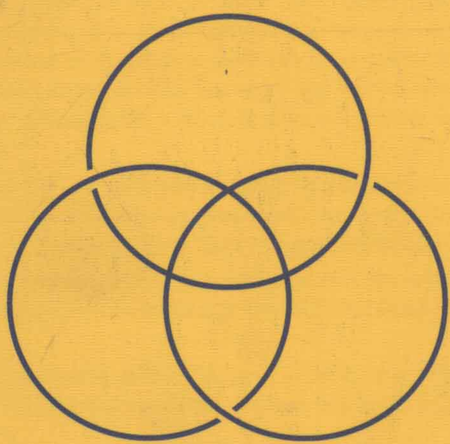
Lecture Notes in Mathematics

1854

Osamu Saeki

Topology of Singular Fibers of Differentiable Maps



Springer

Osamu Saeki

Topology of Singular Fibers of Differentiable Maps

Springer

Author

Osamu Saeki

Faculty of Mathematics
Kyushu University
Hakozaki, Fukuoka 812-8581, Japan
e-mail: saeki@math.kyushu-u.ac.jp

Library of Congress Control Number: 2004110903

Mathematics Subject Classification (2000): 57R45, 57N13

ISSN 0075-8434
ISBN 3-540-23021-1 Springer Berlin Heidelberg New York
DOI 10.1007/b100393

Springer is part of Springer Science+Business Media
springeronline.com

Typesetting: Camera-ready TEX output by the author

41/3142-543210 - Printed on acid-free paper

To Célia

Preface

In 1999, a friend of mine, Kazuhiro Sakuma, kindly asked me to give a series of lectures in the Kwansai Seminar on Differential Analysis, held at the Kinki University, Japan. At that time, I was studying the global topology of differentiable maps of 4-dimensional manifolds into lower dimensional manifolds. Sakuma and I had obtained a lot of interesting results concerning the relationship between the singularities of such maps and the differentiable structures of 4-dimensional manifolds; however, our results were not based on a systematic theory and were not satisfactory in a certain sense. So I was trying to construct such a systematic theory when I was asked to give lectures.

I wondered what kind of objects can reflect the *global* properties of manifolds. "Singularity" of a differentiable map can be such an object, but it is *local* in nature. I already knew that the notion of the Stein factorization played an important role in the global study of such maps; for example, refer to the works of Burlet–de Rham [7] or Kushner–Levine–Porto [28, 30]. Stein factorization is constructed by considering the connected components of the fibers of a given map.

This inspired me to consider singular fibers of differentiable maps. I promptly started the classification of singular fibers of stable maps of orientable 4-manifolds into 3-manifolds. It was not a difficult task, though quite tedious. Then I obtained the modulo two Euler characteristic formula in terms of the number of a certain singular fiber, by using Szűcs's formula [55], which Nuño Ballesteros and I had also obtained independently [36, 37]. The formula on the number of singular fibers was so beautiful that I was very happy to be able to present such a result in the Kwansai Seminar, in November 1999.

After attending my lectures, one of the participants, Toru Ohmoto, gave me a very important remark. He said "Your argument is closely related to Vassiliev's universal complex of multi-singularities. You just increased the number of generators for each cochain complex using the topology of singular fibers".

So I began to study Vassiliev's work and at the same time began to elaborate my results. It took a long time to write down all the details. A preprint version of the whole work was finished only in the middle of 2003, when I was staying in Strasbourg, France.

Now the acknowledgment follows. First of all, I would like to thank Kazuhiro Sakuma and the co-organizer Shuzo Izumi for kindly asking me to give a series of lectures at the Kwansai Seminar. I would like to thank Toru Ohmoto for his important remark at the seminar. Without these people, this work would have never appeared.

I would like to thank Minoru Yamamoto and Takahiro Yamamoto for carefully reading several earlier versions of the manuscript and for pointing out some important errors. I would also like to thank Goo Ishikawa, who gave me invaluable comments through his student Takahiro Yamamoto. I would also like to thank Jorge T. Hiratuka for stimulating discussions concerning Stein factorizations of stable maps of 4-dimensional manifolds.

I would like to express my sincere gratitude to András Szűcs for thoroughly reading an earlier version of the manuscript and for giving me many invaluable comments, which improved the manuscript considerably.

