FOUNDATIONS OF DIFFERENTIAL GEOMETRY

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VOLUME I

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PREFACE

Differential geometry has a long history as a field of mathematics and yet its rigorous foundation in the realm of contemporary mathematics is relatively new. We have written this book, the first of the two volumes of the *Foundations of Differential Geometry*, with the intention of providing a systematic introduction to differential geometry which will also serve as a reference book.

Our primary concern was to make it self-contained as much as possible and to give complete proofs of all standard results in the foundation. We hope that this purpose has been achieved with the following arrangements. In Chapter I we have given a brief survey of differentiable manifolds, Lie groups and fibre bundles. The readers who are unfamiliar with them may learn the subjects from the books of Chevalley, Montgomery-Zippin, Pontrjagin, and Steenrod, listed in the *Bibliography*, which are our standard references in Chapter I. We have also included a concise account of tensor algebras and tensor fields, the central theme of which is the notion of derivation of the algebra of tensor fields. In the *Appendices*, we have given some results from topology, Lie group theory and others which we need in the main text. With these preparations, the main text of the book is self-contained.

Chapter II contains the connection theory of Ehresmann and its later development. Results in this chapter are applied to linear and affine connections in Chapter III and to Riemannian connections in Chapter IV. Many basic results on normal coordinates, convex neighborhoods, distance, completeness and holonomy groups are proved here completely, including the de Rham decomposition theorem for Riemannian manifolds.

In Chapter V, we introduce the sectional curvature of a Riemannian manifold and the spaces of constant curvature. A more complete treatment of properties of Riemannian manifolds involving sectional curvature depends on calculus of variations and will be given in Volume II. We discuss flat affine and Riemannian connections in detail.

In Chapter VI, we first discuss transformations and infinitesimal transformations which preserve a given linear connection or a Riemannian metric. We include here various results concerning Ricci tensor, holonomy and infinitesimal isometries. We then treat the extension of local transformations and the so-called equivalence problem for affine and Riemannian connections. The results in this chapter are closely related to differential geometry of homogeneous spaces (in particular, symmetric spaces) which are planned for Volume II.

In all the chapters, we have tried to familiarize the readers with various techniques of computations which are currently in use in differential geometry. These are: (1) classical tensor calculus with indices; (2) exterior differential calculus of E. Cartan; and (3) formalism of covariant differentiation $\nabla_X Y$, which is the newest among the three. We have also illustrated, as we see fit, the methods of using a suitable bundle or working directly in the base space.

The *Notes* include some historical facts and supplementary results pertinent to the main content of the present volume. The *Bibliography* at the end contains only those books and papers which we quote throughout the book.

Theorems, propositions and corollaries are numbered for each section. For example, in each chapter, say, Chapter II, Theorem 3.1 is in Section 3. In the rest of the same chapter, it will be referred to simply as Theorem 3.1. For quotation in subsequent chapters, it is referred to as Theorem 3.1 of Chapter II.

We originally planned to write one volume which would include the content of the present volume as well as the following topics: submanifolds; variations of the length integral; differential geometry of complex and Kähler manifolds; differential geometry of homogeneous spaces; symmetric spaces; characteristic classes. The considerations of time and space have made it desirable to divide the book in two volumes. The topics mentioned above will therefore be included in Volume II.

In concluding the preface, we should like to thank Professor L. Bers, who invited us to undertake this project, and Interscience Publishers, a division of John Wiley and Sons, for their patience and kind cooperation. We are greatly indebted to Dr. A. J. Lohwater, Dr. H. Ozeki, Messrs. A. Howard and E. Ruh for their kind help which resulted in many improvements of both the content and the presentation. We also acknowledge the grants of the National Science Foundation which supported part of the work included in this book.

