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Edited by A. Dold and B. Eckmann

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Wilhelm Plesken

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PREFACE

In the present notes the theory of orders over Dedekind domains is applied to study group rings of finite groups over the p-adic integers. The presentation grew out of my Habilitationsschrift at the Rheinisch-Westfälische Technische Hochschule Aachen, but goes far beyond it. The major part of the material is accessible to anyone who knows the definition of a maximal order and is familiar with the elements of modular representation theory of finite groups.

It was Professor H. Zassenhaus who introduced me to the subject a couple of years ago, and it is fair to say that these notes would have never been written without him. I acknowledge with pleasure that I greatly profited from discussions with H. Benz, H. Jacobinski, G. Michler, H. Pahlings, and K. Roggenkamp. I am very grateful to W. Rump for reading the manuskript and to a referee for pointing out an error in an earlier version, the correction of which lead to generalizations of some results. For typing the manuscript I would like to thank Mrs. D. Burkel, Miss A. Nijenhuis, and Mrs. C. Schneider. Part of the work was done while I held a Heisenberg scholarship. I would like to thank the Deutsche Forschungsgemeinschaft for the opportunities given to me by this grant. Finally I would like to thank the Sonderforschungsbereich für theoretische Mathematik, Bonn, for their hospitality during part of the preparation of the manuscript.

W. Plesken

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CONTENTS

I. Introduction

The knowledge of the structure of the group ring RG of a finite group G over the ring R of integers in a finite extension field K of the p-adic number field $\mathbb{Q}_p$ yields insight into the possible actions of G on abelian groups. Compared with the matrix ring $\Omega^{n\times n}$ over the maximal R-order Ω in a K-division algebra D the group ring RG is little understood in case p divides the order $|G|$ of the group G . The present paper makes one step towards a description of RG in terms of such matrix rings, their twosided ideals, and isomorphisms between certain (finite) factor rings. Such a description more or less allows to read off the isomorphism types of the irreducible RG-lattices and their possible embeddings into one another, as well as a way the projective indecomposable RG-lattices are built up form certain irreducible ones. (An RG-lattice L is called irreducible if $KL := K\otimes_R L$ is an irreducible KG-module.) Among other examples, the principal 2-block of the group ring of $SL_2(q)$, q an odd prime power, over the 2-adic integers, and blocks with cyclic defect groups over the ring of p-adic integers are treated as applications of the general theory.

As BRAUER [Bra 56] points out, the most interesting arithmetic properties of $\Lambda = RG$ are lost when one passes from RG to a maximal R-order in $A = KG$ which contains RG . Since hereditary orders are equally well understood as maximal orders, JACOBINSKI, cf. [Jac 66], [Jac 81], suggests to embed Λ into a certain hereditary R-order in A , which he calls a hereditary hull of Λ . In Chapter II of this paper the hereditary orders are replaced by the considerably more general graduated orders or even graduable orders, cf. ZASSENHAUS [Zas 75], as the "well-known" R-orders in which Λ might be embedded and a graduated hull of Λ is defined. Of course Λ need not be a group ring for this but only an arbitrary R-order in a semisimple K-algebra A . Graduated orders in A are essentially defined by the property that they contain a full set of primitive orthogonal idempotents of A (cf. (II.1) for the exact definition) and are distinguished by the property that they are determined by their irreducible lattices, cf. (II.8).

