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Introduction

This book is based on a set of lectures that I gave at the Institute of Advanced Studies in Vienna in 1992, and subsequently taught in the Economics Department, Queen's University. My brief was to provide a series of ten lectures that surveyed and introduced recent asset-pricing models in Finance, using mathematical techniques and microeconomic theory at the level of Varian's *Microeconomic Analysis*, or Kreps's *A Course in Microeconomic Theory*. With these prerequisites the book should be accessible to any first-year graduate student who has a good grounding in microeconomics. Necessarily this book is not complete, and omits much that is important in terms of generality and detail. To make the book complete in that sense would triple its length and increase the level of mathematical difficulty substantially.

The first chapter provides a brief history of modern finance theory, emphasizing the main contributions and sketching the role of application in the development of the theory. Chapter 2 introduces the two-date model with complete markets and uncertainty. The chapter recalls standard microeconomic arguments and introduces some of the geometric arguments that are developed more fully later in the book. Chapter 3 generalizes the model by allowing for incomplete markets and non-trivial asset markets. In Chapter 4 the incomplete market economy equilibrium is analysed using the idea of induced preferences and production sets over assets. This idea allows us to construct geometric proofs of arbitrage results and relate them to familiar microeconomic theoretical arguments. The Modigliani–Miller arguments on capital structure and the binomial option-pricing model are introduced as illustrations of the general argument. Chapter 5 covers the same ground but uses the technique of personalized martingale pricing as an alternative method for analysing the same problems. Chapter 6 considers asset-pricing models with consumer aggregation when arbitrage arguments are not possible. The arguments are geometric and avoid functional forms except as illustrations from the literature. Chapter 7 discusses diversification arguments using induced preferences, and establishes the equilibrium arbitrage-pricing theory when there is a finite number of assets. The capital asset-pricing model is deduced as a special case of the general theorem. Chapter 8 extends the two-date model to a multi-date complete market structure, and introduces some preliminary results. Chapter 9 explores arbitrage pricing in the complete market model and illustrates the ideas with the multiperiod binomial option-pricing model. The last chapter allows for incomplete markets and shows how previous results can be extended into a multi-period incomplete-markets framework. The book ends with a brief conclusion discussing extensions of the models and recent developments in asset-pricing models.

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1

A Brief History of Finance Theory

The history of finance theory is an interesting example of the interaction between abstract theorizing and practical application. Many of the original contributions in finance theory began as theoretical abstractions that appeared to be of limited or no practical use. But with additional assumptions and restrictions, these same theories have become commonplace in the major financial markets as standard frames of reference in analysing financial decisions and the functioning of markets. In addition, what had once been seen as a group of related theories can now be unified within a general framework. These developments have taken place in a relatively short space of time: the original ideas were developed in the 1950s, and culminated in the general theoretical structures published in the 1980s.

THE IMPORTANT CONTRIBUTIONS OF THE 1950s

To understand the current state of finance theory, we should go back to the fundamental contributions of Arrow (1963)—first published in French in 1953—and Debreu (1959). Their contribution was fundamental in showing how the economic model under certainty could be adapted to incorporate uncertainty. The basic idea was very simple: the commodity space was expanded to incorporate possible future states of the world. The market system was complete in the sense that there was a set of contingent markets for all commodities. Standard theorems on the existence and Pareto optimality of competitive equilibria could be reinterpreted, so that one could have an efficient allocation of resources under uncertainty. Although not recognized at the time, this abstract economy was the foundation for much of what was to follow.

Two other important theoretical developments occurred in the 1950s. In 1958, Modigliani and Miller published a controversial paper arguing that the financial structure of firms was a matter of indifference for all agents in the economy. Their proof relied upon the idea that individuals could employ a riskless arbitrage to undo the variation in the firm's financial structure. Although originally couched in terms of the firm's choice over debt and equity, it became apparent that the argument was general and could be applied to changes in dividend policy, debt structure, or other financial decisions. (See Miller, 1988 for a detailed account of these ideas.) The major novelty in the Modigliani–Miller paper was the use of financial arbitrage. In the coming decades, arbitrage arguments were to play an important role in understanding a whole array of complex asset-pricing problems.

The other major development was the publication of Markowitz's (1959) monograph on mean-variance portfolio selection. The basic idea was quite straightforward: if consumers were concerned about the average, and variability of portfolio returns, then one could obtain a simple analysis of portfolio choice in terms of the means and covariances of the original assets. This contribution was the first step in the development of portfolio analysis and asset pricing based on mean-variance analysis.

