that $0 \neq b \in B$, a subgroup of a rank-1 torsion free group A. Then $A/B \simeq \oplus_p Z(p^{i_p})$ where $h^A(b) = (\ell_p)$, $h^B(b) = (k_p)$, and $i_p = \ell_p - k_p$ (agreeing that $\infty - \infty = 0$ and $\infty - k = \infty$ if $k < \infty$).

Proof: In view of the exact sequence $0 \to B/Zb \to A/Zb \to A/B \to 0$ it is sufficient to prove that $A/Zb \simeq \oplus_p Z(p^{\ell_p})$. As noted in Section 0, A/Zb is isomorphic to a subgroup of Q/Z so that $A/Zb \simeq \oplus_p C_p$ with $C_p \subseteq Z(p^\infty)$. But $C_p \simeq Z(p^{\ell_p})$ since $p^i x = nb$ with g.c.d. $(p,n) = 1$ has a solution $x \in A$ iff $i \leq h_p^A(b) = h_p^A(nb) = \ell_p$. ///

The types of $\mathrm{Hom}(A,B)$ and $A \otimes_Z B$, where A and B are rank-1 groups, may be computed from the types of A and B. A type τ is non-nil if there is $\alpha = (\alpha_p) \in \tau$ with $\alpha_p = 0$ or ∞ for each p.

Theorem 1.5: Suppose that A and B are torsion free groups of rank-1.

(a) If $\mathrm{type}(A) \leq \mathrm{type}(B)$ then $\mathrm{Hom}(A,B)$ is a rank-1 torsion free group with type $= [(m_p)]$, where $0 \neq a \in A$, $0 \neq b \in B$, $h^A(a) = (k_p) \leq h^B(b) = (\ell_p)$, $m_p = \infty$ if $\ell_p = \infty$, and $m_p = \ell_p - k_p$ if $\ell_p < \infty$.

(b) If $\mathrm{type}(A) = [(k_p)]$ then $\mathrm{type}(\mathrm{Hom}(A,A)) = [(m_p)]$ is non-nil, where $m_p = \infty$ if $k_p = \infty$ and $m_p = 0$ if $k_p < \infty$.

(c) $A \otimes_Z B$ is a torsion free group of rank 1 with $\mathrm{type}(A \otimes_Z B) = \mathrm{type}(A) + \mathrm{type}(B)$.

Proof:

(a) As noted in Section 0, the group $\mathrm{Hom}(A,B)$ is torsion free of rank 1. Define $\phi : \mathrm{Hom}(A,B) \to B$ by $\phi(f) = f(a)$. Then Image $(\phi) \subseteq G = \{x \in B \mid h^B(x) \geq h^A(a)\}$ since $h^B(f(a)) \geq h^A(a)$ for each $f \in \mathrm{Hom}(A,B)$. On the other hand, $G \subseteq \mathrm{Image}(\phi)$ for if $x \in B$ and $h^B(x) \geq h^A(a)$ then there is $f : A \to B$ with $f(a) = x$. Hence $\mathrm{type}(\mathrm{Hom}(A,B)) = \mathrm{type}(G) = [(m_p)]$ as desired.

(b) is a consequence of (a).

(c) There is an embedding $A\otimes_Z B \to Q\otimes_Z Q \simeq Q$ so that $A\otimes_Z B$ is torsion free of rank 1. If $0 \neq a \in A$ and $0 \neq b \in B$ then $h^{A\otimes B}(a\otimes b) \geq h^A(a) + h^B(b)$ since $a = ma'$ and $b = nb'$ imply that $a\otimes b = mn(a'\otimes b')$. Therefore, $\text{type}(A\otimes_Z B) \geq \text{type}(A) + \text{type}(B)$.

To show that $\text{type}(A\otimes_Z B) \leq \text{type}(A) + \text{type}(B)$, it suffices to prove that if $p^i x = a\otimes b$, with $h_p^A(a) = h_p^B(b) = 0$, has a solution x in $A\otimes_Z B$ then $i = 0$. By Exercise 1.4, $x = a'\otimes b'$ for some $a' \in A$, $b' \in B$. Thus $p^i(a'\otimes b') = a\otimes b$. Choose non-zero integers k and ℓ with $ka' = \ell a$ and p prime to k. Then $k(a\otimes b) = (p^i ka')\otimes b' = (p^i \ell a)\otimes b'$ so that $a\otimes(kb) = a\otimes(p^i \ell b')$. Thus $kb = p^i \ell b'$, hence $i = 0$, since g.c.d. $(p,k) = 1$ and $h_p^B(b) = 0$. ///

Corollary 1.6. Assume that A is a torsion free group of rank-1. The following are equivalent:

(a) A has non-nil type;

(b) $\text{type}(A) + \text{type}(A) = \text{type}(A)$;

(c) $A \simeq \text{Hom}(A,A)$;

(d) A is isomorphic to the additive group of a subring of Q;

(e) $A\otimes_Z A \simeq A$;

(f) If $0 \neq a \in A$ then $A/Za = T \oplus D$, where T is a finite torsion group and $D \simeq \oplus_{p\in S} Z(p^\infty)$ for some subset S of Π.

Proof: Exercise 1.9. ///

The preceding results, due essentially to Baer [2], demonstrate that the set of types is a useful set of invariants for torsion-free groups of rank 1. In general, however, the type of a group provides little information about the structure of the group.

The remainder of this section is devoted to several generalizations of the notion of type, due to Warfield [1] and Richman [1].