There are several advantages of the replacement of hereditary hulls by graduated hulls. Graduated orders can still be described by comparatively few invariants, cf. (II.2) and (II.3). A graduated hull is generally a better approximation (from above) to the order Λ than a hereditary hull. Indeed, it follows from (II.8) that there exists only one unique graduated hull under certain conditions in which case it is the intersection of all hereditary hulls or equivalently of all maximal R-orders containing Λ . Thirdly JACOBINSKI's conductor formula for a hereditary R-order containing a group ring $\Lambda = RG$, cf. [Jac 66], [Jac 81], can be generalized in two respects: The hereditary order can be replaced by a graduated order containing Λ and Λ need not be a group ring but only a "selfdual order" with respect to a generalized trace bilinear form on the separable K-algebra A , (cf. (III.1) for a proper definition). The conductor of some overorder Γ of a selfdual order Λ , i.e. the biggest Γ-ideal contained in Λ , turns out to be the dual Γ^* of Γ with respect to the generalized trace bilinear form of A belonging to Λ , cf. (III.7). The conductor formula (III.8) gives an explicit description of the conductor of Γ in Λ in terms of the structural invariants of Γ in case Γ is a graduated order. The idea is that a graduated hull Γ of Λ restricts the possibilities for Λ considerably more than a hereditary hull, since $\Gamma^* \subseteq \Lambda \subseteq \Gamma$, and that the conductor formula makes it possible to discuss these inclusions. Note, if $\Gamma = \bigoplus_{s=1}^{h} \varepsilon_s \Lambda$, where $\varepsilon_1, \ldots, \varepsilon_h$ are the central primitive idempotents of A , then the conductor is also equal to $\bigoplus_{s=1}^{h} (\varepsilon_s \Lambda \cap \Lambda)$. Other examples of selfdual orders apart from group rings are twisted group rings, cf. (III.2), (III.4), and orders of the form $\varepsilon \Lambda \varepsilon$ where ε is an idempotent in a selfdual R-order Λ .

These ideas can most successfully be applied to group rings RG where R is a sufficiently large unramified (finite) extension of the p-adic integers $\mathbb{Z}_p$. But of course the primary group theoretical interest is concentrated in the group rings $\mathbb{Z}_p G$ over the p-adic integers, for instance because they determine the possible actions of G on finite abelian p-groups. To develop the tools for the Galois descent from RG to $\mathbb{Z}_p G$ Chapter II.b discusses graduable orders, i.e. orders which become graduated orders after a sufficiently big unramified ground ring extension, cf. Definition (II.13). Among other characterizations Theorem (II.16), cf. also (III.12), gives an easily applicable criterion for $\varepsilon_s \Lambda$ to be a graduable order: Certain modified (cf.(III.10)) decomposition numbers have to be equal to zero or one. As for the Galois descent Theorem (II.20) gives a satisfactory answer: A graduable order Γ in a central simple algebra is determined up to isomorphism by $\Gamma' = R' \otimes_R \Gamma$ and the natural embedding $\Gamma/\mathrm{Jac}(\Gamma) \hookrightarrow \Gamma'/\mathrm{Jac}(\Gamma')$ for some arbitrary unramified extension R' of R . Many things done with graduated orders in later chapters, could also be done with graduable orders, e.g. the conductor formula could be proved for graduable overorders. Since this can easily be obtained from the stated results and (II.20) it is usually not mentioned explicitely. This is about as far as the general theory is developed in Chapter II and III, the main results being the characterizations of graduated and graduable orders in (II.8) and (II.16), the essential uniqueness for the Galois descent for graduated to graduable orders in (II.20) and of course the conductor formula (III.8). It should be noted, however, that Chapter II.b can be skipped upon first reading, since the other chapters are kept essentially independent of this part.

Chapter IV discusses selfdual orders Λ for which $\bigoplus_{s=1}^{h} \varepsilon_s \Lambda$ (ε_s as above) is a graduated R-order in A . This is essentially tantamount to demanding that the decomposition numbers of Λ are all equal to 0 and 1 , (and R "big enough") cf. (III.12) or Chapter IIb for the exact conditions. In this situation, the projective indecomposable Λ-lattices P_i have the property that $\varepsilon_s P_i$ is irreducible or 0 by Brauer's reciprocity. The major part of Chapter IV is a careful analysis of the embedding of P_i in the completely decomposable Λ-lattice $\bigoplus_{s=1}^{h} \varepsilon_s P_i$ by investigating the "amalgamating factors" $(\bigoplus_{s=1}^{h} \varepsilon_s P_i)/P_i$ and $P_i/\bigoplus_{s=1}^{h} (\varepsilon_s P_i \cap P_i)$ of P_i . For most applications in later chapters (IV.1), (IV.7), and (IV.10) to (IV.12) are sufficient.