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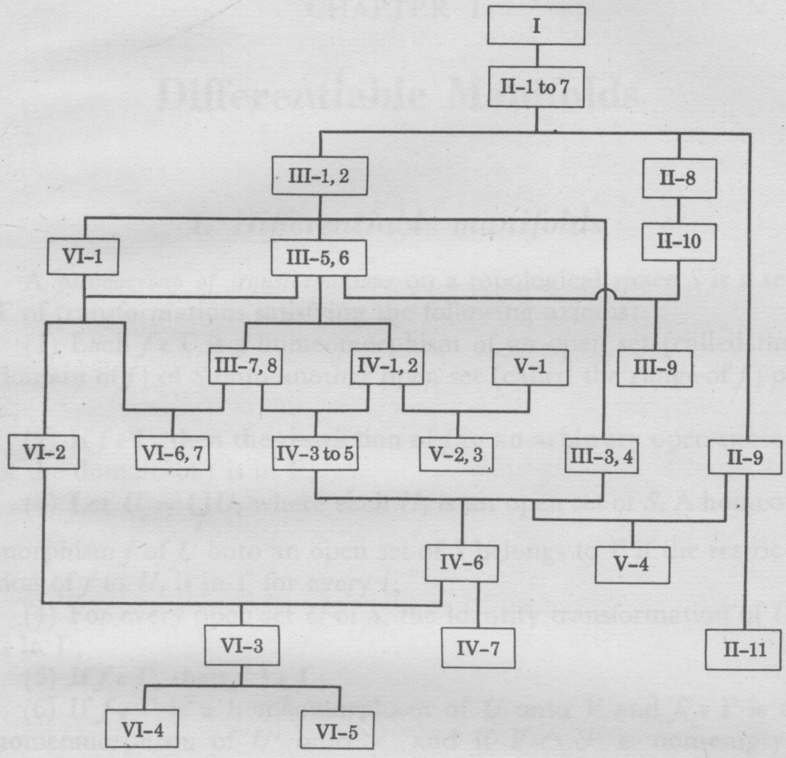
Interdependence of the Chapters and the Sections



Exceptions

Chapter	II:	Theorem 11.8 requires Section II-10.
Chapter	III:	Proposition 6.2 requires Section III-4.
Chapter	IV:	Corollary 2.4 requires Proposition 7.4 in Chapter III.
Chapter	IV:	Theorem 4.1, (4) requires Section III-4 and Proposition 6.2 in Chapter III.
Chapter	V:	Proposition 2.4 requires Section III-7.
Chapter	VI:	Theorem 3.3 requires Section V-2.
Chapter	VI:	Corollary 5.6 requires Example 4.1 in Chapter V.
Chapter	VI:	Corollary 6.4 requires Proposition 2.6 in Chapter IV.
Chapter	VI:	Theorem 7.10 requires Section V-2.

CONTENTS

NOTES

CHAPTER I

Differentiable Manifolds

1. *Differentiable manifolds*

A *pseudogroup of transformations* on a topological space S is a set Γ of transformations satisfying the following axioms:

(1) Each $f \in \Gamma$ is a homeomorphism of an open set (called the domain of f) of S onto another open set (called the range of f) of S;

(2) If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

(3) Let $U = \bigcup_i U_i$ where each U_i is an open set of S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i;

(4) For every open set U of S, the identity transformation of U is in Γ;

(5) If $f \in \Gamma$, then $f^{-1} \in \Gamma$;

(6) If $f \in \Gamma$ is a homeomorphism of U onto V and $f' \in \Gamma$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \circ f$ of $f^{-1}(V \cap U')$ onto $f'(V \cap U')$ is in Γ.

