Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Diophantine Approximations and Value Distribution Theory

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Introduction

Finding all solutions of a given system of diophantine equations has been shown to be an unsolvable problem, in general. More tractable, although still difficult, is the problem of determining whether the system has a finite number of solutions over every ring of integers of every number number field, possibly localized at a finite number of places. Or, one might ask the same question about k-rational solutions of the system.

The answer to both questions is known if the system of equations defines a curve, in the sense of algebraic geometry. Indeed, if the genus of the projective closure of the curve is zero, then there are always an infinite number of k-rational points for sufficiently large k, and similarly for integral points if there are at most two points at infinity. If there are three or more points at infinity, however, then finiteness always holds. If the genus is equal to one, then over a sufficiently large number field the curve is an elliptic curve with an infinite number of rational points, but finiteness always holds for integral points. Finally, if the genus is greater than one, then finiteness holds for rational as well as integral points.

In each case the answer depended only on algebraic-geometric invariants of the curve, and in fact, the "function field case" of the above questions has provided much insight.

More classically, though, the above invariants are also invariants of the associated Riemann surfaces, and the above answers have close parallels in the theory of holomorphic maps to Riemann surfaces. Indeed, there exists a nonconstant holomorphic map from $\mathbf{C}$ to a given compact Riemann surface of genus g if and only if $g \leq 1$ (Picard's theorem), and in the non-compact case, a Riemann surface of genus g missing $s > 0$ points admits such a map if and only if $g = 0$ and $s < 3$. Thus a curve has an infinite set of rational (resp. integral) points if and only if the associated compact (resp. non-compact) Riemann surface is the image of at least <u>one</u> non-constant holomorphic map.

These statements on holomorphic mappings can be proved using hyperbolic geometry; in fact Lang ([**L 5**], [**L 8**], see also §4.3) has posed a number of conjectures as higher dimensional equivalents of the above. For example:

CONJECTURE. *An algebraic variety V has the property that $V(k)$ is finite for all number fields k if, and only if, the associated complex space is Kobayashi hyperbolic.*

In this work we present a quantitative version of the above conjecture (the "Main Conjecture"). Of the above finiteness statements, those dealing with integral points can be proved using Roth's theorem, a statement in diophantine approximations. Moreover, the proofs give estimates on the minimum growth of the denominators of rational solutions. On the analytic side, the theorems dealing with the nonexistence of holomorphic functions all can be proved using a theory known as Nevanlinna theory; in the case of noncompact Riemann surfaces V, it provides asymptotic estimates for how many times a holomorphic map to $\overline{V}$ must meet $\overline{V} \setminus V$. Up to a point, the proofs of either set of theorems and the resulting estimates are very similar. This leads us to translate versions of Nevanlinna theory for higher dimensional varieties into quantitative versions of the above-mentioned conjectures of Lang. It should be noted that C. Osgood [**Os 1**] has also, previously, noted a Nevanlinna-Roth connection.

We also obtain a conjecture on all algebraic points of a variety (the "General Conjecture"); it is similar to the Main Conjecture, with an added term for the discriminant of the number field over which the point is defined. For example, Roth's theorem extends to a conjecture that, given $\epsilon > 0$ and points $\alpha_v \in k$ for finitely many places $v \in S$ of a number field k, then there exists a constant $c = c(k, S, \epsilon, \alpha_v)$ such that,

$$\prod_{v \in S} \prod_{\substack{w \in M_{k(x)} \\ w | v}} \min(1, \|x - \alpha_v\|_w) > \frac{c^{[k(x):k]}}{H(x)^{2+\epsilon} D_{k(x)/\mathbf{Q}}}$$

holds for almost all $x \in \mathbf{Q}$. Or, if C is a curve of genus ≥ 2 with canonical divisor K, then we conjecture that,

$$h_K(P) < \frac{1+\epsilon}{[k(P) : \mathbf{Q}]} \log D_{k(P)/\mathbf{Q}} + O(1).$$

This special case of the General conjecture already implies a number of other conjectures–see Section 5.5.

Finally, in the last chapter we compare the proofs of theorems of Ahlfors and Schmidt, which give bounds on approximation to hyperplanes in projective space. The proofs are very similar; in particular, the use of differentiation in Ahlfors' proof is similar to the use of successive minima in Schmidt's.