The last four chapters contain applications of the theory developed in the first chapters to the explicite computation of group rings in the sense described at the beginning of this introduction. The general procedure consists of two steps: At first one determines $\bigoplus_{s=1}^{h} \varepsilon_s \Lambda$ which essentially amounts to finding the sublattices of all irreducible Λ-lattices. Then one has to find the embedding of Λ into $\bigoplus_{s=1}^{h} \varepsilon_s \Lambda$, i.e. to see how the $\varepsilon_s \Lambda$ (resp. $\varepsilon_s P_i$) are amalgamted to Λ (resp. P_i). In practice these two steps are not performed one after the other. Indeed, it is part of the idea to use step 2 to compare the various epimorphic images $\varepsilon_s \Lambda$ of Λ and thereby getting information to perform step 1. Of course, step 2 can only be completed after step 1 is fully carried out.

Chapter V discusses block ideals of group rings with all diagonal Cartan numbers equal to 2. In case all Frobenius characters are real a complete description of the ringtheoretical structure of these blocks is given. In Chapter VI step 1 of the procedure outlined above is carried out for the principal blocks of various group rings with decomposition numbers 0 and 1 (i.e. the sublattices of the irreducible lattices in the block are determined). The examples are the symmetric group S_{10} at the prime 5, $SL_3(3)$ at the prime 2, $SL_3(4)$ at the prime 3, and the Mathieu group M_{11} at the primes 2 and 3. The usual information one starts out with in Chapter VI and VII is the character table of the group and the decomposition numbers. Of course, some insight in the subgroup structure is always a help. In Chapter VII the principal blocks of RG, $G = SL_2(q)$ with q an odd prime power and R the ring of 2-adic integers are completely determined in the above sense. For instance for $q \equiv \pm 1 \pmod 8$, it turns out that all irreducible lattices of this block are uniserial except possibly for the ones of R-rank q belonging to the Steinberg character.

Finally blocks with cyclic defect groups are described in Chapter VIII, where the Dedekind ring R is first assumed to be a sufficiently large unramified extension of the p-adic integers. In particular a generalization of Brauer's Theorem 11 in [Bra 41] on the sublattices of the irreducible lattices in a block of defect 1 is proved, cf. (VIII.3). This was mentioned as an open problem at the end of Dade's basic paper [Dad 66] on blocks with cyclic defect groups, cf. also [Fei 82]. As a corollary one obtains linear congruences for the central characters modulo $|D|R$ where D is a cyclic defect group of the block. In a final step the description is extended to blocks with cyclic defect groups over the ring of p-adic integers.

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Some comments on the earlier literature might be useful. Graduated orders turn up at various places under various names; e.g. in [Jat 74] their global dimensions are investigated, cf. also [WiR 82]; in [ZaK 77] the graduated orders of finite representation type were characterized, cf. also [Rum 81a], [Rum 81b]; the connection of graduated orders with orders Λ whose irreducible lattices have a distributive lattice of Λ-sublattices was observed in [Ple 77], cf. also [Zas 69], [Ple 80a], [Rum 81b]. The major part of the theory developed here is already contained in the author's Habilitationsschrift [Ple 80a] for group rings RG under the two additional assumptions that the quotient field K of R is a splitting field for the group G and that the decomposition numbers are all equal to 0 and 1; cf. also [Ple 80b] for a short summary of those results. The expression for $\bigoplus_{s=1}^{h} (\varepsilon_s RG \cap RG)$ there was obtained as a consequence of Schur's relations, cf. [Hup 67] page 477, which allow to express matrix units in the group algebra KG as linear combinations of the group elements with coefficients coming from irreducible matrix representations of G, cf. also [Ser 77]. The presentation given here was influenced (at a late stage) by Jacobinski's lectures on hereditary orders, cf. [Jac 81] and his discussion of blocks of defect 1 . For other approaches to special blocks of group rings cf. also [Rog 80a], [Rog 80b], [Rog 81].

The used notation is standard. Groups usually act from the left side, module homomorphisms are written on the left, right resp. left side for right, left, resp. bi-modules. All rings have a unit and all modules are unital. If not stated otherwise, all modules are left finitely generated modules (and hence also viewed as right modules over their endomorphism rings). A general reference for orders over Dedekind domains is [Rei 75].

II. Graduated and graduable orders

In this chapter R is a complete local Dedekind domain with quotient field K and maximal ideal p. Furthermore D is a (finite dimensional) separable division algebra over K, Ω the maximal R-order of D, $\mathfrak{P} = \mathrm{Jac}(\Omega)$ the maximal ideal of Ω (cf. e.g. [Jac 81], [Rei 75]). A will denote a separable K-algebra containing an R-order Λ such that $K\Lambda = A$.

