Lecture Notes in Mathematics

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Ofer Gabber
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Almost Ring Theory

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Contents

Le bruit des vagues était encore plus paresseux, plus étale qu'a midi. C'était le même soleil, la même lumière sur le même sable qui se prolongeait ici.

A. Camus – *L'étranger*

1. Introduction

1.1. Motivations and a little history. Almost mathematics made its official debut in Faltings' fundamental article [33], the first of a series of works on the subject of p-adic Hodge theory, culminating with [34]. Although almost ring theory is developed here as an independent branch of mathematics, stretching somewhere in between commutative algebra and category theory, the original applications to p-adic Hodge theory still provide the main motivation and largely drive the evolution of the subject.

Indeed, one of the chief aims of our monograph is to supply adequate foundations for [34], and to pave the way to further extensions of Faltings' methods (especially, of his deep "almost purity" theorem), that we plan to present in a future work. For these reasons, it is fitting to begin this introduction with some background, leading up to a review of the results of [33]. (Besides, we suspect that all but the most dedicated expert of p-adic Hodge theory will require some inducement before deciding to plunge into close to 300 pages of foundational arcana.)

The starting point of p-adic Hodge theory can be located in Tate's paper [74] on p-divisible groups. An important example of p-divisible group is the p-primary torsion subgroup A_{p^∞} of an abelian scheme A defined over the valuation ring K^+ of a complete discretely valued field K of characteristic zero. We assume that the residue field κ of K is perfect of characteristic $p > 0$; also, let π be a uniformizer of K^+, K^{a} the algebraic closure of K and denote by C the completion of K^{a}; the Galois group $G := \mathrm{Gal}(K^{\mathrm{a}}/K)$ acts linearly on the étale cohomology of A, and actually A_{p^∞} and the Galois module $H^1_{\text{ét}}(A_{K^{\mathrm{a}}}, \mathbb{Z}_p)$ determine each other. G also acts semilinearly on C, whence a natural *continuous semilinear* action of G on the tensor product of Galois modules

$$H^\bullet_{\text{ét}}(A_{K^{\mathrm{a}}}, C) := H^\bullet_{\text{ét}}(A_{K^{\mathrm{a}}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C.$$

At first sight, it would seem that, in replacing a linear Galois representation by a semilinear one, we are trading a simpler object by a more complicated one. In fact, the opposite holds: as a consequence of his general study of p-divisible groups, Tate showed that for every $i \leq 2\dim(A)$ there exists a natural equivariant isomorphism

$$H^i_{\text{ét}}(A_{K^{\mathrm{a}}}, C) \simeq \bigoplus_{j+k=i} H^j(A, \Omega^k_A) \otimes_K C(-k) \tag{1.1.1}$$

where, for every integer $j \in \mathbb{Z}$, we define $C(j) := \mathbb{Q}_p(j) \otimes_{\mathbb{Q}_p} C$, and $\mathbb{Q}_p(j)$ is the j-th tensor power of the one-dimensional p-adic representation $\mathbb{Q}_p(1)$ on which G acts as the p-primary cyclotomic character. Tate conjectured that an equivariant decomposition such as (1.1.1) should exist for any smooth projective variety defined over K. To put things in perspective, let us turn to consider the archimedean counterpart of (1.1.1): if X is a smooth, proper complex algebraic variety, one can combine deRham's theorem with Grothendieck's theorem on algebraic deRham cohomology [42], to deduce a natural isomorphism

$$H^\bullet(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^\bullet_{\mathrm{dR}}(X) \tag{1.1.2}$$

between the singular and deRham cohomologies. The two sides of (1.1.2) contribute complementary information on X; namely, singular cohomology supplies an integral structure for $H^\bullet(X^{\mathrm{an}}, \mathbb{R})$ (the lattice of periods) and deRham cohomology gives the Hodge filtration: neither of these two structures is reducible to the other.

The above conjecture of Tate is rather startling because it implies that in the non-archimedean case, étale cohomology and deRham cohomology should not be complementary: rather, étale cohomology, viewed as a Galois module, would already detect, if not quite the Hodge filtration, at least its associated graded subquotients, each of them clearly recognizable by the different weight (or Tate twist) with which it appears in $H^\bullet_{\text{ét}}(A_{K^{\mathrm{a}}}, C)$ (now this graded Galois module is known as the Hodge-Tate cohomology and often denoted $H_{\mathrm{HT}}(-)$).