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Notations

$\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$	The fields of complex, real, and rational numbers, respectively.
$\mathbf{Z}$	The ring of rational integers.
k	Usually a number field. When discussing the function field case, this will also denote the field of constants.
M_k	The set of all (inequivalent) absolute values of the global field k.
S_∞	The set of archimedean absolute values of k.
S	A finite set of absolute values containing S_∞. Except for in Section 5.7, this does not necessarily contain the places of bad reduction.
v	An element of M_k; if it is non-archimedean, the associated prime ideal is denoted $\wp$.
e	The index of ramification of the non-archimedean place v.
f	The degree of the residue class field extension of the local field k_v over $\mathbf{Q}_p$.
$\Vert\cdot\Vert_v$	$\vert\cdot\vert$, if $k_v = \mathbf{R}$; $\vert\cdot\vert^2$, if $k_v = \mathbf{C}$, or $p^{-f \operatorname{ord}_\wp(\cdot)}$, if v is non-archimedean.
$\mathcal{O}_k$	The ring of integers of the number field k.
$\mathcal{O}_{k,S}$	The ring O_k with primes in S inverted.
λ_v	Weil function, normalized as in §1.3.
$\vert D\vert$	The support of a divisor D; i. e. the divisor obtained by changing all nonzero multiplicities to one.
$\vert S\vert$	The number of elements of the set S.
$D \ge E$	When D and E are divisors, this means that $D-E$ is effective.

Also, the notations of algebraic geometry are those of [**H**], except that varieties are often taken as defined over number fields.

Other notations appear in the index.

Contents

Chapter 1
Heights and Integral Points

§1. Absolute Values and the Product Formula

Let k be a number field. An absolute value on k is a real-valued function $|\cdot|$ on k satisfying the following properties:

(1). $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$;
(2). $|xy| = |x||y|$;
(3). $|x+y| \leq 2\max(|x|, |y|)$.

Absolute values are also sometimes called places or primes. An embedding $\phi\colon k \to \mathbf{C}$ causes the absolute value on $\mathbf{C}$ to induce an absolute value on k; such absolute values are called infinite, or archimedean, primes.

Finite, or non-archimedean, places come from a prime ideal $\wp$ of $\mathcal{O}_k$, the ring of integers of k. Such places are defined by,

$$|x|_\wp = c^{-\operatorname{ord}_\wp(x)}$$

where $c > 1$ is a real constant and $\operatorname{ord}_\wp(x)$ is the power of $\wp$ appearing in the prime factorization of the principal ideal (x). For the field $\mathbf{Q}$, it is customary to take $c = p$; for finite extensions of $\mathbf{Q}$, two options are possible, denoted $|\cdot|$ and $\|\cdot\|$. They are defined such that, if p is the rational prime associated to $\wp$,

$$(1.1.1) \qquad |p| = p^{-1} \quad \text{and} \quad \|p\| = p^{-[k_\wp : \mathbf{Q}_p]}.$$

Instead of condition (3) above, non-archimedean places are characterized by the fact that they satisfy the stronger inequality,

(3′). $|x+y| \leq \max(|x|, |y|)$.

If v is an absolute value, it induces on k the structure of a metric space, and thus a topology. Two absolute values are said to be <u>equivalent</u> if they induce the same topology; in that case one is a power of the other. The completion of k relative to the topology induced by v is a field which is denoted k_v. If v is archimedean, then $k_v = \mathbf{R}$ or $\mathbf{C}$; if it is non-archimedean, then k_v is a finite extension of $\mathbf{Q}_p$, which explains the notation $[k_\wp : \mathbf{Q}_p]$ in (1.1.1).

For all absolute values v, we define the normed absolute value $\|x\|_v$ for $x \in k$ by,

$$\|x\|_v = \begin{cases} |x|, & \text{if } k_v = \mathbf{R}; \\ |x|^2, & \text{if } k_v = \mathbf{C}; \\ \text{See (1.1.1)}, & \text{if } v \text{ is non-archimedean.} \end{cases} \tag{1.1.2}$$

Under these conventions there exists a set M_k of inequivalent primes of k satisfying a product formula,

$$\prod_{v \in M_k} \|x\|_v = 1 \qquad \text{for all } x \in k^* . \tag{1.1.3}$$

This set M_k consists of one v for each prime ideal of $\mathcal{O}_k$ and one prime for each real or conjugate pair of complex embeddings. This product formula is often stated with multiplicities, but by the choices (1.1.2), these multiplicities are unnecessary.

