Fuzzy Sets, Uncertainty, and Information

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Preface

It has increasingly been recognized that our society is undergoing a significant transformation, usually described as a transition from an industrial to an information society. There is little doubt that this transition is strongly connected with the emergence and development of computer technology and with the associated intellectual activities resulting in new fields of inquiry such as systems science, information science, decision analysis, or artificial intelligence.

Advances in computer technology have been steadily extending our capabilities for coping with systems of an increasingly broad range, including systems that were previously intractable to us by virtue of their nature and complexity. While the level of complexity we can manage continues to increase, we begin to realize that there are fundamental limits in this respect. As a consequence, we begin to understand that the necessity for simplification of systems, many of which have become essential for characterizing certain currently relevant problem situations, is often unavoidable. In general, a good simplification should minimize the loss of information relevant to the problem of concern. Information and complexity are thus closely interrelated.

One way of simplifying a very complex system—perhaps the most significant one—is to allow some degree of uncertainty in its description. This entails an appropriate aggregation or summary of the various entities within the system. Statements obtained from this simplified system are less precise (certain), but their relevance to the original system is fully maintained. That is, the information loss that is necessary for reducing the complexity of the system to a manageable level is expressed in uncertainty. The concept of uncertainty is thus connected with both complexity and information.

It is now realized that there are several fundamentally different types of uncertainty and that each of them plays a distinct role in the simplification problem. A mathematical formulation within which these various types of uncertainty can be properly characterized and investigated is now available in terms of the theory of fuzzy sets and fuzzy measures.

The primary purpose of this book is to bring this new mathematical formalism into the education system, not merely for its own sake, but as a basic framework for characterizing the full scope of the concept of uncertainty and its relationship to the increasingly important concepts of information and complexity. It should be stressed that these concepts arise in virtually all fields of inquiry; the usefulness of the mathematical framework presented in this book thus transcends the artificial boundaries of the various areas and specializations in the sciences and professions. This book is intended, therefore, to make an understanding of this mathematical formalism accessible to students and professionals in a broad range of disciplines. It is written specifically as a text for a one-semester course at the graduate or upper division undergraduate level that covers the various issues of uncertainty, information, and complexity from a broad perspective based on the formalism of fuzzy set theory. It is our hope that this book will encourage the initiation of new courses of this type in the various programs of higher education as well as in programs of industrial and continuing education. The book is, in fact, a by-product of one such graduate level course, which has been taught at the State University of New York at Binghamton for the last three years.

No previous knowledge of fuzzy set theory or information theory is required for an understanding of the material in this book, thus making it a virtually self-contained text. Although we assume that the reader is familiar with the basic notions of classical (nonfuzzy) set theory, classical (two-valued) logic, and probability theory, the fundamentals of these subject areas are briefly overviewed in the book. In addition, the basic ideas of classical information theory (based on the Hartley and Shannon information measures) are also introduced. For the convenience of the reader, we have included in Appendix B a glossary of the symbols most frequently used in the text.

Chapters 1–3 cover the fundamentals of fuzzy set theory and its connection with fuzzy logic. Particular emphasis is given to a comprehensive coverage of operations on fuzzy sets (Chap. 2) and to various aspects of fuzzy relations (Chap. 3). The concept of general fuzzy measures is introduced in Chap. 4, but the main focus of this chapter is on the dual classes of belief and plausibility measures along with some of their special subclasses (possibility, necessity, and probability measures); this chapter does not require a previous reading of Chapters 2 and 3. Chapter 5 introduces the various types of uncertainty and discusses their relation to information and complexity. Measures of the individual types of uncertainty are investigated in detail and proofs of the uniqueness of some of these are included in Appendix A. The classical information theory (based on the Hartley and Shannon measures of uncertainty) is overviewed, but the major emphasis is given to the new measures of uncertainty and information that have emerged from fuzzy set theory. While Chapters 1–5 focus on theoretical developments, Chap. 6 offers a brief look at some of the areas in which successful applications of this mathematical formalism have been made. Each section of Chap. 6 gives a brief overview of a major area of application along with some specific illustrative examples. We