On the other hand, working around the same time as Tate, Grothendieck realized that the deRham cohomology of an abelian scheme carries more structure than it would appear at first sight: using his crystalline Dieudonné theory he showed that $H^1_{\mathrm{dR}}(A)$ comes with a canonical K_0-structure (where K_0 is the field of fractions of the ring $W(\kappa)$ of Witt vectors of κ), namely the K_0-vector space $M \otimes_{W(\kappa)} K_0$ where M is the Dieudonné module of the special fibre of A_{p^∞} (see [43]). Furthermore, this K_0-vector space is endowed with an automorphism ϕ which is semilinear, *i.e.* compatible with the Frobenius automorphism of K_0. Grothendieck even proved that A_{p^∞} is determined up to isogeny by $H^1_{\mathrm{dR}}(A)$ together with its Hodge filtration, K_0-structure and automorphism ϕ. Taking into account the above theorem of Tate, he was then led to ask the question of describing an algebraic procedure that would allow to pass directly from $H^1_{\mathrm{dR}}(A)$ to $H^1_{\text{ét}}(A_{K^{\mathrm{a}}}, \mathbb{Q}_p)$ without the intermediary of the p-divisible group A_{p^∞}; he also expected that such a procedure should exist for the cohomology in arbitrary degree (he baptized[1] this as the problem of the "mysterious functor").

The question in degree one was finally solved by Fontaine, several years later ([37]); he actually constructed a functor in the opposite direction, *i.e.* from the category of p-adic Galois representations to the category of filtered K-modules with additional structure as above. The construction of Fontaine hinges on a remarkable ring (actually a whole hierarchy of increasingly complex rings), endowed with both a Galois action and a filtration (and eventually, additional subtler structures). The simplest of such rings is the graded ring $B_{\mathrm{HT}} := \oplus_{i \in \mathbb{Z}} C(i)$, with its obvious multiplication; with its help, Tate's decomposition can be rewritten as an isomorphism of graded K-vector spaces:

[1]The name entered the folklore, even though Grothendieck apparently only ever used it orally, and we could find no trace of it in his writings

$$(H^i_{\text{ét}}(A_{K^a}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}})^G \simeq \bigoplus_{j+k=i} H^j(A, \Omega^k_A).$$

Fontaine proved that his functor solves Grothendieck's problem for the H^1, and proposed a precise conjecture (the C_{cris} conjecture) in arbitrary degree, for schemes X that are proper and smooth over K and have good reduction over K^+.

1.2. **The method of almost étale extensions.** The C_{cris} conjecture is now completely proved, as well as some later extensions (*e.g.* to schemes with not necessarily good reduction, or not necessarily proper). There are actually at least two very different methods that both have led to a proof: one – due to Fontaine, Kato, Messing and crowned by Tsuji's work [75] – relies on so-called syntomic cohomology and a delicate study of vanishing cycles; the other, due to Faltings, is based on his theory of almost étale extensions (for the case of varieties of good reduction, Niziol has found yet another method, that uses a comparison theorem from étale cohomology to K-theory as a go-between: see [63]).

We won't say anything about the first and the third approaches, but we wish to give a rough overview of the method of almost étale extensions, which was first presented in [33], where Faltings used it to prove the sought comparison with Hodge-Tate cohomology; in subsequent papers the method has been refined and amplified, and its latest incarnation is contained in [34]. However, many important ideas are already found in [33], so it is on the latter that we will focus in this introduction.

For simplicity we will assume that our varieties have good reduction over K^+, hence we let X be a smooth, connected and projective K^+-scheme. The idea is to construct an intermediate cohomology $\mathscr{H}(X)$, with values in C-vector spaces, receiving maps from both étale and Hodge-Tate cohomology, and prove that the resulting natural transformations

$$H_{\text{ét}}(X_{K^a}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to \mathscr{H}(X) \quad \text{and} \quad H_{\text{HT}}(X_K) \to \mathscr{H}(X)$$

are isomorphisms of functors. In order to motivate the definition of $\mathscr{H}(X)$, it is instructive to consider first the case of a point, *i.e.* $X = \operatorname{Spec} K^+$. In this case étale cohomology reduces to Galois cohomology, and the calculation of the latter was the main technical result in [74].

Tate's calculation can be explained as follows. The valuation v of K extends uniquely to any algebraic extension, and we want to normalize the value group in such a way that $v(p) = 1$ in every such extension. Let E be a finite Galois extension of K, with Galois group G_E. Typically, one is given a discrete $E^+[G_E]$-module M (such that the Γ-action on M is *semilinear*, that is, compatible with the G_E-action on E^+), and is interested in studying the (modified) Tate cohomology $\widehat{H}^i := \widehat{H}^i(G_E, M)$ (for $i \in \mathbb{Z}$). (Recall that $\widehat{H}^i$ agrees with Galois cohomology $R^i\Gamma^{G_E} M$ for $i > 0$, with Galois homology for $i < -1$, and for $i = 0$ it equals $M^{G_E}/\operatorname{Tr}_{E/K}(M)$, the G_E-invariants divided by the image of the trace map).