The subset of archimedean places $v \in M_k$ is denoted S_∞. If F is a finite extension of k, then places $w \in M_F$ restrict to places $v \in M_k$ (up to a fixed power). Denote this by writing $w \mid v$. The product formulas of F and k correspond term by term in the sense that,

$$\prod_{\substack{w \mid v \\ w \in M_F}} \|x\|_v = \|x\|_v^{[F:k]}. \tag{1.1.4}$$

The product formula (1.1.3) can be used to axiomatically describe a global field; i. e. a number field or the function field of a variety. In the sequel, however, function fields will play only a minor role and therefore we consider primarily number fields.

§2. Height Functions

The so-called naïve height, or the Weil height, is defined on $\mathbf{P}^n(k)$ by choosing homogeneous coordinates $(x_0, \ldots, x_n)$ for P and letting

$$H(P) = \prod_{v \in M_k} \max(\|x_0\|_v, \ldots, \|x_n\|_v). \tag{1.2.1}$$

By the product formula, $H(P)$ is independent of the choice of homogeneous coordinates for P.

Note, however, that $H(P)$ does depend on the number field in question; in fact, if F is a finite extension of k, then by (1.1.4),

$$H_F(P) = H_k(P)^{[F:k]}.$$

Thus, it is called the relative height. It is possible to make a definition of height independent of the number field; it is convenient to also take the logarithm, defining,

$$h(P) = \frac{1}{[k:\mathbf{Q}]} \log H(P),$$

where the height on the right is defined over k. This is called the (absolute) logarithmic height. By contrast, the height $H(P)$ is called the (relative) multiplicative height.

DEFINITION 1.2.2. *Let f and g be two non-negative functions. If $f < cg$ for some positive constant c, then we write $f \ll g$ or $g \gg f$. The notation $f \gg\ll g$ means that both $f \ll g$ and $f \gg g$ hold.*

DEFINITION 1.2.3. *Two heights H_1 and H_2 (resp. logarithmic heights h_1 and h_2) are called equivalent if $H_1 \gg\ll H_2$ (resp. $h_1 = h_2 + O(1)$).*

The remainder of this section will use only the logarithmic height; similar comments obviously hold for the multiplicative height.

We now wish to define a notion of height on a projective variety V defined over k. (For the remainder of this section, assume that all varieties, divisors, functions, etc. on V are defined over k.) To define the height, we use a projective embedding of V, together with (1.2.1). Therefore the definition of the height depends on choices of a very ample divisor D and functions in the linear system $\mathcal{L}(D)$. Up to equivalence, however, the height depends only on D and furthermore,

LEMMA 1.2.4. *If D and D' are two very ample divisors on V, then*

$$h_{D+D'} = h_D + h_{D'} + O(1).$$

PROOF: See [**L 7**]. □

Now, given any divisor D, we can write $D = E - E'$ where E and E' are very ample, and define

$$h_D = h_E - h_{E'}.$$

This definition depends on the choices of E and E', but by Lemma 1.2.4 h_D is well defined up to equivalence. Furthermore, it then follows that (1.2.4) holds for arbitrary divisors D and D'.

In order to summarize the properties of heights it will be necessary to define some terminology, starting with a definition due to Iitaka [**Ii**, Ch. 10, Theorem 10.2]:

DEFINITION 1.2.5.

(a). *Let D be a divisor on a nonsingular variety V. The dimension of D is the integer $d = \dim D$ such that*

$$\ell(nD) \gg\ll n^d.$$

for n sufficiently divisible. If $\mathcal{L}(nD)$ is always empty, then let $d = -\infty$.

(b). *D is almost ample if $\dim D = \dim V$.*

Thus, for example, a variety of general type is one for which the canonical divisor is almost ample.

LEMMA 1.2.6. *The dimension has the following properties:*

(a). *If D, E are divisors and if D is effective, then $\dim(D+E) \geq \dim E$; in particular, $\dim D \geq 0$.*
(b). $\dim D \leq \dim V$.
(c). *If D is ample then $\dim D = \dim V$.*
(d). *The dimension depends only on the linear equivalence class of D; in particular it can be defined for a line bundle.*
(e). *If $f\colon V \to W$ is a proper surjective morphism of nonsingular complete varieties and $\mathcal{L}$ is a line bundle on W, then $\dim f^*\mathcal{L} = \dim \mathcal{L}$.*

PROOF: (a)–(d) are trivial; for (e) see [**Ii**, Theorem 10.5.] □

EXAMPLE. Let V be a surface and let E be a curve on V with negative self intersection. Let D be an ample divisor on V. Then $D+mE$ is almost ample for all m, but it is not ample if $(E^2)m > (E.D)$. Thus not all almost ample divisors are ample.