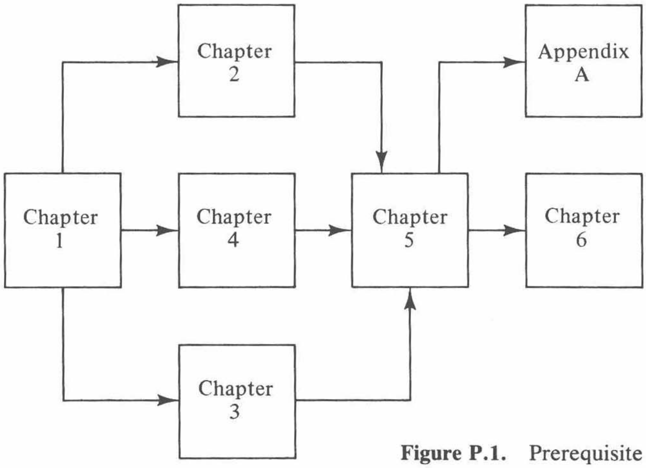

Figure P.1. Prerequisite dependencies among chapters of this book.

have attempted to provide the reader with a flavor of the numerous and diverse areas of application of fuzzy set theory and information theory without attempting an exhaustive study of each one. Ample references are included, however, which will allow the interested reader to pursue further study in the application area of concern.

The prerequisite dependencies among the individual chapters are expressed by the diagram in Fig. P.1. It is clear that the reader has some flexibility in studying the material; for instance, the chapters may be read in order, or the study of Chap. 4 may preceed that of Chaps. 2 and 3.

In order to avoid interruptions in the main text, virtually all bibliographical, historical, and other side remarks are incorporated into the notes that follow each individual chapter. These notes are uniquely numbered and are only occasionally referred to in the text.

When the book is used at the undergraduate level, coverage of some or all of the proofs of the various mathematical theorems may be omitted, depending on the background of the students. At the graduate level, on the other hand, we encourage coverage of most of these proofs in order to effect a deeper understanding of the material. In all cases, the relevance of the material to the specific area of student interest or study can be emphasized with additional application-oriented readings; the notes to Chap. 6 contain annotated references to guide in the selection of such readings from the literature.

Chapters 1–5 are each followed by a set of exercises, which are intended to enhance an understanding of the material presented in the chapter. The solutions to a selected subset of these exercises are provided in the instructor's manual; the remaining exercises are left unanswered so as to be suitable for examination use. Further suggestions for the use of this book in the teaching context can be found in the instructor's manual.

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The emergence of this book was greatly influenced by contacts with many colleagues all over the world. In the area of fuzzy sets, the main influence was Lotfi A. Zadeh, the founder of fuzzy set theory as well as some of the key contributors to the theory, particularly Wyllis Bandler, Didier Dubois, Brian R. Gaines, Ladislav J. Kohout, Maria Nowakowska, Henri Prade, Ronald R. Yager, and Hans-J. Zimmermann. As far as the concepts of uncertainty, information, and complexity are concerned, the main influence was the late W. Ross Ashby, a pioneer of the use of information theory in the investigation of complex systems, and Ronald Christensen, who achieved a great mastery in the use of the minimum and maximum entropy principles. In addition, some material in Chapter 5 was influenced by contacts with Gerrit Broekstra, Roger Conant, and Klaus Krippendorff.

The book is based upon class notes prepared for a course offered by the Systems Science Department of the Thomas J. Watson School of Engineering, Applied Science, and Technology, State University of New York at Binghamton during the last few years. We are grateful to the faculty of the department for supporting this innovative course.

A draft of the book manuscript was critically evaluated by students in the Spring 1986 class. In particular, we received useful comments from Angela Bartlett, Kevin Hufford, Marlene Kaye, Efiok Otudor, Kathy Pendleton, and Dennis Rookwood. We are especially grateful to Mark Wierman; in addition to his many constructive comments, he solved almost all exercises in the book and prepared a few useful APL programs for solving some of the problems involved in the exercises. These solutions as well as programs are included in the Instructor's Guide. Our special thanks also go to Marlene Kaye; she was extremely helpful in proofreading the manuscript and Instructor's Guide and in making creative suggestions for the cover design of this book. Finally, we are grateful to Bonnie Cornick for her excellent typing of the Instructor's Guide.