II.a. Definition and characterization of graduated orders

(II.1) Definition. Λ *is called a graduated order if there exist orthogonal (primitive) idempotents* $\varepsilon_1,\ldots,\varepsilon_t$ *in* Λ *with* $1 = \varepsilon_1+\ldots+\varepsilon_t$ $(1=1_\Lambda)$ *such that* $\varepsilon_i\Lambda\varepsilon_i$ *is a maximal order in* $\varepsilon_i A\varepsilon_i$ *for* $i = 1,\ldots,t$.

In particular, the maximal orders and the hereditary orders are among the graduated orders. Note, if K is a splitting field of A, then Λ is a graduated order iff Λ contains a complete set of orthogonal idempotents of A. If A decomposes into a direct sum of minimal twosided ideals $A_1,\ldots,A_h$, then Λ is a graduated order if and only if Λ is the direct sum of the $\Lambda_s = \Lambda \cap A_s$ and each of the Λ_s is a graduated order in A_s $(s = 1,\ldots,h)$. Therefore it suffices to investigate graduated orders in simple algebras. Let A be the matrix ring $D^{n\times n}$ of degree n over D in the sequel. To describe the obvious examples of graduated orders, the following notation is useful: For $\tilde{n} = (n_1,\ldots,n_t) \in \mathbb{N}^{1\times t}$ with $n = \sum_{i=1}^{t} n_i$ and $M = (m_{ij}) \in \mathbb{Z}^{t\times t}$ let

$\Lambda(\Omega,\tilde{n},M) = \{ (a_{ij}) \mid a_{ij} \in (\mathfrak{p}^{m_{ij}})^{n_i \times n_j},\ 1 \leq i,j \leq t\} \subseteq D^{n\times n} = A.$

I.e., the elements of $\Lambda(\Omega,\tilde{n},M)$ are $n\times n$-matrices over D partitioned into $n_i \times n_j$-submatrices a_{ij} the entries of which lie in $\mathfrak{p}^{m_{ij}}$. If $\Omega = R$, one writes simply $\Lambda(\tilde{n},M)$ instead of $\Lambda(R,\tilde{n},M)$. Obviously $\Lambda = \Lambda(\Omega,\tilde{n},M)$ is an order in A, iff $M = (m_{ij})$ satisfies $m_{ii} = 0$ and $m_{ij} + m_{jk} \geq m_{ik}$ for $1 \leq i,j,k \leq t$. In this case Λ is already a graduated order, the matrix graduation being induced by the standard diagonal idempotents.

(II.2) Definition. *A graduated order Λ in $A = D^{n\times n}$ is said to be in standard form, if there exist $\tilde{n} = (n_1,\ldots,n_t) \in \mathbb{N}^{1\times t}$ with $n_1 + \ldots + n_t = n$, $M = (m_{ij}) \in \mathbb{Z}_{\geq 0}^{t\times t}$ such that $\Lambda = \Lambda(\Omega,\tilde{n},M)$ and*

$$(*)\quad \begin{cases} m_{ij} + m_{jk} \geq m_{ik} \\ m_{ii} = 0 \\ m_{ij} + m_{ji} > 0 \quad (i \neq j) \end{cases}$$

for $i \leq i,j,k \leq t$. In this case M is called the exponent matrix of Λ and $\tilde{n}$ the dimension type of Λ.

Since any two complete sets of primitive idempotents of A can be conjugated by nonsingular matrices of A into each other and since the ideals of Ω are the powers of $\mathfrak{p}$, one easily verifies (II.3), cf. e.g. [Zas 75].

(II.3) Remark. *Each graduated order Λ in $A = D^{n\times n}$ is isomorphic to a graduated order in standard form.*

For a graduated order in standard form it is easy to compute the irreducible lattices. Therefore the proof of (II.4) is left to the reader, cf. e.g. [Zas 75], [Ple 77], [Ple 80a], [Rum 81b].

(II.4) Remark. Let $\Lambda = \Lambda(\Omega,\tilde{n},M)$ be a graduated order of standard type in A .

(i) The Jacobson radical of Λ is given by $Jac(\Lambda) = \Lambda(\Omega,\tilde{n},M+I_t)$, where I_t is the $t\times t$-unit matrix.