In such a situation, the scalar multiplication map $E^+ \otimes_{\mathbb{Z}} M \to M$ induces natural cup product pairings

$$\widehat{H}^i(G_E, E^+) \otimes_{\mathbb{Z}} \widehat{H}^j \to \widehat{H}^{i+j}.$$

Especially, the action of $(E^+)^{G_E} = K^+$ on $\widehat{H}^i$ factors through $K^+/\mathrm{Tr}_{E/K}(E^+)$; in other words, the image of E^+ under the trace map annihilates the modified Tate cohomology.

If now the extension E is *tamely ramified* over K, then $\mathrm{Tr}_{E/K}(E^+) = K^+$, so the $\widehat{H}^i$ vanish for all $i \in \mathbb{Z}$. Even sharper results can be achieved when the extension is *unramified*. Indeed, in such case E^+ is a G_E-torsor for the étale topology of K^+, hence, some basic descent theory tells us that the natural map

$$E^+ \otimes_{K^+} R\Gamma^{G_E} M \to M[0]$$

is an isomorphism in the derived category of the category of $E^+[G_E]$-modules (where we have denoted by $M[0]$ the complex consisting of M placed in degree zero).

In Tate's paper [74] there occurs a variant of the above situation : instead of the finite extension E one considers the algebraic closure K^{a} of K, so that $G_{K^{\mathrm{a}}} = G$ is the absolute Galois group of K, and the discrete G-module M is replaced by the *topological* module $C(\chi)$, obtained by "twisting" the natural G-action on C via a continuous character $\chi : G \to K^\times$. Then the relevant $H^\bullet$ is the *continuous* Galois cohomology $H^\bullet_{\mathrm{cont}}(G, C(\chi))$, which is defined in general as the homology of a complex of continuous cochains. Under the present assumptions, H^i can be computed by the formula:

$$H^i_{\mathrm{cont}}(G, C(\chi)) := (\lim_{\substack{\longleftarrow \\ n}} H^i(G, K^{\mathrm{a}+}(\chi) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let now K_∞ be a totally ramified Galois extension with Galois group H isomorphic to $\mathbb{Z}_p$. Tate realized that, for cohomological purposes, the extension K_∞ plays the role of a maximal totally ramified Galois extension of K. More precisely, let L be any finite extension of K, and set $L_n := L \cdot K_n$, where K_n is the subfield of K_∞ fixed by $H^{p^n} \simeq p^n \cdot \mathbb{Z}_p$. The extension $K_n \subset L_n$ is unramified if and only if the different ideal $\mathscr{D}_n := \mathscr{D}_{L_n^+/K_n^+}$ equals L_n^+. In case this fails, the valuation $v(\delta_n)$ of a generator δ_n of $\mathscr{D}_n$ will be a strictly positive rational number, giving a quantitative measure for the ramification. With this notation, [74, §3.2, Prop.9] reads

$$\lim_{n\to\infty} v(\delta_n) = 0 \tag{1.2.1}$$

(indeed, $v(\delta_n)$ approaches zero about as fast as p^{-n}). In this sense, one can say that the extension $K_\infty \subset L_\infty := L \cdot K_\infty$ is *almost unramified*. One immediate consequence is that the maximal ideal $\mathfrak{m}$ of K_∞^+ is contained in $\mathrm{Tr}_{L_\infty/K_\infty}(L_\infty^+)$. If, additionally, L is a Galois extension of K, we can consider the subgroup

$$G_\infty := \mathrm{Gal}(L_\infty/K_\infty) \subset \mathrm{Gal}(L/K)$$

and the foregoing implies that $\mathfrak{m}$ annihilates $H^i(G_\infty, M)$, for every $i > 0$, and every $L_\infty^+[G_\infty]$-module M. More precisely, the homology of the cone of the natural morphism

$$L_\infty^+ \otimes_{K_\infty^+} R\Gamma^{G_\infty} M \to M[0] \tag{1.2.2}$$

is annihilated by $\mathfrak{m}$ in all degrees, *i.e.* it is *almost zero*. Equivalently, one says that the maps on homology induced by (1.2.2) are *almost isomorphisms* in all degrees.