We give a few examples of pseudogroups which are used in this book. Let $\mathbf{R}^n$ be the space of n-tuples of real numbers $(x^1, x^2, \ldots, x^n)$ with the usual topology. A mapping f of an open set of $\mathbf{R}^n$ into $\mathbf{R}^m$ is said to be of *class* C^r, $r = 1, 2, \ldots, \infty$, if f is continuously r times differentiable. By class C^0 we mean that f is continuous. By class C^ω we mean that f is real analytic. The *pseudogroup* $\Gamma^r(\mathbf{R}^n)$ *of transformations of class* C^r of $\mathbf{R}^n$ is the set of homeomorphisms f of an open set of $\mathbf{R}^n$ onto an open set of $\mathbf{R}^n$ such that both f and f^{-1} are of class C^r. Obviously $\Gamma^r(\mathbf{R}^n)$ is a pseudogroup of transformations of $\mathbf{R}^n$. If $r < s$, then $\Gamma^s(\mathbf{R}^n)$ is a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. If we consider only those $f \in \Gamma^r(\mathbf{R}^n)$ whose Jacobians are positive everywhere, we obtain a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. This subpseudogroup, denoted by $\Gamma^r_o(\mathbf{R}^n)$, is called the *pseudogroup of orientation-preserving transformations of class C^r of* $\mathbf{R}^n$. Let $\mathbf{C}^n$ be the space of n-tuples of complex numbers with the usual topology. The *pseudogroup of holomorphic* (i.e., complex analytic) *transformations of* $\mathbf{C}^n$ can be similarly defined and will be denoted by $\Gamma(\mathbf{C}^n)$. We shall identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, when necessary, by mapping $(z^1, \ldots, z^n) \in \mathbf{C}^n$ into $(x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbf{R}^{2n}$, where $z^j = x^j + iy^j$. Under this identification, $\Gamma(\mathbf{C}^n)$ is a subpseudogroup of $\Gamma^r_o(\mathbf{R}^{2n})$ for any r.

An *atlas* of a topological space M compatible with a pseudogroup Γ is a family of pairs (U_i, φ_i), called *charts*, such that

(a) Each U_i is an open set of M and $\bigcup_i U_i = M$;

(b) Each φ_i is a homeomorphism of U_i onto an open set of S;

(c) Whenever $U_i \cap U_j$ is non-empty, the mapping $\varphi_j \circ \varphi_i^{-1}$ of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$ is an element of Γ.

A *complete* atlas of M compatible with Γ is an atlas of M compatible with Γ which is not contained in any other atlas of M compatible with Γ. Every atlas of M compatible with Γ is contained in a unique complete atlas of M compatible with Γ. In fact, given an atlas $A = \{(U_i, \varphi_i)\}$ of M compatible with Γ, let $\tilde{A}$ be the family of all pairs (U, φ) such that φ is a homeomorphism of an open set U of M onto an open set of S and that

$$\varphi_i \circ \varphi^{-1}\colon \varphi(U \cap U_i) \to \varphi_i(U \cap U_i)$$

is an element of Γ whenever $U \cap U_i$ is non-empty. Then $\tilde{A}$ is the complete atlas containing A.

If Γ' is a subpseudogroup of Γ, then an atlas of M compatible with Γ' is compatible with Γ.

A *differentiable manifold* of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma^r(\mathbf{R}^n)$. The integer n is called the dimension of the manifold. Any atlas of a Hausdorff space compatible with $\Gamma^r(\mathbf{R}^n)$, enlarged to a complete atlas, defines a differentiable structure of class C^r. Since $\Gamma^r(\mathbf{R}^n) \supset \Gamma^s(\mathbf{R}^n)$ for $r < s$, a differentiable structure of class C^s defines uniquely a differentiable structure of class C^r. A differentiable manifold of class C^ω is also called a *real analytic manifold.* (Throughout the book we shall mostly consider differentiable manifolds of class C^∞. By a *differentiable manifold* or, simply, *manifold*, we shall mean a differentiable manifold of class C^∞.) A *complex (analytic) manifold* of complex dimension n is a Hausdorff space with a fixed complete atlas compatible with $\Gamma(\mathbf{C}^n)$. An *oriented* differentiable manifold of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma_o^r(\mathbf{R}^n)$. An oriented differentiable structure of class C^r gives rise to a differentiable structure of class C^r uniquely. Not every differentiable structure of class C^r is thus obtained; if it is obtained from an oriented one, it is called *orientable*. An orientable manifold of class C^r admits exactly two orientations if it is connected. Leaving the proof of this fact to the reader, we shall only indicate how to *reverse the orientation* of an oriented manifold. If a family of charts (U_i, φ_i) defines an oriented manifold, then the family of charts (U_i, ψ_i) defines the manifold with the reversed orientation where ψ_i is the composition of φ_i with the transformation $(x^1, x^2, \ldots, x^n) \rightarrow (-x^1, x^2, \ldots, x^n)$ of $\mathbf{R}^n$. Since $\Gamma(\mathbf{C}^n) \subset \Gamma_o^r(\mathbf{R}^{2n})$, every complex manifold is oriented as a manifold of class C^r.