The book contains some original results, particularly in the areas of uncertainty and information measures. Some of the research that led to these results was supported by the National Science Foundation under Research Grants IST-8401220, IST-8544191, and IST-8644676.

CONTENTS

1

CRISP SETS AND FUZZY SETS

1.1 INTRODUCTION

The process and progress of knowledge unfolds into two stages: an attempt to know the character of the world and a subsequent attempt to know the character of knowledge itself. The second reflective stage arises from the failures of the first; it generates an inquiry into the possibility of knowledge and into the limits of that possibility. It is in this second stage of inquiry that we find ourselves today. As a result, our concerns with knowledge, perceptions of problems and attempts at solutions are of a different order than in the past. We want to know not only specific facts or truths but what we can and cannot know, what we do and do not know, and how we know at all. Our problems have shifted from questions of how to cope with the world (how to provide ourselves with food, shelter, and so on), to questions of how to cope with knowledge (and ignorance) itself. Ours has been called an ''information society,'' and a major portion of our economy is devoted to the handling, processing, selecting, storing, disseminating, protecting, collecting, analyzing, and sorting of information, our best tool for this being, of course, the computer.

Our problems are seen in terms of decision, management, and prediction; solutions are seen in terms of faster access to more information and of increased aid in analyzing, understanding and utilizing the information that is available and in coping with the information that is not. These two elements, large amounts of information coupled with large amounts of uncertainty, taken together constitute the ground of many of our problems today: complexity. As we become aware of how much we know and of how much we do not know, as information and uncertainty themselves become the focus of our concern, we begin to see our problems as centering around the issue of complexity.

The fact that complexity itself includes both the element of how much we know, or how well we can describe, and the element of how much we do not know, or how uncertain we are, can be illustrated with the simple example of driving a car. We can probably agree that driving a car is (at least relatively) complex. Further, driving a standard transmission or stick-shift car is more complex than driving a car with an automatic transmission, one index of this being that more description is needed to cover adequately our knowledge of driving in the former case than in the latter. Thus, because more knowledge is involved in the driving of a standard-transmission car (we must know, for instance, the revolutions per minute of the engine and how to use the clutch), it is more complex. However, the complexity of driving also involves the degree of our uncertainty; for example, we do not know precisely when we will have to stop or swerve to avoid an obstacle. As our uncertainty increases—for instance, in heavy traffic or on unfamiliar roads—so does the complexity of the task. Thus, our perception of complexity increases both when we realize how much we know and when we realize how much we do not know.

How do we manage to cope with complexity as well as we do, and how could we manage to cope better? The answer seems to lie in the notion of simplifying complexity by making a satisfactory trade-off or compromise between the information available to us and the amount of uncertainty we allow. One option is to increase the amount of allowable uncertainty by sacrificing some of the precise information in favor of a vague but more robust summary. For instance, instead of describing the weather today in terms of the exact percentage of cloud cover (which would be much too complex), we could just say that it is sunny, which is more uncertain and less precise but more useful. In fact, it is important to realize that the imprecision or vagueness that is characteristic of natural language does not necessarily imply a loss of accuracy or meaningfulness. It is, for instance, generally more meaningful to give travel directions in terms of city blocks than in terms of inches, although the former is much less precise than the latter. It is also more accurate to say that it is usually warm in the summer than to say that it is usually 72° in the summer. In order for a term such as *sunny* to accomplish the desired introduction of vagueness, however, we cannot use it to mean precisely 0 percent cloud cover. Its meaning is not totally arbitrary, however; a cloud cover of 100 percent is not sunny and neither, in fact, is a cloud cover of 80 percent. We can accept certain intermediate states, such as 10 or 20 percent cloud cover, as sunny. But where do we draw the line? If, for instance, any cloud cover of 25 percent or less is considered sunny, does this mean that a cloud cover of 26 percent is not? This is clearly unacceptable since 1 percent of cloud cover hardly seems like a distinguishing characteristic between sunny and not sunny. We could, therefore, add a qualification that any amount of cloud cover 1 percent greater than a cloud cover already considered to be sunny (that is, 25 percent or less) will also be labeled as sunny. We can see, however, that this definition eventually leads us to accept all degrees of cloud cover as sunny, no matter how gloomy the weather looks! In order to resolve this paradox, the term *sunny* may introduce vagueness by allowing some sort of gradual transition from degrees of cloud cover that are considered to be sunny and those that are not. This is, in fact, precisely the basic concept of the *fuzzy set,* a concept that is both simple and intuitively pleasing and that forms, in essence, a generalization of the classical or *crisp set.*