A first generalization of (1.2.1) can be found in the work [38] by Fresnel and Matignon; one interesting aspect of this work is that it does away with any consideration of local class field theory (which was used to get the main estimates in [74]); instead, Fresnel and Matignon write a general extension L as a tower of monogenic subextensions, whose structure is sufficiently well understood to allow a direct and very explicit analysis. The main tool in [38] is a notion of different ideal $\mathscr{D}_{E^+/K^+}$ for a possibly infinite algebraic field extension $K \subset E$; then the extension K_∞ considered in [74] is replaced by any extension E of K such that $\mathscr{D}_{E^+/K^+} = (0)$, and (1.2.1) is generalized by the claim that $\mathscr{D}_{F^+/E^+} = F^+$, for every finite extension $E \subset F$.

In some sense, the arguments of [38] anticipate those used by Faltings in the first few paragraphs of [33]. There we find, first of all, a further extension of (1.2.1): the residue field of K is now not necessarily perfect, instead one assumes only that it admits a finite p-basis; then the relevant K_∞ is an extension whose residue field is perfect, and whose value group is p-divisible. The assertion (1.2.1) under such assumptions represents the one-dimensional case of the almost purity theorem. In order to state and prove the higher dimensional case, Faltings invents the method of "almost étale extensions", and indeed sketches in a few pages a whole program of "almost commutative algebra", with the aim of transposing to the almost context as much as possible of the classical theory. So, for instance, if A is a given K^+_∞-algebra, and M is an A-module, one says that M is *almost flat* if, for every A-module N, the natural map of complexes

$$M \overset{\mathbf{L}}{\otimes}_A N \to M \otimes_A N[0]$$

induces almost isomorphisms on homology in all degrees. Similarly, M is *almost projective* if the same holds for the map of complexes

$$\mathrm{Hom}_A(M,N)[0] \to R\mathrm{Hom}_A(M,N).$$

Then, according to [33], a map $A \to B$ of K^+_∞-algebras is called *almost étale* if B is almost projective as an A-module and as a $B \otimes_A B$-module (moreover, B is required to be *almost finitely generated* : the discussion of finiteness conditions in almost ring theory is a rather subtle business, and we dedicate the better part of section 2.3 to its clarification).

With this new language, the almost purity theorem should be better described as an almost version of Abhyankar's lemma, valid for morphisms $A \to B$ of K^+-algebras that are étale in characteristic zero and possibly wildly ramified on the locus of positive characteristic. The actual statement goes as follows. Suppose that A admits global étale coordinates, that is, there exists an étale map $K^+[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to A$ (following Faltings, one calls *small* such an algebra); whereas in the tamely ramified case a finite ramified base change $K^+ \to K^+[\pi^{1/n}]$ (with $(p,n)=1$) suffices to kill all ramification, the infinite extension

$$A \to A_\infty := A[T_1^{\pm 1/p^\infty},\dots,T_d^{\pm 1/p^\infty}] \otimes_{K^+} K^+_\infty$$

is required in the wildly ramified case, to kill *almost all* ramification, which means that the normalization B_∞ of $A_\infty \otimes_A B$ is almost étale over A_∞.

Faltings has proposed two distinct strategies for the proof of his theorem : the first one, presented in [33], consists in adapting Grothendieck's proof of Zariski-Nagata's purity[2]; a more recent one ([34]) uses the action of Frobenius on some local cohomology modules, and is actually valid under more general assumptions (one does not require the existence of étale coordinates, but only a weaker semi-stable reduction hypothesis on the special fibre).

As a corollary, one deduces cohomological vanishings generalizing the foregoing : indeed, suppose that the extension of fraction fields $\mathrm{Frac}(A) \subset \mathrm{Frac}(B)$ is Galois with group G_B; then, granting almost purity, B_∞ is an "almost G_B-torsor" over A_∞, therefore, for any $B_\infty[G_B]$-module M, the natural map of complexes

$$B_\infty \otimes_{A_\infty} R\Gamma^{G_B} M \to M[0]$$

induces almost isomorphisms on homology.