PROPOSITION 1.2.7 (KODAIRA, [**K-O**, APPENDIX]). *Let D be a divisor on a projective variety V. Then D is almost ample if and only if*

nD can be written as a sum of an ample divisor and an effective divisor, for some sufficiently large integer n.

PROOF: If D can be written as a sum of an ample divisor and an effective divisor, then it is almost ample, by Lemma 1.2.6. Conversely, let A be any ample divisor, and assume that it is irreducible and smooth. Then we have an exact sequence,

$$0 \to \mathcal{O}(-A) \to \mathcal{O}_V \to \mathcal{O}_A \to 0$$

where $\mathcal{O}(-A)$ is the invertible sheaf associated to the divisor $-A$. Tensoring with $\mathcal{O}(nD)$ gives a long exact sequence in cohomology,

$$0 \to H^0(V, \mathcal{O}(nD - A)) \to H^0(V, \mathcal{O}(nD)) \to H^0(A, \mathcal{O}(nD)|_A)) \to \dots$$

Counting dimensions, the second term grows like $n^{\dim V}$, whereas the third grows only at most as fast as $n^{\dim V - 1}$. Thus, $h^0(V, \mathcal{O}(nD - A)) > 0$ for some n sufficiently large, as was to be shown. □

DEFINITION 1.2.8. *Let h_1 and h_2 be two logarithmic height functions on V, and let D be any ample divisor. If for any $\epsilon > 0$ there exists a constant $c = c(\epsilon)$ such that,*

$$|h_1 - h_2| < \epsilon h_D + c,$$

then we say that h_1 and h_2 are quasi-equivalent.

This notion is independent of the choice of D, by (f), below. We can now summarize the properties of heights:

PROPOSITION 1.2.9.

(a). *For all divisors D, D' on V,*

$$h_{D+D'} = h_D + h_{D'} + O(1).$$

(b). *If D and D' are linearly equivalent, then*

$$h_D = h_{D'} + O(1).$$

(c). *If $f\colon V \to W$ is an algebraic map defined over k, and D is a divisor on W, then*

$$h_{V,f^*D} = h_{W,D} \circ f + O(1).$$

(d). *If D and D' are two numerically equivalent divisors, then h_D and $h_{D'}$ are quasi-equivalent.*
(e). *If D is effective, then $h_D \geq O(1)$ outside of the base locus of D.*
(f). *If D is ample then h_D is the largest possible height function, up to a constant; i. e. for any other divisor E,*

$$h_D \gg h_E + O(1).$$

(g). *(Northcott) if D is ample then there are only finitely many k-rational points of V for which h_D is below a given bound.*
(h). *If D is almost ample, then (f) and (g) hold outside of a Zariski-closed subset of V.*

PROOF: (a) has been discussed. For (b), (c), (e), and (f) see [**L 7**]; Ch. 4; 2.1, 5.1, 5.2, and 5.4, respectively. For (g), see [**L 7**, Ch. 3, Theorem 2.6.]

(d). By (a) it will suffice to show that if D is numerically equivalent to zero and L is an ample divisor, then for all $\epsilon > 0$ there exists a constant c such that,

$$|h_D| \leq \epsilon h_L + c.$$

But, for any positive integer n, $L - nD$ is ample. Indeed, by the Nakai-Moishezon criterion [**H**, A.5.1], ampleness depends only on the numerical equivalence class of V; since $L-nD$ is numerically equivalent to L, $L-nD$ is ample. Thus by (b) and (e),

$$h_{L-nD} \geq O(1);$$

$$h_D \leq \frac{1}{n} h_L + c.$$

By the same argument applied to $-D$, $-h_D \leq \frac{1}{n} h_L + c$. This concludes the proof of (d).

(h). Follows from (f), (e), (c), and Proposition 1.2.7. □

REMARK. At first glance, (d) may appear more general than the usual property of heights being quasi-equivalent if their divisors are algebraically equivalent [**L 2**]. However, by [**Mat**], numerical equivalence is the same as algebraic equivalence up to torsion, so the results are actually identical.

By (h), an almost ample divisor will give a height function which is "largest," even on a complete variety which is not projective.