(ii) $\Lambda/Jac(\Lambda) \cong \bigoplus_{i=1}^{t} (\Omega/\mathfrak{P})^{n_i\times n_i}$.

(iii) Let $V = D^{n\times 1}$ be the standard irreducible module of $A = D^{n\times n}$. The set $\mathfrak{Z}(V)$ of all Λ-lattices $\neq 0$ in V is given by all

$$L(\tilde{m}) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix} \mid a_i \in (\mathfrak{P}^{m_i})^{n_i\times 1} \right\} \subseteq V$$

with $\tilde{m} = (m_1,\ldots,m_t)^{tr} \in \mathbb{Z}^{n\times 1}$ satisfying

(**) $m_{ij} + m_j \geq m_i \quad (1 \leq i,j \leq t)$

(iv) Two Λ-lattices $L(\tilde{m}_1)$, $L(\tilde{m}_2) \in \mathfrak{Z}(V)$ are isomorphic if and only if $\tilde{m}_1 - \tilde{m}_2$ is a multiple of $(1,\ldots,1)^{tr} \in \mathbb{Z}^{t\times 1}$, i.e. $L(\tilde{m}_1) = L(\tilde{m}_2)\mathfrak{P}^{\alpha}$ for some $\alpha \in \mathbb{Z}$.

(v) Each projective indecomposable Λ-lattice is irreducible and isomorphic to $L(M_i)$ for some $i = 1,\ldots,t$, where M_i is the i-th column of M .

(vi) Define $S_i = L(M_i)/Jac(\Lambda)L(M_i)$ for $i = 1,\ldots,t$. Then $S_1,\ldots,S_t$ form a set of representatives of the simple Λ-(torsion)modules. For solutions $\tilde{m}_1,\tilde{m}_2$ of (**) with $L(\tilde{m}_1) \subseteq L(\tilde{m}_2)$ the i-th coefficient of $\tilde{m}_1 - \tilde{m}_2$ is the multiplicity of S_i in a composition series of the Λ-module $L(\tilde{m}_2)/L(\tilde{m}_1)$ for $i = 1,\ldots,t$.

(vii) Each injective indecomposable Λ-lattice is irreducible and isomorphic to $L({}_iM)$ for some $i = 1,\ldots,t$, where ${}_iM$ is the i-th column of $-M^{tr}$. (A Λ-lattice L is injective, if $\mathrm{Hom}_R(L,R)$ is a projective right Λ-lattice.)

(viii) The two-sided (fractional) ideals of Λ in A are given by $\Lambda(\Omega,\tilde{n},N)$, where $N = (n_{ij}) \in \mathbb{Z}^{t\times t}$ satisfies $m_{ij} + n_{jk} \geq n_{ik}$ and $n_{ij} + m_{jk} \geq n_{ik}$ for $1 \leq i,j,k \leq t$.

Note, by part (ii) of this remark the dimension type $\tilde{n}$ depends only on the isomorphism type of Λ (up to the order of the n_i). By (II.4) (iii) and (iv) the irreducible Λ-lattices of the graduated order $\Lambda = \Lambda(\Omega,\tilde{n},M)$ fall into finitely many isomorphism classes the number of which only depends on M . A convenient set of representatives can be produced as follows: For $\tilde{m} = (m_1,\dots,m_t)^{tr} \in \mathbb{Z}^{t\times 1}$ satisfying (**) let

$$\mathfrak{R}(\tilde{m}) = \{L(\tilde{l}) \mid \tilde{l} = (l_1,\dots,l_t)^{tr} \in \mathbb{Z}^{t\times 1} \text{ satisfying } (**), \; l_i \geq m_i \text{ for } i=1,\dots,t\,;\; l_j = m_j \text{ for one } j=1,\dots,t\} =$$
$$= \{L \in \mathfrak{J}(V) \mid L \subseteq L(\tilde{m}),\; L \not\subseteq L(\tilde{m})\mathfrak{P}\} .$$

This set $\mathfrak{R}(\tilde{m})$ can conveniently be computed "layer by layer" as follows:
Compute the (at most t) maximal Λ-sublattices of $L(\tilde{m})$ and drop those which are not in $\mathfrak{R}(\tilde{m})$. Continue with the maximal Λ-sublattices of $L(\tilde{m})$ in $\mathfrak{R}(\tilde{m})$ which are contained in $\mathfrak{R}(\tilde{m})$ to get the second maximal Λ-sublattices of $L(\tilde{m})$ in $\mathfrak{R}(\tilde{m})$ etc.. (Note, $L(\tilde{m})$ contains a maximal sublattice L with $L(\tilde{m})/L \cong A_i$ iff $m_{ij} + m_j > m_i$ for $j = 1,\dots,t$, $j \neq i$.)