For any structure under consideration (e.g., differentiable structure of class C^r), an *allowable* chart is a chart which belongs to the fixed complete atlas defining the structure. From now on, by a chart we shall mean an allowable chart. Given an allowable chart (U_i, φ_i) of an n-dimensional manifold M of class C^r, the system of functions $x^1 \circ \varphi_i, \ldots, x^n \circ \varphi_i$ defined on U_i is called a *local coordinate system* in U_i. We say then that U_i is a *coordinate neighborhood*. For every point p of M, it is possible to find a chart (U_i, φ_i) such that $\varphi_i(p)$ is the origin of $\mathbf{R}^n$ and φ_i is a homeomorphism of U_i onto an open set of $\mathbf{R}^n$ defined by $|x^1| < a, \ldots, |x^n| < a$ for some positive number a. U_i is then called a *cubic neighborhood* of p.

In a natural manner $\mathbf{R}^n$ is an oriented manifold of class C^r for any r; a chart consists of an element f of $\Gamma_o^r(\mathbf{R}^n)$ and the domain of f. Similarly, $\mathbf{C}^n$ is a complex manifold. Any open subset N of a manifold M of class C^r is a manifold of class C^r in a natural manner; a chart of N is given by $(U_i \cap N, \psi_i)$ where (U_i, φ_i) is a chart of M and ψ_i is the restriction of φ_i to $U_i \cap N$. Similarly, for complex manifolds.

Given two manifolds M and M' of class C^r, a mapping $f: M \rightarrow M'$ is said to be differentiable of class C^k, $k \leqq r$, if, for every chart (U_i, φ_i) of M and every chart (V_j, ψ_j) of M' such that

$f(U_i) \subset V_j$, the mapping $\psi_j \circ f \circ \varphi_i^{-1}$ of $\varphi_i(U_i)$ into $\psi_j(V_j)$ is differentiable of class C^k. If $u^1, \dots, u^n$ is a local coordinate system in U_i and $v^1, \dots, v^m$ is a local coordinate system in V_j, then f may be expressed by a set of differentiable functions of class C^k:

$$v^1 = f^1(u^1, \dots, u^n), \dots, v^m = f^m(u^1, \dots, u^n).$$

By a *differentiable mapping* or simply, a *mapping*, we shall mean a mapping of class C^∞. A differentiable function of class C^k on M is a mapping of class C^k of M into $\mathbf{R}$. The definition of a *holomorphic* (or *complex analytic*) mapping or function is similar.

By a *differentiable curve* of class C^k in M, we shall mean a differentiable mapping of class C^k of a closed interval $[a, b]$ of $\mathbf{R}$ into M, namely, the restriction of a differentiable mapping of class C^k of an open interval containing $[a, b]$ into M. We shall now define a *tangent vector* (or simply a *vector*) at a point p of M. Let $\mathfrak{F}(p)$ be the algebra of differentiable functions of class C^1 defined in a neighborhood of p. Let $x(t)$ be a curve of class C^1, $a \leqq t \leqq b$, such that $x(t_0) = p$. The vector tangent to the curve $x(t)$ at p is a mapping X: $\mathfrak{F}(p) \to \mathbf{R}$ defined by

$$Xf = (df(x(t))/dt)_{t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $x(t)$ at $t = t_0$. The vector X satisfies the following conditions:

(1) X is a linear mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$;

(2) $X(fg) = (Xf)g(p) + f(p)(Xg) \quad \text{for } f, g \in \mathfrak{F}(p)$.