§3. Weil Functions

Let k, M_k, and S_∞ be as above; let D be a divisor on a nonsingular variety V. Extend $\|\cdot\|_v$ to an absolute value on the algebraic closure $\overline{k}_v$. Then a local Weil function for D relative to v is a function $\lambda_{D,v}\colon V(\overline{k}_v)\setminus|D|\to\mathbf{R}$ such that if D is represented locally by (f) on an open set U, then

$$\lambda_{D,v}(P)=-\frac{1}{[k:\mathbf{Q}]}\log\|f(P)\|_v+\alpha(P)$$

where $\alpha(P)$ is a continuous function on $U(\overline{k}_v)$. We sometimes think of $\lambda_{D,v}$ as a function of $V(k)\setminus|D|$ or $V(\overline{k})\setminus|D|$ by (implicitly) choosing an embedding $\overline{k}\to\overline{k}_v$.

This is very similar to the definition for a metric. Indeed, let (U_i,f_i) be a Cartier divisor representing D. Let the associated line bundle $[D]$ be defined by trivializations $\phi_i\colon [D]|_{U_i}\to U_i\times\mathbf{C}$ such that

$$\phi_j\circ\phi_i^{-1}\colon (u,x)\mapsto (u,f_j(u)/f_i(u)\cdot x)$$

on $[D]|_{U_i\cap U_j}$. Then a metric on the line bundle $[D]$ is a set of C^∞ functions,

$$\rho_i\colon U_i\to\mathbf{R}$$

satisfying,

$$\rho_i/\rho_j=|f_i/f_j|^2\qquad\text{on } U_i\cap U_j. \tag{1.3.1}$$

The functions f_i give a section s of the line bundle $[D]$ whose divisor is D. Then, given $P\in V$, we have

$$|s(P)|=\frac{|f_i(P)|}{\rho_i(P)}$$

so that

$$\begin{aligned}\lambda_{D,v}(P)&=-\frac{1}{[k:\mathbf{Q}]}\log\|s(P)\|_v\\&=-\frac{1}{[k:\mathbf{Q}]}\log\|f_i(P)\|_v+\frac{1}{[k:\mathbf{Q}]}\log\rho_{i,v}(P)\end{aligned} \tag{1.3.2}$$

can be taken to be a local Weil function for D at v.

A global Weil function for D over k is a collection $\{\lambda_D\}$ of local Weil functions, as v ranges through M_k, subject to an additional continuity constraint. This constraint is fairly hard to state (see [**L 7**, Ch. 10, §1, §2]); however, for our purposes it suffices to know that changing a finite number of local Weil functions will not affect the continuity condition. Note also that we normalize the Weil functions differently here.

LEMMA 1.3.3. *Weil functions satisfy the following properties:*

(a). *If* λ_D *and* $\lambda_{D'}$ *are (local or global) Weil functions for* D *and* D', *then* $\lambda_D + \lambda_{D'}$ *is a Weil function for* $D + D'$ *and* $-\lambda_D$ *is a Weil function for* $-D$.

(b). *If* D *is an effective divisor then* $\lambda_{D,v}(P) \geq c_v$ *for some constant* c_v; *furthermore* $c_v = 0$ *for almost all* v.

(c). $-\frac{1}{[k:\mathbf{Q}]}\log\|f\|_v$ *is a global Weil function for the principal divisor* (f).

(d). *Let* $\phi: V \to W$ *be a morphism of nonsingular varieties and let* D *be a divisor on* W *not containing the image of* ϕ. *If* λ_D *is a Weil function for* D *on* W *then* $\lambda_D \circ f$ *is a Weil function for* f^*D *on* V.

(e). *If* E *is a finite extension of the number field* k *and* λ_D *is a Weil function for* D *over* k *then*

$$\lambda_{D,w}(P) = \frac{[E_w : k_v]}{[E : k]}\lambda_{D,v}(P)$$

is a Weil function for D *over* E. *Thus, if* $P \in V(k)$, *then*

$$\lambda_{D,v}(P) = \sum_{w|v} \lambda_{D,w}(P) + O(1).$$

(f). *Let* $D_1, \ldots, D_n$ *and* D *be divisors on* V *such that* $D_i - D$ *are effective divisors with no geometric point in common. Then*

$$\inf_i \lambda_{D_i}$$

is a Weil function for D.

(g). *The height can be expressed in terms of Weil functions as,*

$$h_D(P) = \sum_{v \in M_k} \lambda_{D,v}(P) + O(1), \tag{1.3.4}$$

for all $P \in V(k) \setminus |D|$.

PROOF: See [**L 7**]. □

For example, (c) and (f) imply that,

$$\lambda_{D,v}(P) = \frac{1}{[k:\mathbf{Q}]}\log\max(1, \|x_1/x_0\|_v, \ldots, \|x_n/x_0\|_v) \tag{1.3.5}$$