We are now ready to return to the construction of $\mathscr{H}(X)$. Let A be a K^+-algebra that is small in the above sense; let d be the relative dimension of A over K^+. We denote by $\overline{A}$ the integral closure of A in a maximal algebraic extension of the field of fractions of A which is unramified over $A[p^{-1}]$; also let $\overline{A}^\wedge$ be the π-adic completion of $\overline{A}$: this is an algebra over the π-adic completion C^+ of $K^{\mathrm{a}+}$. The fundamental group

$$\Delta(A) := \pi_1(\mathrm{Spec}\, A \otimes_{K^+} K^{\mathrm{a}})$$

is the subgroup of the Galois group of $\mathrm{Frac}(\overline{A})$ over $\mathrm{Frac}(A)$ that fixes $A_{K^{\mathrm{a}+}} := A \otimes_{K^+} K^{\mathrm{a}+}$. The (continuous) Galois cohomology

$$\mathscr{H}^\bullet(A) := H^\bullet_{\mathrm{cont}}(\Delta(A), \overline{A}^\wedge)$$

can be computed by the functorial simplicial complex $\mathscr{C}^\bullet(\Delta(A), \overline{A}^\wedge)$ such that

$$\mathscr{C}^n(\Delta(A), \overline{A}^\wedge) := \text{continuous maps } \Delta(A)^n \to \overline{A}^\wedge.$$

Let $\Delta(A)_\infty$ be the kernel of the natural surjective homomorphism:

$$\Delta(A) \to \mathrm{Gal}(A_\infty / A_{K^{\mathrm{a}+}}) \simeq \mathbb{Z}_p^{\oplus d}.$$

As a consequence of the foregoing almost vanishings, the Hochschild-Serre spectral sequence

$$E_2^{pq}: \quad H^p_{\mathrm{cont}}(\mathbb{Z}_p^{\oplus d}, H^q_{\mathrm{cont}}(\Delta(A)_\infty, \overline{A}^\wedge)) \Rightarrow \mathscr{H}^{p+q}(A)$$

degenerates up to $\mathfrak{m}$-torsion, and one obtains a natural $\mathrm{Gal}(K^{\mathrm{a}}/K)$-equivariant isomorphism

$$\mathscr{H}^\bullet(A) \simeq A^\wedge_{K^{\mathrm{a}+}} \otimes_{\mathbb{Z}_p} \Lambda^\bullet_{\mathbb{Z}_p}((\mathbb{Z}_p(-1))^d) \oplus (\text{rest}) \tag{1.2.3}$$

where $A^\wedge_{K^{\mathrm{a}+}}$ is the π-adic completion of $A_{K^{\mathrm{a}+}}$ and $\Lambda^\bullet_{\mathbb{Z}_p}$ denotes the exterior algebra of the free $\mathbb{Z}_p$-module that is the sum of d copies of $\mathbb{Z}_p(-1)$. The (rest) is a module annihilated by $p^{1/(p-1)}$.

Moreover, for any étale map $A \to B$ of small K^+-algebras, the induced map:

$$\mathscr{H}^\bullet(A) \otimes_{A_{K^{\mathrm{a}+}}} B_{K^{\mathrm{a}+}} \to \mathscr{H}^\bullet(B) \tag{1.2.4}$$

[2] At the time of writing, there are still some obscure points in this proof

is an almost isomorphism.

Let X be a smooth projective K^+-scheme; we take an arbitrary (Zariski) hypercovering $U_\bullet \to X$ consisting of *small* affine open subschemes, meaning that each U_i is the spectrum of a small K^+-algebra. By applying termwise the functor $\mathscr{C}^\bullet$ we deduce a bicosimplicial K^+-module $\mathscr{C}^\bullet(\Delta(U_\bullet), U_\bullet)$, whose diagonal is a cosimplicial complex that we denote by $\mathscr{D}^\bullet(X)$. The intermediate cohomology $\mathscr{H}^\bullet(X)$ is defined as the homology of $\mathscr{D}^\bullet(X) \otimes_{K^+} K$.

Using the fact that (1.2.4) is an almost isomorphism, a standard argument shows that $\mathscr{D}^\bullet(X)$ is independent – up to natural almost isomorphisms – on the choice of hypercovering. Functoriality on X is also clear, and one can even define cup products, Kunneth isomorphisms for products of varieties, as well as versions with torsion coefficients.

Finally, since étale cohomology is a globalization of Galois cohomology, it is not difficult to construct a natural transformation

$$R\Gamma_{\text{ét}}(X_{K^a}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C^+ \to \mathscr{D}^\bullet(X) \tag{1.2.5}$$

(for this, one needs to know that X admits a basis of open subsets consisting of étale $K(\pi, 1)$-spaces, and this is also shown in [33]). The proof that (1.2.5) induces an almost isomorphism on cohomology is laborious, but not exceedingly difficult.

The relationship with Hodge-Tate cohomology is very direct, and can already be scented from (1.2.3); Faltings also spends a little extra effort to wring out some integral refinements (*i.e.*, to control the powers of p appearing in the denominators of the isomorphism map).

