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Edited by A. Dold and B. Eckmann

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Proceedings, Clausthal 1981

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Federal Republic of Germany, 1981

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Editors

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PREFACE

Non-linear physical systems and their mathematical structure form one of the most active fields in present mathematics and mathematical physics. This volume covers parts of that topic. It reports on differential geometrical and topological properties of those non-linear systems, which can be viewed physically as models for quantized non-relativistic particles constrained, i.e. localized, on a (smooth) manifold or as classical or quantized fields with non-linear field equations. The contributions of this volume show how to deal with these different types of non-linearities. There are various physically motivated approaches to both of them. For systems constrained on a manifold generically geometric methods are used with promising mathematical and physical results. Now that the feeling has dissipated, that global solutions of non-linear field equations are"extra - terrestrial beasts" (see the contribution of I.E. SEGAL), also here a more global and geometrical approach is applied with extreme success, we refer e.g. to the application of twistor geometry or to the analysis of solution manifolds of non-linear equations. The structures of both types of non-linearities are deeply related.

A summer workshop in connection with the above programme was held in July 1981 at the Technical University in Clausthal, Institute for Theoretical Physics and an international conference on mathematical physics was organized parallel to the workshop. The lectures at the workshop and some of the contributions to the conference are collected and edited in an updated version in this volume.

Quantization Procedures

Quantizations of non-relativistic (mechanical) systems constrained on a smooth manifold are discussed. The method of geometrical quantization is justified on more physical grounds and presented in a new context by R.J. BLATTNER. The kinematics of such systems is described with the notion of a "quantized Borel kinematics" without using the phase space and its symplectic structure by B. ANGERMANN, H.D. DOEBNER and J. TOLAR. A method for the quantization of constrained systems is proposed by J. SNIATICKY and is based on aspects of Dirac's theory and on a reduced phase space. The late S. PANEITZ defined "stable subvarieties" of solution manifolds of a class of time dependent Hamiltonian systems and "stable polarizations" and shows how these notions apply to certain systems with non-linear scattering. The Frobenius reciprocity theorem is discussed by V. GUILLEMIN and S. STERNBERG from the symplectic point of view and is linked to some structures of the geometric quantization method and to induced representations of symmetry groups.

Non-Linear Field Equations

The general properties of solution "manifolds" of non-linear field equations are discussed. I.E. SEGAL reviews authoritatively historical aspects and part of the present status of this field. R.O. WELLS describes with details and applications the twistor geometric approach to classical field equations. One of the physically interesting non-linear systems with a genuine geometry is the non-linear sigma model. A comprehensive report on this model is given by M. FORGER. F.B. PASEMANN describes a quantization of gauge theories based on their geometrical structure as Kaluza-Klein theories on a principle bundle and on de Rham - p - currents as fields and potentials.

From the Clausthal workshop and Conference on "Non-Linear Partial Differential Operators and Quantization Procedures" this volume contains only part of the lectures presented there. The editors agree with the general editorial requirements that a lecture notes volume should be homogenous and that papers presenting mainly already known results or having the character of a research announcement should not be included. Some manuscripts were not received in time. The articles in Part I and II are arranged in alphabetical order.

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The Editors.

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EINSTEIN'S EVOLUTION EQUATION FOR THE VACUUM FORMULATED ON A SPACE OF DIFFERENTIALS OF IMMERSIONS

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Dedicated to H.H. Keller

0. Introduction

Einstein's equation for the vacuum on a four dimensional Lorentz manifold reads as $\mathrm{Ric}\,{}^4G = 0$, where $\mathrm{Ric}\,{}^4G$ means the Ricci tensor of the Lorentz metric ${}^4G$.

One way to construct a special type of solutions to this equation is beautifully described in [16] and is as follows:

Let M be an oriented compact three dimensional C^∞-manifold and $I \subset \mathbb{R}$ an open interval centered around zero. On $M \times I$, a manifold of dimension four, consider a Lorentz metric ${}^4G$ of the so called (3+1)-type: Along I this metric is determined by ${}^4G(p,t)(N,N) = -1$ where $N: M \times I \longrightarrow \mathbb{R}$ maps (p,t) to 1 for all $p \in M$ and all $t \in I$ and is otherwise characterized by assumption that ${}^4G$ restricted to the tangent bundle $T(M \times \{t\})$ of $M \times \{t\}$ is a Riemannian metric, called $G(t)$, for each $t \in I$. Thus aside of N, the Lorentz metric ${}^4G$ on $M \times I$ is characterized by a curve γ from I into the collection

$\mathcal{M}(M)$ of all Riemannian metrics on M. On each $t \in I$, γ assumes the value $G(t)$. Any such curve, vice versa, determines a Lorentz metric on $M \times I$ of the above described type.

The variational principle of Hilbert for Einstein's vacuum equation formulated for a Lorentz metric of type 3+1 yields a Lagrangian L on the tangent space of $\mathcal{M}(M)$, a.e. on $\mathcal{M}(M) \times S_2(M)$. Here $S_2(M)$ denotes the Fréchet space of all symmetric two tensors of M (which contains (M) as an open subset).

This Lagrangian splits into what is known as the De Witt metric and a potential term. The De Witt metric is directly related with the second fundamental form of $M \times \{t\} \subset M \times I$ for all $t \in I$. The potential is defined by the scalar curvature of $G(t)$ for all $t \in I$.