The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and nonmembers (those that certainly do not). A sharp, unambiguous distinction exists between the members and nonmembers of the class or category represented by the crisp set. Many of the collections and categories we commonly employ, however (for instance, in natural language), such as the classes of tall people, expensive cars, highly contagious diseases, numbers much greater than 1, or sunny days, do not exhibit this characteristic. Instead, their boundaries seem vague, and the transition from member to nonmember appears gradual rather than abrupt. Thus, the fuzzy set introduces vagueness (with the aim of reducing complexity) by eliminating the sharp boundary dividing members of the class from nonmembers. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set. Thus, individuals may belong in the fuzzy set to a greater or lesser degree as indicated by a larger or smaller membership grade. These membership grades are very often represented by real-number values ranging in the closed interval between 0 and 1. Thus, a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0 percent, .8 to a cloud cover of 20 percent, .4 to a cloud cover of 30 percent and 0 to a cloud cover of 75 percent. These grades signify the degree to which each percentage of cloud cover approximates our subjective concept of *sunny,* and the set itself models the semantic flexibility inherent in such a common linguistic term. Because full membership and full nonmembership in the fuzzy set can still be indicated by the values of 1 and 0, respectively, we can consider the crisp set to be a restricted case of the more general fuzzy set for which only these two grades of membership are allowed.

Research on the theory of fuzzy sets has been abundant, and in this book we present an introduction to the major developments of the theory. There are, however, several types of uncertainty other than the type represented by the fuzzy set. The classical probability theory, in fact, represents one of these alternative and distinct forms of uncertainty. Understanding these various types of uncertainty and their relationships with information and complexity is currently an area of active and promising research. Therefore, in addition to offering a thorough introduction to the fuzzy set theory, this book provides an overview of the larger framework of issues of uncertainty, information, and complexity and places the fuzzy set theory within this framework of mathematical explorations.

In addition to presenting the theoretical foundations of fuzzy set theory and associated measures of uncertainty and information, the last chapter of this book offers a glimpse at some of the successful applications of this new conceptual framework to real-world problems. As general tools for dealing with complexity independent of the particular content of concern, the theory of fuzzy sets and the various mathematical representations and measurements of uncertainty and information have a virtually unrestricted applicability. Indeed, possibilities for application include any field that examines how we process or act on information, make decisions, recognize patterns, or diagnose problems or any field in which the complexity of the necessary knowledge requires some form of simplification. Successful applications have, in fact, been made in fields as numerous and diverse as engineering, psychology, artificial intelligence, medicine, ecology, decision theory, pattern recognition, information retrieval, sociology, and meteorology. Few fields remain, in fact, in which conceptions of the major problems and obstacles have not been reformulated in terms of the handling of information and uncertainty. While the diversity of successful applications has thus been expanding rapidly, the theory of fuzzy sets in particular and the mathematics of uncertainty and information in general have been achieving a secure identity as valid and useful extensions of classical mathematics. They will undoubtedly continue to constitute an important framework for further investigations into rigorous representations of uncertainty, information, and complexity.