THE 1960s: THEORY AND THE BEGINNINGS OF APPLICATION

There were two major developments in finance theory in the 1960s. The first extended the Arrow–Debreu theory to explore financial markets in more detail. Hirshleifer (1965, 1966) made an important contribution by showing how the Arrow–Debreu theory could be applied to basic finance problems. In particular, he proved the Modigliani–Miller financial irrelevance result in the Arrow–Debreu framework. This was the first time that Arrow–Debreu had been linked to arbitrage theory.

These papers were followed quickly by Diamond's (1967) paper investigating the implications of incomplete asset markets. Diamond showed, in a two-date model under uncertainty, that with exogenously specified asset markets, the competitive equilibrium is a constrained optimum. Furthermore, he showed that one could obtain the Modigliani–Miller theorem so long as the bonds did not have default risk.

The second major development in the 1960s was the extension of the Markowitz mean-variance analysis to a competitive economy. Sharpe (1964), Lintner (1965), and Mossin (1966) observed that, with market clearance, all consumers would choose portfolios that were a linear combination of the risk-free asset and the market portfolio. A direct consequence of that observation is that equilibrium asset prices can be written as a linear combination of the bond price and the market value of the market portfolio. Or, in more familiar terms, the expected rate of return on any asset can be written as the risk-free rate of interest plus the asset's normalized covariance with the market times the difference between market's expected rate of return and the risk-free rate. This model and the pricing result became known as the capital asset-pricing model (CAPM). For the first time finance theory had created a simple model relating asset returns that could (in principle) be tested with econometric methods. By the late 1960s these tests were being carried out at the University of Chicago using the newly acquired CRSP share price data. The full flowering of this empirical research was to come in the next decade.

THE 1970s: THEORETICAL AND EMPIRICAL FINANCE COME OF AGE

There were a number of major developments in finance theory in the 1970s. The first was a continuation of the CAPM research programme, extending the model to a multiperiod economy (Merton, 1973*a*), introducing restrictions on borrowing (Black, 1972), introducing transaction costs (Milne and Smith, 1980), and applying it to a range of empirical problems in finance. As an empirical model CAPM began to have a major impact on the way investors and mutual fund managers controlled portfolios and assessed their performance. (For an informal discussion of the impact of these ideas see Bernstein, 1992.)

The second major contribution grew out of dissatisfaction with empirical tests of the CAPM. Although initial testing of CAPM appeared to show that the theory provided good fits to the data, subsequent work (Roll, 1977) showed that the predictive power of CAPM was exaggerated by the test methodology. Ross (1976) introduced the arbitrage-pricing theory (APT) as a generalized competitor to CAPM. By amalgamating pure arbitrage and diversification arguments he showed that one could obtain asset prices as a linear function of a few basic factors. Potentially, the model appeared more flexible and robust than CAPM, and possibly immune to the testing problems associated with CAPM. As we shall see, the APT played a more important role in asset-pricing theory in the following decade.

The third advance in finance theory has had a dramatic impact on theory, and practical financial decisions in capital markets. Black and Scholes (1972) and Merton (1973*b*) showed that one could exploit an arbitrage argument to obtain a relatively simple formula for a call stock option. This result led to the rapid development of a whole range of variations on this model. (See Smith, 1976 for a survey of the advances of that period.) Finance traders and bankers were interested in the models for providing pricing formulae for an ever-increasing array of derivative financial assets being traded in financial markets. Because these models exploited techniques used in physics (i.e. stock returns follow a diffusion process, Ito's lemma is used to obtain the arbitrage hedge, and the solution to a heat exchange equation is employed to derive the formula) there arose a mystique about derivative asset-pricing associated with a popular 'rocket scientist' image. In an important contribution Cox, Ross, and Rubinstein (1979) showed that the Black–Scholes logic and pricing derivation could be greatly simplified. Assuming an elementary binomial stochastic process for the stock it is easy to use arbitrage arguments to derive a binomial option-pricing formula. In addition they showed that by taking appropriate limits, one could obtain the Black–Scholes formula. Although not stressed in the paper, the underlying model used arbitrage arguments to derive Arrow–Debreu prices, so that the pricing formula was a discounted martingale with Arrow–Debreu prices acting as probabilities.

Another interesting development was the derivation by Rubinstein (1976) of the Black–Scholes formula from a discrete-time incomplete markets equilibrium model. By assuming consumer aggregation, the economy achieved a trivial Pareto optimal allocation and the Arrow–Debreu prices supported the consumer optimum. This was the first representative consumer model where the martingale pricing result was obtained, albeit in a restricted form. In the next decade this general insight was exploited in finance, and particularly in macroeconomic representative consumer models following Lucas (1978).