In January 2004, the results in this book were presented in a mini-course given at the University of Tokyo, Japan. I would like to thank all the participants at the mini-course, who attended it with enthusiasm and posed a lot of questions. In particular, I would like to thank Mikio Furuta for his excellent questions with fantastic ideas: in fact, I included some of the results based on his ideas in this book. I would also like to thank Masamichi Takase and Keiichi Suzuoka for their invaluable comments on my mini-course. I would like to thank Yukio Matsumoto, my ex-supervisor, for inviting me to give such a mini-course.

I would like to thank Vincent Blanlœil for inviting me to Strasbourg in 2003, where I could finish the first draft of this work. I would also like to express my thanks to Rustam Sadykov for posing many interesting questions concerning the book.

Finally, I would like to thank all the members of my family, especially to Célia, for their patience and support during the preparation of the book.

The author has been supported in part by Grant-in-Aid for Scientific Research (No. 16340018), Japan Society for the Promotion of Science.

Fukuoka,
July, 2004 *Osamu Saeki*

Contents

Introduction

Let $f : M \to N$ be a proper differentiable[1] map of an n-dimensional manifold M into a p-dimensional manifold N. When the codimension $p - n$ is nonnegative, for any point y in the target N, the inverse image $f^{-1}(y)$ consists of a finite number of points, provided that f is generic enough. Hence, in order to study the semi-local behavior of a generic map f around (the inverse image of) a point $y \in N$, we have only to consider the multi-germ $f : (M, f^{-1}(y)) \to (N, y)$. Therefore, we can use the well-developed theory of multi-jet spaces and their sections in order to study such semi-local behaviors of generic maps.

However, if the codimension $p - n$ is strictly negative, then the inverse image $f^{-1}(y)$ is no longer a discrete set. In general, $f^{-1}(y)$ forms a complex of positive dimension $n - p$. Hence, we have to study the map germ $f : (M, f^{-1}(y)) \to (N, y)$ along a set $f^{-1}(y)$ of positive dimension and the theory of multi-jet spaces is not sufficient any more. Surprisingly enough, there has been no systematic study of such map germs in the literature, as long as the author knows, although we can find some studies of the multi-germ of f at the singular points of f contained in $f^{-1}(y)$.

In this book, we consider the codimension -1 case, i.e. the case with $n - p = 1$, and classify the right-left equivalence classes of generic map germs $f : (M, f^{-1}(y)) \to (N, y)$ for $n = 2, 3, 4$. For the case $n = 3$, Kushner, Levine and Porto [28, 30] classified the singular fibers of C^∞ stable maps of 3-manifolds into surfaces up to *diffeomorphism*; however, they did not mention a classification up to *right-left equivalence* (for details, see Definition 1.1 (2) in Chap. 1). In this book, we clarify the difference between the classification up to diffeomorphism and that up to right-left equivalence by completely classifying the singular fibers up to these two equivalences.

Given a generic map $f : M \to N$ of negative codimension, the target manifold N is naturally stratified according to the right-left equivalence classes

[1]In this book, "differentiable" means "differentiable of class C^∞". We also use the term "smooth".

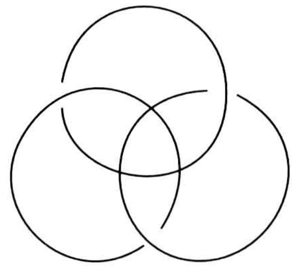


Fig. 0.1. The singular fiber whose number has the same parity as the Euler characteristic of the source 4-manifold M

of f-fibers. By carefully investigating how the strata are incident to each other, we get some information on the homology class represented by a set of the points in the target whose associated fibers are of certain types. This leads to some limitations on the co-existence of singular fibers. For example, we show that for a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold, the number of singular fibers containing both a cusp point and a fold point is always even.

As an interesting and very important consequence of such co-existence results, we show that for a C^∞ stable map $f : M \to N$ of a closed orientable 4-manifold M into a 3-manifold N, the Euler characteristic of the source manifold M has the same parity as the number of singular fibers as depicted in Fig. 0.1 (Theorem 5.1). Note that this type of result would be impossible if we used the multi-germs of a given map at the singular points contained in a fiber instead of considering the topology of the fibers. In other words, our idea of essentially using the topology of singular fibers leads to new information on the global structure of generic maps.

Furthermore, the natural stratification of the target manifold according to the fibers enables us to generalize Vassiliev's universal complex of multi-singularities [58] to our case. In this book, we define such universal complexes of singular fibers and compute the corresponding cohomology groups in certain cases. It turns out that cohomology classes of such complexes give rise to cobordism invariants for maps with a given set of singularities in the sense of Rimányi and Szűcs [40].