1.2 CRISP SETS: AN OVERVIEW

This text is devoted to an examination of fuzzy sets as a broad conceptual framework for dealing with uncertainty and information. The reader's familiarity with the basic theory of crisp sets is assumed. Therefore, this section is intended to serve simply to refresh the basic concepts of crisp sets and to introduce notation and terminology useful for our discussion of fuzzy sets.

Throughout this book, sets are denoted by capital letters and their members by lower-case letters. The letter X denotes the universe of discourse, or *universal set*. This set contains all the possible elements of concern in each particular context or application from which sets can be formed. Unless otherwise stated, X is assumed in this text to contain a finite number of elements.

To indicate that an individual object x is a *member* or *element* of a set A, we write

$$x \in A.$$

Whenever x is not an element of a set A, we write

$$x \notin A.$$

A set can be described either by naming all its members (the *list method*) or by specifying some well-defined properties satisfied by the members of the set (the *rule method*). The list method, however, can be used only for finite sets. The set A whose members are $a_1, a_2, \ldots, a_n$ is usually written as

$$A = \{a_1, a_2, \ldots, a_n\},$$

and the set B whose members satisfy the properties $P_1, P_2, \ldots, P_n$ is usually

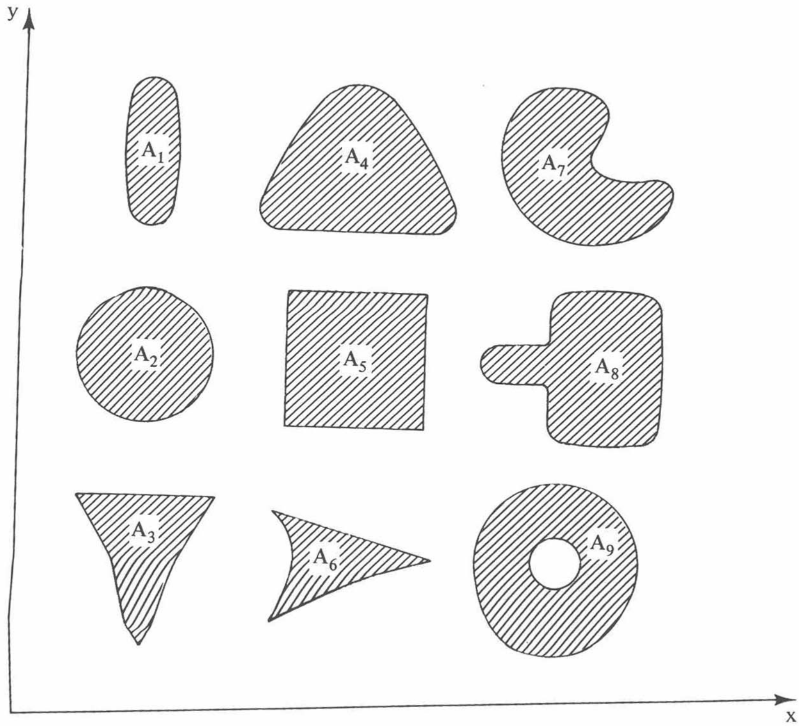

Figure 1.1. Example of sets in $\mathbb{R}^2$ that are either convex (A_1–A_5) or nonconvex (A_6–A_9).

written as

$$B = \{b \mid b \text{ has properties } P_1, P_2, \ldots, P_n\},$$

where the symbol | denotes the phrase "such that."

An important and frequently used universal set is the set of all points in the n-dimensional Euclidean vector space $\mathbb{R}^n$ (i.e., all n-tuples of real numbers). Sets defined in terms of $\mathbb{R}^n$ are often required to possess a property referred to as convexity. A set A in $\mathbb{R}^n$ is called *convex* if, for every pair of points*

$$\mathbf{r} = (r_i \mid i \in \mathbb{N}_n) \quad \text{and} \quad \mathbf{s} = (s_i \mid i \in \mathbb{N}_n)$$

in A and every real number λ between 0 and 1, exclusively, the point

$$\mathbf{t} = (\lambda r_i + (1 - \lambda)s_i \mid i \in \mathbb{N}_n)$$

is also in A. In other words, a set A in $\mathbb{R}^n$ is convex if, for every pair of points **r** and **s** in A, all points located on the straight line segment connecting **r** and **s** are also in A. Examples of convex and nonconvex sets in $\mathbb{R}^2$ are given in Fig. 1.1.