The idea of martingale pricing was exploited in detail by Harrison and Kreps (1979). They showed that the martingale binomial logic could be generalized to a more abstract setting with continuous or discrete asset-price processes.

This abstract approach was to have a big impact on finance theory in the following decade in sorting out ambiguity that had arisen over the efficient-markets hypothesis (EMH). The idea of the EMH was first introduced by Fama (1970). Building on the earlier work of Samuelson (1965) and earlier writers, he argued that, in financial markets with free entry, no agent could make abnormal returns by exploiting publicly available information. This simple idea was to have a profound impact on empirical finance and the way agents in financial markets viewed their role and performance (see Bernstein, 1992). One of the early problems with the theory was its lack of coherence in making a link with asset-pricing models. This ambiguity was clarified in the 1980s using the theoretical ideas of martingale pricing.

There were two further significant developments. The first was the elaboration and analysis of complete and incomplete asset markets with multiple commodities and finite and infinite time-horizons. The work of Radner (1972) and Hart (1974, 1975) was important in clarifying the properties of incomplete markets. Unfortunately this work and related work on transaction costs in asset trading, introducing money into the model, the objective function of the firm with incomplete markets, and other generalizations, were largely ignored by finance theorists for nearly two decades.

The other major innovation was the introduction of recently developed ideas in asymmetric information into finance theory. Grossman (1976) analysed stock markets where agents had asymmetric information, and explored the idea that stock prices could completely or partially reveal private information. These ideas were explored in detail by a number of writers. (See Huang and Litzenberger, 1988 for a brief review.)

Asymmetric information ideas were introduced to explore the theory of corporate finance when there were differences in information between shareholders and management. These theories examined the robustness of the Modigliani–Miller theorem, when financial structure could act as a signal, or as an incentive mechanism. (See Huang and Litzenberger, 1988; or Bhattacharya and Constantinides, 1989: ii for a review of this literature.)

Because this book concentrates on competitive symmetric information models we will not discuss this large and interesting research topic of asymmetric information and game-theoretic models in finance.

THE 1980s AND BEYOND: THEORETICAL CONSOLIDATION AND UNIFICATION

In the 1980s the advances in theory were largely unifying and extending the existing theories. The various ideas were unified under the general Arrow–Debreu framework, and shown to be very flexible in application. This flexibility proved to be important in understanding the rapidly expanding market in derivative securities. In particular, the hedging and pricing of a whole array of securities became a major industry. Perhaps the most spectacular example of a derivative security was the development of portfolio insurance. This was an application of option-hedging ideas to portfolio management. Although simple in principle, the idea was developed into a significant financial product by two Berkeley finance theorists—Hayne Leland and Mark Rubinstein (see Bernstein, 1992).

On the theoretical front, the martingale idea became a central tool in characterizing asset-pricing in arbitrage or

Arrow–Debreu economies. Using the general idea of stochastic integrals, the models of Black–Scholes and Merton were generalized significantly by Harrison and Pliska (1981), Duffie and Huang (1985), and Duffie (1986).

A more specialized version of those models was introduced by Cox, Ingersoll, and Ross (1985*a*, 1985*b*) to explore the implications of stochastic interest rates for asset-pricing. This model stimulated a series of papers extending the hedging idea to derivative securities defined over bonds, or associated with bonds—see Heath, Jarrow, and Morton (1992) or Jarrow (1992).

Recalling the Rubinstein (1976) equilibrium approach to the Black–Scholes pricing formula, Turnbull and Milne (1991) were able to construct an equilibrium (possibly incomplete market) model that paralleled the Heath, Jarrow, and Morton results and applications. This provided a striking illustration of a more general idea that martingale asset-pricing could be obtained via equilibrium or arbitrage arguments (see Milne and Turnbull, 1994). For practical asset-pricing it is important to construct an argument (either arbitrage or equilibrium) that reduces the general martingale measure to a simpler density that can be written as a function of a small number of observable variables, simulated numerically on a lattice (for a survey see Jarrow, 1992), or approximated by polynomial methods (Madan and Milne, 1992).

Another advance was the clarification of Ross's APT. Two alternative approaches were taken: the first exploited an approximation argument (Chamberlain, 1983; Chamberlain and Rothschild, 1983; Huberman, 1983); the second used general equilibrium arguments to provide an exact or approximate APT (Connor, 1984; Milne 1988).