This is the basic outline of Faltings' proof; the method can even be extended to treat cohomology with not necessarily constant coefficients (see [34]), thereby providing the most comprehensive approach to p-adic Hodge theory found so far.

1.3. **Contents of this book.** Since each chapter is preceded by its own detailed introductory remarks, we will bound ourselves to a general overview of the organization of the monograph. The purpose of chapters 2 through 5 is to fully work out the foundations of "almost commutative algebra" outlined by Faltings; in the process we generalize and simplify considerably the theory, and also extend it in directions that were not explored in [33], [34].

It turns out that most of almost ring theory can be built up satisfactorily from a very slim and general set of assumptions: our basic setup, introduced in section 2.1, consists of a ring V and an ideal $\mathfrak{m} \subset V$ such that $\mathfrak{m} = \mathfrak{m}^2$; starting from (2.5.14) we also assume that $\mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module : simple considerations show this to be a natural hypothesis, often verified in practice.

The V-modules killed by $\mathfrak{m}$ are the objects of a (full) Serre subcategory Σ of the category V-**Mod** of all V-modules, and the quotient V^a-**Mod** $:= V$-**Mod**$/\Sigma$ is an abelian category which we call the category of *almost V-modules*. It is easy to check that the usual tensor product of V-modules descends to a bifunctor $\otimes$ on almost V-modules, so that V^a-**Mod** is a monoidal abelian category in a natural way. Then an *almost ring* is just an almost V-module A endowed with a "multiplication" morphism $A \otimes A \to A$ satisfying certain natural axioms. Together with the obvious morphisms, these gadgets form a category V^a-**Alg**. Given any almost V-algebra A, one can then define the notion of A-module and A-algebra, just like for usual rings. The purpose of the game is to reconstruct in this new framework as much as possible (and useful) of classical linear and commutative algebra.

Essentially, this is the same as the ideology informing Deligne's paper [23], which sets out to develop algebraic geometry in the context of abstract tannakian categories. We could also claim an even earlier ancestry, in that some of the leading motifs resonating throughout our text, can be traced as far back as Gabriel's memoir [40] "Des catégories abéliennes". Furthermore, it has been recently pointed out to us that Roos, in a series of works dating from the mid-sixties, had already discovered much of the homological algebra that forms the backbone for our more systematic study of almost modules (see *e.g.* [68], [69]).

A pervasive theme – recurring throughout the text – is the study of deformations of various interesting objects, may they be almost algebras, almost modules, or almost group schemes. Especially, the analysis of *nilpotent* deformations of étale almost algebras and of almost projective modules is important for the proof of the almost purity theorem; whereas Faltings used Hochschild cohomology to this aim, we employ the cotangent complex; this gives us shorter proofs of essentially stronger results (and answers some questions of Faltings). Nilpotent deformations usher the way to *formal* deformations, hence to the definition of adic topologies on almost rings and modules; this in turns leads straight to the study of *henselian* deformations and to the notion of henselian pair in almost ring theory.

Another important thread is *descent theory* for almost algebras; faithfully flat descent is easy, but in [34] one needs also some cases of non-flat descent, so we give a comprehensive treatment of the latter.

Closely related to descent is the problem of constructing quotients under flat equivalence relations, and we dedicate section 4.5 to this question.

The third main ingredient in the newer proof of the almost purity theorem is the Frobenius endomorphism of almost algebras of positive characteristic: this is investigated in section 3.5.

In many instances, our results go well beyond what is strictly necessary in order to justify Faltings' proof of almost purity; this is mainly because our emphasis is on supplying natural frameworks and – as much as possible – pure thought arguments, rather than choosing the most economical presentation. Another reason is that we plan to discuss and extend the almost purity theorem in a future work: some of the extra generality gained here will pay off then.

Chapters 6 and 7 are dedicated to applications, respectively to valuation theory and to p-adic analytic geometry: especially the reader will find there our own contributions to almost purity. Much in these two chapters pertains to subjects that border on – but have, strictly speaking, no intersection with – almost ring theory; however, at some crucial junctures, the methods developed in the previous chapters intervene in an essential way and link up the discussion with almost mathematics. Two notable examples are:

(a) theorem 6.3.23, that generalizes the relationship between the module of differentials and the different ideal; this is classical in the case of a finite separable extension of discrete valuation rings, but new for general valuations of rank one, where the map is not any longer finite, but only *almost* finite;
(b) proposition 7.5.15 on the semicontinuity of the discriminant function, for a finite étale map of rings; again, this is a result of (usual) commutative algebra, whose statement and proof would both be very ackward without the machinery of almost rings.