* $\mathbb{N}$ subscripted by a positive integer is used in this text to denote the set of all integers from 1 through the value of the subscript; that is, $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

A set whose elements are themselves sets is often referred to as a *family of sets*. It can be defined in the form

$$\{A_i \mid i \in I\},$$

where i and I are called the *set identifier* and the *identification set,* respectively. Because the index i is used to reference the sets A_i, the family of sets is also called an *indexed set*.

If every member of set A is also a member of set B—that is, if $x \in A$ implies $x \in B$—then A is called a *subset* of B, and this is written as

$$A \subseteq B.$$

Every set is a subset of itself and every set is a subset of the universal set. If $A \subseteq B$ and $B \subseteq A$, then A and B contain the same members. They are then called *equal* sets; this is denoted by

$$A = B.$$

To indicate that A and B are not equal, we write

$$A \neq B.$$

If both $A \subseteq B$ and $A \neq B$, then B contains at least one individual that is not a member of A. In this case, A is called a *proper subset* of B, which is denoted by

$$A \subset B.$$

The set that contains no members is called the *empty set* and is denoted by $\varnothing$. The empty set is a subset of every set and is a proper subset of every set except itself.

The process by which individuals from the universal set X are determined to be either members or nonmembers of a set can be defined by a *characteristic,* or *discrimination, function*. For a given set A, this function assigns a value $\mu_A(x)$ to every $x \in X$ such that

$$\mu_A(x) = \begin{cases} 1 & \text{if and only if } x \in \text{A}, \\ 0 & \text{if and only if } x \notin \text{A}. \end{cases}$$

Thus, the function maps elements of the universal set to the set containing 0 and 1. This can be indicated by

$$\mu_A : X \rightarrow \{0, 1\}.$$

The number of elements that belong to a set A is called the *cardinality* of the set and is denoted by $|A|$. A set that is defined by the rule method may contain an infinite number of elements.

The family of sets consisting of all the subsets of a particular set A is referred to as the *power set* of A and is indicated by $\mathcal{P}(A)$. It is always the case that

$$|\mathcal{P}(A)| = 2^{|A|}.$$

The *relative complement* of a set A with respect to set B is the set containing all the members of B that are not also members of A. This can be written $B - A$. Thus,

$$B - A = \{x \mid x \in B \quad \text{and} \quad x \notin A\}.$$

If the set B is the universal set, the complement is absolute and is usually denoted by $\overline{A}$. Complementation is always *involutive*; that is, taking the complement of a complement yields the original set, or

$$\overline{\overline{A}} = A.$$

The absolute complement of the empty set equals the universal set, and the absolute complement of the universal set equals the empty set. That is,

$$\overline{\varnothing} = X,$$

and

$$\overline{X} = \varnothing.$$

The *union* of sets A and B is the set containing all the elements that belong either to set A alone, to set B alone, or to both set A and set B. This is denoted by $A \cup B$. Thus,

$$A \cup B = \{x \mid x \in A \quad \text{or} \quad x \in B\}.$$

The union operation can be generalized for any number of sets. For a family of sets $\{A_i \mid i \in I\}$, this is defined as

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}.$$

The union of any set with the universal set yields the universal set, whereas the union of any set with the empty set yields the set itself. We can write this as

$$A \cup X = X$$

and

$$A \cup \varnothing = A.$$

Because all the elements of the universal set necessarily belong either to a set A or to its absolute complement, $\overline{A}$, the union of A and $\overline{A}$ yields the universal set. Thus,

$$A \cup \overline{A} = X.$$

This property is usually called the *law of excluded middle*.

The *intersection* of sets A and B is the set containing all the elements belonging to both set A and set B. It is denoted by $A \cap B$. Thus,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The generalization of the intersection for a family of sets $\{A_i \mid i \in I\}$ is defined as

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$