The book is organized as follows.

In Part I, we define and study equivalence relations for singular fibers of generic differentiable maps and carry out the classification of singular fibers for some specific classes of maps. We use these classifications to obtain some results on the co-existence of singular fibers, and on the relationship between the numbers of certain singular fibers of stable maps and the topology of the source manifolds. We also give explicit concrete examples of such stable maps exhibiting typical singular fibers.

In Part II, we formalize the idea used to obtain the co-existence results of singular fibers in Part I in a more general setting. This leads to the notion of the universal complex of singular fibers, which is a refinement of Vassiliev's universal complex of multi-singularities. We develop a rather detailed theory of universal complex of singular fibers, and at the same time we give explicit calculations based on Part I. We will see that the cohomology classes of the universal complex of singular fibers give rise to invariants of cobordisms of singular maps in the sense of Rimányi and Szűcs [40] in the negative codimension case.

In Part III, we give some applications of our theory to the global topology of differentiable maps and present some further developments of the theory given in this book.

Part I consists of six chapters, which are organized as follows.

In Chap. 1, we give precise definitions of certain equivalence relations among the fibers of proper smooth maps, which will play essential roles in this book.

In Chap. 2, in order to clarify our idea, we classify the fibers of proper Morse functions on surfaces. The result itself should be folklore; however, we give a rather detailed argument, since similar arguments will be used in subsequent chapters.

In Chap. 3, we classify the fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds up to right-left equivalence. Our strategy is to use a combinatorial argument, for obtaining all possible 1-dimensional complexes, together with a classification up to right equivalence of certain multi-germs due to [11, 61]. After the classification, we will see that the equivalence up to diffeomorphism and that up to right-left equivalence are almost equivalent to each other in our case. Furthermore, as another consequence of the classification, we will see that two fibers of such stable maps are C^0 right-left equivalent if and only of they are C^∞ right-left equivalent. This is an analogy of Damon's result [10] for C^∞ stable map germs in nice dimensions. Furthermore, we give similar results for proper C^∞ stable maps of (not necessarily orientable) 3-manifolds into surfaces and for proper C^∞ stable Morse functions on surfaces. For Morse functions on surfaces, we prove the following very important result: for two proper C^∞ stable Morse functions on surfaces, they are C^0 equivalent if and only if they are C^∞ equivalent.

In Chap. 4, we investigate the stratification of the target 3-manifold of a C^∞ stable map of a closed orientable 4-manifold as mentioned above and obtain certain relations among the numbers (modulo two) of certain singular fibers.

In Chap. 5, we combine the results obtained in Chap. 4 with the following two results. One is a result of Fukuda [14] and the author [45] about the Euler characteristics of the source manifold and the singular point set, and the other is Szűcs' formula [55] on the number of triple points of a generic surface in 3-space (see also [36, 37]). As a result, we obtain a congruence modulo two between the Euler characteristic of the source 4-manifold and the number of singular fibers as depicted in Fig. 0.1.

In Chap. 6, we construct explicit examples of C^∞ stable maps of closed orientable 4-manifolds into $\mathbf{R}^3$. Since $(4, 3)$ is a nice dimension pair in the sense of Mather [32], given a 4-manifold M and a 3-manifold N, we have a plenty of C^∞ stable maps of M into N. However, it is surprisingly difficult to give an *explicit* example and to give a detailed description of the structure of the fibers. Here, we carry this out, and at the same time we explicitly construct infinitely many closed orientable 4-manifolds with odd Euler characteristics which admit smooth maps into $\mathbf{R}^3$ with only fold singularities. In the subsequent chapters, we will see that such explicit examples are essential and very important in the study of singular fibers of generic maps.

Part II consists of eight chapters as follows.

In Chap. 7, we generalize the idea given in Chaps. 4 and 5 in a more general setting to obtain certain results on the co-existence of singular fibers.

In Chap. 8, we define the universal complexes of singular fibers for proper Thom maps with coefficients in $\mathbf{Z}_2$, using an idea similar to Vassiliev's [58] (see also [23, 38]). Our universal complexes of singular fibers are very similar to Vassiliev's universal complexes of multi-singularities. In fact, we construct the complexes using the right-left equivalence classes of fibers instead of multi-singularities, and this corresponds to increasing the generators of each cochain group according to the topological structures of fibers. In order to use such universal complexes in several situations, we will develop a rather detailed theory of universal complexes of singular fibers. Here, given a set of generic maps and a certain equivalence relation among their fibers, we will define the corresponding universal complex of singular fibers.