We hope to demonstrate with these samples that almost ring theory has something to offer even to mathematicians that are not directly involved with p-adic Hodge theory.

We close with an appendix collecting some miscellanea: a sketch of a theory of the fundamental group for almost algebras, and the construction of the derived functors of some standard non-additive functors defined on almost modules.

1.4. **The view from above.** In evoking Deligne's and Gabriel's works, we have unveiled another source of motivation whose influence has steadily grown throughout the long gestation of our paper. Namely, we have come to view almost ring theory as a contribution to that expanding body of research of still uncertain range and shifting boundaries, that we could call "abstract algebraic geometry". We would like to encompass under this label several heterogeneous developments: notably, it should include various versions of non-commutative geometry that have been proposed in the last twenty years (*e.g.* [70]), but also the relative schemes of [47], as well as Deligne's ideas for algebraic geometry over symmetric monoidal categories.

The running thread loosely unifying these works is the realization that "geometric spaces" do not necessarily consist of set-theoretical points, and – perhaps more importantly – functions on such "spaces" do not necessarily form (sheaves of) commutative rings. Much effort has been devoted to extending the reach of geometric intuition to non-commutative algebras; alternatively, one can retain commutativity, but allow "structure sheaves" which take values in tensor categories other than the category of rings.

As a case in point, to any given almost ring A one can attach its *spectrum* Spec A, which is just A viewed as an object of the opposite of the category V^a-**Alg**. Spec A has even a natural *flat topology*, which allows to define more general *almost schemes* by gluing (*i.e.* taking colimits of) diagrams of affine spectra; all this is explained in section 5.7, where we also introduce *quasi-projective* almost schemes and investigate some basic properties of the *smooth locus* of a quasi-projective almost scheme.

By way of illustration, these generalities are applied in section 5.8 in order to solve a deformation problem for torsors over affine almost group schemes; let us stress that the problem in question is stated purely in terms of affine objects (*i.e.* almost rings and "almost Hopf algebras"), but the solution requires the introduction of certain auxiliary almost schemes that are not affine.

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2. HOMOLOGICAL THEORY

As explained in the introduction, in order to define a category of almost modules one requires a pair $(V, \mathfrak{m})$ consisting of a ring V and an ideal $\mathfrak{m} \subset V$ such that $\mathfrak{m} = \mathfrak{m}^2$. In section 2.1 we collect a few useful ring-theoretic preliminaries concerning such pairs. In section 2.2 we introduce the category V^a-**Mod** of *almost modules* : it is a quotient V-**Mod**$/\Sigma$ of the category of V-modules, where Σ is the thick subcategory of the V-modules killed by $\mathfrak{m}$. V^a-**Mod** is an abelian tensor category and its commutative unitary monoids, called *almost algebras*, are the chief objects of study in this work. The first useful observation is that the localization functor V-**Mod** $\to$ V^a-**Mod** admits both left and right adjoints. Taken together, these functors exhibit the kind of exactness properties that one associates to open embeddings of topoi, perhaps a hint of some deeper geometrical structure, still to be unearthed.

After these generalities, we treat in section 2.3 the question of finiteness conditions for almost modules. Let A denote an almost algebra, fixed for the rest of this introduction. It is certainly possible to define as usual a notion of finitely generated A-module, however this turns out to be too restrictive a class for applications. The main idea here is to define a uniform structure on the set of isomorphism classes of A-modules; then we will say that an A-module is *almost finitely generated* if its isomorphism class lies in the topological closure of the subspace of finitely generated A-modules. Similarly we define *almost finitely presented* A-modules. The uniform structure also comes handy when we want to construct operators on almost modules: if one can show that the operator in question is uniformly continuous on a class $\mathscr{C}$ of almost modules, then its definition extends right away by continuity to the topological closure $\overline{\mathscr{C}}$ of $\mathscr{C}$. This is exemplified by the construction of the (almost) Fitting ideals for A-modules, at the end of section 2.3.

In section 2.4 we introduce the basic toolkit of homological algebra, beginning with the notion of flat almost module, which poses no problem, since we do have a tensor product in our category. The notion of projectivity is more subtle : it turns out that the category of A-modules usually does *not* have enough projectives. The useful notion is *almost projectivity*: simply one uses the standard definition, except that the role of the Hom functor is played by the internal alHom functor. The scarcity of projectives should not be regarded as surprising or pathological: it is quite analogous to the lack of enough projective objects in the category of quasi-coherent $\mathscr{O}_X$-modules on a non-affine scheme X.