Unlike the dimension type $\tilde{n}$ the exponent matrix M of a graduated order Λ is not uniquely defined by the isomorphism type of Λ , because Λ can well be isomorphic to more than one graduated order in standard form. Therefore ZASSENHAUS [Zas 75] introduced structural invariants m_{ijk} of a graduated order Λ in $A = D^{n\times n}$ as follows:
Lift the central primitive idempotents $\bar{\varepsilon}_1,\dots,\bar{\varepsilon}_t$ of $\Lambda/\mathrm{Jac}(\Lambda)$ to orthogonal idempotents $\varepsilon_1,\dots,\varepsilon_t$ of Λ with $1 = \varepsilon_1 + \dots + \varepsilon_t$. Let $\Lambda_{ij} = \varepsilon_i \Lambda \varepsilon_j$ for $i,j = 1,\dots,t$. Then there are nonnegative numbers $m_{ijk} \in \mathbb{Z}$ satisfying

$$\Lambda_{ij}\Lambda_{jk} = \mathfrak{p}^{m_{ijk}}\Lambda_{ik} \quad \text{for} \quad i,j,k = 1,\ldots,t \; .$$

(Note Λ_{ii} is a maximal order for $i = 1,\ldots,t$.) The m_{ijk} satisfy

$$(***) \quad \begin{cases} m_{ijk} + m_{ikl} = m_{ijl} + m_{jkl} \\ m_{iii} = 0 \quad (= m_{iij} = m_{ijj}) \\ m_{iji} > 0 \qquad (j \neq i) \end{cases}$$

for $1 \leq i,j,k,l \leq t$. The first equations of (***) follow from associativity: $(\Lambda_{ij}\Lambda_{jk})\Lambda_{kl} = \Lambda_{ij}(\Lambda_{jk}\Lambda_{kl})$, the second from the property of Λ_{ii} to be an order, and the third from the choice of the idempotents ε_i . If $\varepsilon_1',\ldots,\varepsilon_t'$ is a second set of idempotents of Λ such that the $\varepsilon_i' + \mathrm{Jac}(\Lambda)$ are the central primitive idempotents of $\Lambda/\mathrm{Jac}(\Lambda)$ the structural invariants m_{ijk}' of Λ with respect to the ε_i' are obtained by permuting the indices of the m_{ijk} . Namely there exists an inner automorphism α of Λ and a permutation $\pi \in S_n$ with $\alpha(\varepsilon_i) = \varepsilon_{\pi i}'$ for $i = 1,\ldots,t$. Therefore $m_{ijk} = m_{\pi i,\pi j,\pi k}'$, $1 \leq i,j,k \leq t$. Hence the structural invariants (up to order) depend only on the isomorphism type of the graduated order Λ . Call $\tilde{n} = (n_1,\ldots,n_t)$ the dimension type of Λ , where n_i is the unique natural number with $\Lambda_i \cong \Omega^{n_i \times n_i}$, $i = 1,\ldots,t$.

If $\Lambda = \Lambda(\Omega,\tilde{n},M)$ is a graduated order in standard form, then clearly $\tilde{n}$ is also the dimension type of Λ in the sense just defined and the structural invariants of Λ are given by

$$(****) \quad m_{ijk} = m_{ij} + m_{jk} - m_{ik} \qquad \text{for} \quad 1 \leq i,j,k \leq t \; .$$

<u>(II.5) Lemma.</u> Let $M' = (m_{ij}') \in \mathbb{Z}_{\geq 0}^{t\times t}$ be a second solution of (****). Then there are integers $m_1,\ldots,m_t \in \mathbb{Z}$ with $m_{ij}' = m_{ij} + m_i - m_j$ for $1 \leq i,j \leq t$.

Proof: Let $x_{ij} = m_{ij}' - m_{ij}$ for $1 \leq i,j \leq t$. Then (****) implies

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