The APT idea of pricing factors has permeated asset-pricing models, so that many models can be seen as static or dynamic factor-pricing theories. In particular, dynamic asset-pricing models based on diffusion processes can be viewed as a special case of a more general dynamic factor model. Furthermore, by taking an appropriate basis, simple discrete models can mimic their more complex continuous-time counterparts. This discrete model provides an accessible and highly flexible framework for integrating asset-pricing theory (see Milne and Turnbull, 1994 for a detailed discussion of this model and its applications.) In addition the model can be adapted to incorporate fiat money and nominal asset returns, multiple currencies and exchange rates, transaction costs, taxes, and many other features. These variations have been developed recently, or are in the process of development.

This unification of finance theory has found a parallel in modern macroeconomics, where representative agent economies have been analysed to investigate real and pricing variables. Clearly macroeconomics and finance theory exploit the same underlying Arrow–Debreu model. It is hardly surprising that the same Modigliani–Miller type of results reappear in discussions of government financing and open-market operations (in the guise of Ricardian equivalence theorems). Increasingly this literature and finance have become integrated so that the boundaries of the two disciplines are blurred.

SUMMARY

The development of finance theory has been rapid. Not only has it provided highly flexible models, but they have found wide application in financial markets. These developments have been important in providing a coherent framework for thinking about existing financial markets and decision-making; and for creating ways of thinking about new financial products.

It is ironic that abstract ideas developed in the 1950s and 1960s, which once were thought to have limited application, should become the common language of financial markets.

2

Two-Date Models: Complete Markets

In the early 1950s Arrow and Debreu introduced a simple extension to the existing theory of a competitive equilibrium. Consider two dates: today there is certainty and tomorrow there is uncertainty, with $s = 1, \ldots, S$, states of the world. To make life simple, assume there is only one physical commodity, at each date or state. By expanding the definition of the commodity space to include dates and states, we can use all the standard tools of price theory under certainty to analyse an economy with contingent consumption and production. We begin with the

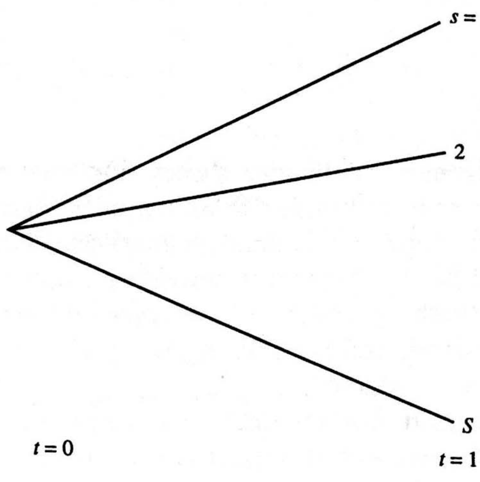

Fig. 2.1

consumer's problem. Each consumer $i = 1, \ldots, I$, has the problem:

$$\max_{\{x_i \in X_i\}} U_i(x_{0i}, x_{1i}, \ldots, x_{Si})$$

$$\text{s.t } p_0 x_{0i} + \sum_s p_s x_{si} \equiv W_{0i}$$

where: (i) the utility function is standard neoclassical (i.e. strictly increasing, quasi-concave, differentiable (if necessary));

(ii) p_0 is the price at $t = 0$ of the commodity; p_s is the price at $t = 0$ of the contingent commodity s and

(iii) W_{0i} is the $t = 0$ wealth of the consumer i.

We can analyse the consumer problem using the same tools as the certainty theory. For example, we can derive an indirect utility function, expenditure function, and obtain a Slutsky decomposition of consumer demand. (For details see Varian, 1992.)

Using the same idea we can analyse the firm's problem. Consider firm $f = 1, \ldots, F$, to have the problem:

$$\max_{\{y_j \in Y_j\}} \sum_{s=1}^{S} p_s y_{sj} - p_0 y_{0j} \equiv \boldsymbol{p} \boldsymbol{y}_j,$$

where $\boldsymbol{p}$ and $\boldsymbol{y}_j$ are the price and production vectors respectively. The firm maximizes its net present value by choosing the most profitable contingent production plan in the production set Y_j, where y_{0j} is the first-date input and y_{sj} the output of contingent commodity s. Again this is identical to the standard theory of the firm, and can be analysed with the same tools (e.g. profit function, cost functions. For details see Varian, 1992).

We can close the system by requiring market-clearing prices for commodity markets.

DEFINITION *A competitive equilibrium for the contingent claims economy is*

a price vector $(p_0^*, p_1^*, \ldots, p_S^*)$;