The set of mappings X of $\mathfrak{F}(p)$ into $\mathbf{R}$ satisfying the preceding two conditions forms a real vector space. We shall show that the set of vectors at p is a vector subspace of dimension n, where n is the dimension of M. Let $u^1, \dots, u^n$ be a local coordinate system in a coordinate neighborhood U of p. For each j, $(\partial/\partial u^j)_p$ is a mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$ which satisfies conditions (1) and (2) above. We shall show that the set of vectors at p is the vector space with basis $(\partial/\partial u^1)_p, \dots, (\partial/\partial u^n)_p$. Given any curve $x(t)$ with $p = x(t_0)$, let $u^j = x^j(t)$, $j = 1, \dots, n$, be its equations in terms of the local coordinate system $u^1, \dots, u^n$. Then

$$(df(x(t))/dt)_{t_0} = \Sigma_j\, (\partial f/\partial u^j)_p \cdot (dx^j(t)/dt)_{t_0}\, ^*,$$

* For the summation notation, see Summary of Basic Notations.

which proves that every vector at p is a linear combination of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Conversely, given a linear combination $\Sigma\, \xi^j(\partial/\partial u^j)_p$, consider the curve defined by

$$u^j = u^j(p) + \xi^j t, \qquad j = 1, \ldots, n.$$

Then the vector tangent to this curve at $t = 0$ is $\Sigma\, \xi^j(\partial/\partial u^j)_p$. To prove the linear independence of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$, assume $\Sigma\, \xi^j(\partial/\partial u^j)_p = 0$. Then

$$0 = \Sigma\, \xi^j(\partial u^k/\partial u^j)_p = \xi^k \qquad \text{for } k = 1, \ldots, n.$$

This completes the proof of our assertion. The set of tangent vectors at p, denoted by $T_p(M)$ or T_p, is called the *tangent space* of M at p. The n-tuple of numbers $\xi^1, \ldots, \xi^n$ will be called the *components* of the vector $\Sigma\, \xi^j(\partial/\partial u^j)_p$ with respect to the local coordinate system $u^1, \ldots, u^n$.

Remark. It is known that if a manifold M is of class C^∞, then $T_p(M)$ coincides with the space of X: $\mathfrak{F}(p) \to \mathbf{R}$ satisfying conditions (1) and (2) above, where $\mathfrak{F}(p)$ now denotes the algebra of all C^∞ functions around p. From now on we shall consider mainly manifolds of class C^∞ and mappings of class C^∞.

A *vector field* X on a manifold M is an assignment of a vector X_p to each point p of M. If f is a differentiable function on M, then Xf is a function on M defined by $(Xf)(p) = X_p f$. A vector field X is called *differentiable* if Xf is differentiable for every differentiable function f. In terms of a local coordinate system $u^1, \ldots, u^n$, a vector field X may be expressed by $X = \Sigma\, \xi^j(\partial/\partial u^j)$, where ξ^j are functions defined in the coordinate neighborhood, called the *components* of X with respect to $u^1, \ldots, u^n$. X is differentiable if and only if its components ξ^j are differentiable.

Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on M. It is a real vector space under the natural addition and scalar multiplication. If X and Y are in $\mathfrak{X}(M)$, define the bracket $[X, Y]$ as a mapping from the ring of functions on M into itself by

$$[X, Y]f = X(Yf) - Y(Xf).$$

We shall show that $[X, Y]$ is a vector field. In terms of a local coordinate system $u^1, \ldots, u^n$, we write

$$X = \Sigma\, \xi^j(\partial/\partial u^i), \qquad Y = \Sigma\, \eta^j(\partial/\partial u^j).$$

Then

$$[X, Y]f = \Sigma_{j,k}(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))(\partial f/\partial u^j).$$