Section 2.5 introduces the cotangent complex of a morphism of almost rings, and establishes its usual properties, such as transitivity and Tor-independent base change theorems. These foundations will be put to use in chapter 3, to study infinitesimal deformations of almost algebras.

2.1. **Some ring-theoretic preliminaries.** Unless otherwise stated, every ring is commutative with unit. This section collects some results of general nature that will be used throughout this work.

2.1.1. Our *basic setup* consists of a fixed base ring V containing an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$. Starting from (2.5.14), we will also assume that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module.

Example 2.1.2. (i) The main example is given by a non-discrete valuation ring $(V, |\cdot|)$ of rank one; in this case $\mathfrak{m}$ will be the maximal ideal.

(ii) Take $\mathfrak{m} := V$. This is the "classical limit". In this case almost ring theory reduces to usual ring theory. Thus, all the discussion that follows specializes to, and sometimes gives alternative proofs for, statements about rings and their modules.

2.1.3. Let M be a given V-module. We say that M is *almost zero* if $\mathfrak{m}M = 0$. A map ϕ of V-modules is an *almost isomorphism* if both $\mathrm{Ker}\,\phi$ and $\mathrm{Coker}\,\phi$ are almost zero V-modules.

Remark 2.1.4. (i) It is easy to check that a V-module M is almost zero if and only if $\mathfrak{m} \otimes_V M = 0$. Similarly, a map $M \to N$ of V-modules is an almost isomorphism if and only if the induced map $\widetilde{\mathfrak{m}} \otimes_V M \to \widetilde{\mathfrak{m}} \otimes_V N$ is an isomorphism. Notice also that, if $\mathfrak{m}$ is flat, then $\mathfrak{m} \simeq \widetilde{\mathfrak{m}}$.

(ii) Let $V \to W$ be a ring homomorphism. For a V-module M set $M_W := W \otimes_V M$. We have an exact sequence

$$0 \to K \to \mathfrak{m}_W \to \mathfrak{m}W \to 0 \tag{2.1.5}$$

where $K := \mathrm{Tor}_1^V(V/\mathfrak{m}, W)$ is an almost zero W-module. By (i) it follows that $\mathfrak{m} \otimes_V K \simeq (\mathfrak{m}W) \otimes_W K \simeq 0$. Then, applying $\mathfrak{m}_W \otimes_W -$ and $- \otimes_W (\mathfrak{m}W)$ to (2.1.5) we derive

$$\mathfrak{m}_W \otimes_W \mathfrak{m}_W \simeq \mathfrak{m}_W \otimes_W (\mathfrak{m}W) \simeq (\mathfrak{m}W) \otimes_W (\mathfrak{m}W)$$

i.e. $\widetilde{\mathfrak{m}}_W \simeq (\mathfrak{m}W)^\sim$. In particular, if $\widetilde{\mathfrak{m}}$ is a flat V-module, then $(\mathfrak{m}W)^\sim$ is a flat W-module. This means that our basic assumptions on the pair $(V, \mathfrak{m})$ are stable under arbitrary base extension. Notice that the flatness of $\mathfrak{m}$ does not imply the flatness of $\mathfrak{m}W$. This partly explains why we insist that $\widetilde{\mathfrak{m}}$, rather than $\mathfrak{m}$, be flat.

2.1.6. Before moving on, we want to analyze in some detail how our basic assumptions relate to certain other natural conditions that can be postulated on the pair $(V, \mathfrak{m})$. Indeed, let us consider the following two hypotheses :

(A) $\mathfrak{m} = \mathfrak{m}^2$ and $\mathfrak{m}$ is a filtered union of principal ideals.

(B) $\mathfrak{m} = \mathfrak{m}^2$ and, for all $k > 1$, the k-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$.

Clearly **(A)** implies **(B)**. Less obvious is the following result.

Proposition 2.1.7. (i) **(A)** *implies that* $\widetilde{\mathfrak{m}}$ *is flat.*

(ii) *If* $\widetilde{\mathfrak{m}}$ *is flat then* **(B)** *holds.*

Proof. Suppose that **(A)** holds, so that $\mathfrak{m} = \operatorname*{colim}_{\alpha \in I} Vx_\alpha$, where I is a directed set parametrizing elements $x_\alpha \in \mathfrak{m}$ (and $\alpha \leq \beta \Leftrightarrow Vx_\alpha \subset Vx_\beta$). For any $\alpha \in I$ we have natural isomorphisms

$$Vx_\alpha \simeq V/\mathrm{Ann}_V(x_\alpha) \simeq (Vx_\alpha) \otimes_V (Vx_\alpha). \tag{2.1.8}$$