In Chap. 9, we apply the general construction introduced in Chap. 8 to a more specific situation, namely in the case of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds. For such maps, we determine the structure of the universal complex of singular fibers with respect to a certain equivalence relation among the fibers and compute its cohomology groups explicitly.

In Chap. 10, we consider co-orientable fibers and construct the corresponding universal complex of co-orientable singular fibers with integer coefficients. We also give some important problems related to the theory of universal complexes of singular fibers.

In Chap. 11, we define a homomorphism induced by a generic map of the cohomology group of the universal complex of singular fibers to that of the target manifold of the map. This corresponds to associating to a cohomology class α of the universal complex the Poincaré dual to the homology class represented by the set of those points over which lies a fiber appearing in a cocycle representing α. We will see that the homomorphisms induced by explicit generic maps will be very useful in the study of the cohomology groups of the universal complexes. This justifies the study developed in Chap. 6.

In Chap. 12, we define a cobordism of smooth maps with a given set of singular fibers. We will see that the homomorphism defined in Chap. 11 restricted to a certain subgroup is an invariant of such a cobordism. Furthermore, we will give a criterion for a certain cochain of the universal complex of singular fibers to be a cocycle in terms of the theory of such cobordisms, and apply it to finding a certain nontrivial cohomology class of a universal complex associated to stable maps of 5-dimensional manifolds into 4-dimensional manifolds.

In Chap. 13, we consider cobordisms of smooth maps with a given set of local singularities in the sense of [40]. We explain how a cohomology class of a universal complex of singular fibers gives rise to a cobordism invariant for such maps. Note that such cobordism relations have been thoroughly studied in [40] in the nonnegative codimension case. Our idea provides a systematic and new method to construct cobordism invariants for negative codimension cases.

In Chap. 14, we give explicit examples of cobordism invariants constructed by using the method introduced in the previous chapters. In particular, we show that this method provides a complete invariant of fold cobordisms of Morse functions on closed oriented surfaces.

Part III consists of two chapters as follows.

In Chap. 15, we give explicit applications of the general idea given in Chap. 7 to the topology of certain generic differentiable maps. For example, we study the homology classes represented by some multiple point sets of certain generic maps. As a corollary, we show the vanishing of the Gysin image of a Stiefel-Whitney class for smooth maps under certain dimensional assumptions.

Finally in Chap. 16, we present some further results (without any details) concerning the topology of singular fibers of generic maps obtained after the first version of this book was written as a preprint.

Throughout this book, all manifolds and maps are differentiable of class C^∞. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects. For a space X, the symbol "id_X" denotes the identity map of X. For other symbols used in this book, refer to the list starting at p. 135.

Part I

Classification of Singular Fibers

1

Preliminaries

In this chapter, we give some fundamental definitions, which will be essential for the classification of singular fibers of generic maps of negative codimensions.

Definition 1.1. (1) Let M_i be smooth manifolds and $A_i \subset M_i$ be subsets, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\widetilde{g} : V \to M_1$ defined on a neighborhood V of q in M_0 such that $\widetilde{g}|_{V \cap A_0} = g|_{V \cap A_0}$. Furthermore, a smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth. When there exists a diffeomorphism between A_0 and A_1, we say that they are *diffeomorphic*.[1]

(2) Let $f_i : M_i \to N_i$ be smooth maps, $i = 0, 1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are *diffeomorphic* (or *homeomorphic*) if $(f_0)^{-1}(y_0) \subset M_0$ and $(f_1)^{-1}(y_1) \subset M_1$ are diffeomorphic in the above sense (resp. homeomorphic in the usual sense). Furthermore, we say that the fibers over y_0 and y_1 are C^∞ *equivalent* (or C^0 *equivalent*), if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array} \tag{1.1}$$

When the fibers over y_0 and y_1 are C^∞ (or C^0) equivalent, we also say that the map germs $f_0 : (M_0, (f_0)^{-1}(y_0)) \to (N_0, y_0)$ and $f_1 : (M_1, (f_1)^{-1}(y_1)) \to (N_1, y_1)$ are smoothly (or topologically) *right-left equivalent*. Note that then

[1]Note that even if A_0 and A_1 are diffeomorphic to each other, the dimensions of the ambient manifolds M_0 and M_1 may be different.