This means that $[X, Y]$ is a vector field whose components with respect to $u^1, \ldots, u^n$ are given by $\Sigma_k(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))$, $j = 1, \ldots, n$. With respect to this bracket operation, $\mathfrak{X}(M)$ is a Lie algebra over the real number field (of infinite dimensions). In particular, we have Jacobi's identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$
$$\text{for } X, Y, Z \in \mathfrak{X}(M).$$

We may also regard $\mathfrak{X}(M)$ as a module over the algebra $\mathfrak{F}(M)$ of differentiable functions on M as follows. If f is a function and X is a vector field on M, then fX is a vector field on M defined by $(fX)_p = f(p)X_p$ for $p \in M$. Then

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$
$$f,g \in \mathfrak{F}(M), \qquad X,Y \in \mathfrak{X}(M).$$

For a point p of M, the dual vector space $T_p^*(M)$ of the tangent space $T_p(M)$ is called the space of *covectors* at p. An assignment of a covector at each point p is called a *1-form* (*differential form of degree* 1). For each function f on M, the *total differential* $(df)_p$ of f at p is defined by

$$\langle (df)_p, X\rangle = Xf \qquad \text{for } X \in T_p(M),$$

where $\langle\ ,\ \rangle$ denotes the value of the first entry on the second entry as a linear functional on $T_p(M)$. If $u^1, \ldots, u^n$ is a local coordinate system in a neighborhood of p, then the total differentials $(du^1)_p, \ldots, (du^n)_p$ form a basis for $T_p^*(M)$. In fact, they form the dual basis of the basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$ for $T_p(M)$. In a neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = \Sigma_j f_j\, du^j,$$

where f_j are functions defined in the neighborhood of p and are called the *components* of ω with respect to $u^1, \ldots, u^n$. The 1-form ω is called *differentiable* if f_j are differentiable (this condition is independent of the choice of a local coordinate system). We shall only consider differentiable 1-forms.

A 1-form ω can be defined also as an $\mathfrak{F}(M)$-linear mapping of the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$ into $\mathfrak{F}(M)$. The two definitions are related by (cf. Proposition 3.1)

$$(\omega(X))_p = \langle \omega_p, X_p \rangle, \qquad X \in \mathfrak{X}(M), \qquad p \in M.$$

Let $\wedge T_p^*(M)$ be the exterior algebra over $T_p^*(M)$. An *r-form* ω is an assignment of an element of degree r in $\wedge T_p^*(M)$ to each point p of M. In terms of a local coordinate system $u^1, \ldots, u^n$, ω can be expressed uniquely as

$$\omega = \Sigma_{i_1 < i_2 < \cdots < i_r} f_{i_1 \ldots i_r}\, du^{i_1} \wedge \cdots \wedge du^{i_r}.$$

The r-form ω is called *differentiable* if the components $f_{i_1 \cdots i_r}$ are all differentiable. By an *r-form* we shall mean a differentiable r-form. An r-form ω can be defined also as a skew-symmetric r-linear mapping over $\mathfrak{F}(M)$ of $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)$ (r times) into $\mathfrak{F}(M)$. The two definitions are related as follows. If $\omega_1, \ldots, \omega_r$ are 1-forms and $X_1, \ldots, X_r$ are vector fields, then $(\omega_1 \wedge \cdots \wedge \omega_r)(X_1, \ldots, X_r)$ is $1/r!$ times the determinant of the matrix $(\omega_j(X_k))_{j,k=1,\ldots,r}$ of degree r.

We denote by $\mathfrak{D}^r = \mathfrak{D}^r(M)$ the totality of (differentiable) r-forms on M for each $r = 0, 1, \ldots, n$. Then $\mathfrak{D}^0(M) = \mathfrak{F}(M)$. Each $\mathfrak{D}^r(M)$ is a real vector space and can be also considered as an $\mathfrak{F}(M)$-module: for $f \in \mathfrak{F}(M)$ and $\omega \in \mathfrak{D}^r(M)$, $f\omega$ is an r-form defined by $(f\omega)_p = f(p)\omega_p$, $p \in M$. We set $\mathfrak{D} = \mathfrak{D}(M) = \Sigma_{r=0}^n \mathfrak{D}^r(M)$. With respect to the exterior product, $\mathfrak{D}(M)$ forms an algebra over the real number field. *Exterior differentiation* d can be characterized as follows:

(1) d is an **R**-linear mapping of $\mathfrak{D}(M)$ into itself such that $d(\mathfrak{D}^r) \subset \mathfrak{D}^{r+1}$;

(2) For a function $f \in \mathfrak{D}^0$, df is the total differential;

(3) If $\omega \in \mathfrak{D}^r$ and $\pi \in \mathfrak{D}^s$, then

$$d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^r \omega \wedge d\pi;$$

(4) $d^2 = 0$.

In terms of a local coordinate system, if $\omega = \Sigma_{i_1 < \cdots < i_r} f_{i_1 \ldots i_r}\, du^{i_1} \wedge \cdots \wedge du^{i_r}$, then $d\omega = \Sigma_{i_1 < \cdots < i_r} df_{i_1 \cdots i_r} \wedge du^{i_1} \cdots \wedge du^{i_r}$.

It will be later necessary to consider differential forms with values in an arbitrary vector space. Let V be an m-dimensional real vector space. A V-valued r-form ω on M is an assignment to each point $p \in M$ a skew-symmetric r-linear mapping of $T_p(M) \times \cdots \times T_p(M)$ (r times) into V. If we take a basis $e_1, \ldots, e_m$ for V, we can write ω uniquely as $\omega = \Sigma_{j=1}^m \omega^j \cdot e_j$, where ω^j are usual r-forms on M. ω is *differentiable*, by definition, if ω^j are all differentiable. The exterior derivative $d\omega$ is defined to be $\Sigma_{j=1}^m d\omega^j \cdot e_j$, which is a V-valued $(r+1)$-form.

Given a mapping f of a manifold M into another manifold M', the *differential* at p of f is the linear mapping f_* of $T_p(M)$ into $T_{f(p)}(M')$ defined as follows. For each $X \in T_p(M)$, choose a curve $x(t)$ in M such that X is the vector tangent to $x(t)$ at $p = x(t_0)$. Then $f_*(X)$ is the vector tangent to the curve $f(x(t))$ at $f(p) = f(x(t_0))$. It follows immediately that if g is a function differentiable in a neighborhood of $f(p)$, then $(f_*(X))g = X(g \circ f)$. When it is necessary to specify the point p, we write $(f_*)_p$. When there is no danger of confusion, we may simply write f instead of f_*. The transpose of $(f_*)_p$ is a linear mapping of $T^*_{f(p)}(M')$ into $T^*_p(M)$. For any r-form ω' on M', we define an r-form $f^*\omega'$ on M by

$$(f^*\omega')(X_1, \ldots, X_r) = \omega'(f_*X_1, \ldots, f_*X_r), \qquad X_1, \ldots, X_r \in T_p(M).$$

The exterior differentiation d commutes with f^*: $d(f^*\omega') = f^*(d\omega')$.

A mapping f of M into M' is said to be of *rank* r at $p \in M$ if the dimension of $f_*(T_p(M))$ is r. If the rank of f at p is equal to $n = \dim M$, $(f_*)_p$ is injective and $\dim M \leqq \dim M'$. If the rank of f at p is equal to $n' = \dim M'$, $(f_*)_p$ is surjective and $\dim M \geqq \dim M'$. By the implicit function theorem, we have

PROPOSITION 1.1. *Let f be a mapping of M into M' and p a point of M.*

(1) *If $(f_*)_p$ is injective, there exist a local coordinate system $u^1, \ldots u^n$ in a neighborhood U of p and a local coordinate system $v^1, \ldots, v^{n'}$ in a neighborhood of $f(p)$ such that*

$$v^i(f(q)) = u^i(q) \quad \textit{for } q \in U \quad \textit{and} \quad i = 1, \ldots, n.$$

In particular, f is a homeomorphism of U onto $f(U)$.