The metric ${}^4G$ of type 3+1 on $M \times I$ obeys Einstein's vacuum equation iff γ is an extremal of L, starting at an arbitrarily given initial metric $G(0) \in \mathcal{M}(M)$ in an appropriate direction. These directions are subjected to certain constraint equations. The Euler-Lagrange-equations for such a γ together with the constraint equations are called Einstein's evolution equation for the vacuum (without shift and with lapse one). Associated with this situation one has the following Cauchy problem: Given $G(0)$ and an initial direction K satisfying the constraint equations. Find an extremal γ of L with $\gamma(0) = G(0)$ and $\dot{\gamma}(0) = K$.

The constraint equations just mentioned are consequences of the C^∞-invariance of L. By C^∞-invariance of L we mean the following: The group Diff M of all C^∞-diffeomorphisms of M operates by pullback on $\mathcal{M}(M)$. The Lagrangian L turns out to be invariant under Diff M. Hence first integrals are available. These determine the constraint equation.

The qualitative properties of ${}^4G$ satisfying Einstein's vacuum equation are thus determined by the qualitative property of γ projected onto the quotient $\mathcal{M}(M)/\text{Diff } M$. The quotient $\mathcal{M}(M)/\text{Diff } M$ is not a manifold.

There is, however, a Frechet manifold which "resolves" the singularities of $\mathcal{M}(M)/\text{Diff } M$. It is constructed as follows:

By the celebrated theorem of Nash, any metric in $\mathcal{M}(M)$ can be obtained by pulling back a fixed scalar product on $\mathbb{R}^n$ via an embedding of M into $\mathbb{R}^n$ (n fixed but large enough).

Hence we have a surjective map m of $E(M,\mathbb{R}^n)$ (the Fréchet manifold of all embeddings of M into $\mathbb{R}^n$) onto $\mathcal{M}(M)$. Again Diff M operators (from the right) by pull back on $E(M,\mathbb{R}^n)$. The quotient $E(M,\mathbb{R}^n)$/Diff M, called $U(M,\mathbb{R}^n)$, is a Frechet manifold [4]. In fact, $E(M,\mathbb{R}^n)$ is a principal bundle over $U(M,\mathbb{R}^n)$ with Diff M as structure group. m yields a projection onto $\mathcal{M}(M)$/Diff M.

The purpose of this note is to formulate Einstein's evolution equation on $E(M,\mathbb{R}^n)$ in order to open up an other to study the metric ${}^4G$ by using the Fréchet manifold $U(M,\mathbb{R}^n)$. More precisely, we formulate the equation on the space of the differentials of all C^∞-immersions of M into $\mathbb{R}^n$ (since these differentials determine the metrics by pulling back the given scalar product in $\mathbb{R}^n$). The differential determines the immersion up to a constant.

The key to our formulation on one hand is the idea of the Cauchy problem for the evolution equation mentioned above. $G(0)$ will be replaced by a fixed initial immersion called i and $\dot{G}(0)$ by an C^∞-map h satisfying the appropriate constraint equations. On the other hand, the following observation will be crucial: If n is large enough (Nash bound will do), the differential dj of an C^∞-immersion $j : M \longrightarrow \mathbb{R}^n$ can be expressed by the differential di of a fixed initial C^∞-immersion $i : M \longrightarrow \mathbb{R}^n$, an appropriate C^∞-bundle map $f : TM \longrightarrow TM$ which is symmetric with respect to the metric $m(i)$ and an "integrating factor" g, which turns the $\mathbb{R}^n$-valued one form $di\cdot f$ into a differential by multiplication from the left. This integrating factor is a C^∞-map $g : M \longrightarrow O(n)$, where $O(n)$ denotes the orthogonal group of $\mathbb{R}^n$ with respect to the fixed scalar product $\langle , \rangle$. In both cases $di\cdot f$ and $g\cdot di\cdot f$, the dots mean the fibrewise formed composition on TM. Hence for any tangent vector v_p at p of M the equation

$$dj(v_p) = g(p)(di(f(p)(v_p)))$$

holds (for any $p \in M$). Clearly, if f is the identity on each fibre of TM and $g(p) \in O(n)$ is the identity for all p, then $dj = di$. (Integrating factors may differ on the normal bundle of $i(M) \subset \mathbb{R}^n$.)

In analogy to the Lagrangian and the constraint equation both mentioned above, we formulate a Lagrangian L (depending on the fixed immersion i) and constraint equation both defined on the tangent bundle of the Frechet manifold which is determined by all pairs (g,f) such that $g \cdot di \cdot f$ is a differential of an immersion. The projections onto $\mathcal{M}(M)$ of those extremals α of L which obey the constraint equation satisfy Einstein's evolution equation for the vacuum mentioned earlier. The differential calculus for Frechet manifolds is the one given by Gutknecht in [9]. Consult also [13],[8]. By C^∞ or by the vervally equivalent term "smooth" we always mean the notion defined in [9]. Let us remark that this note still has preliminary character. I am thankful to H.R. Fischer for introducing me to [1] and [16].

1) A review of the formulation of Einstein's evolution equation for the vacuum (without shift and with lapse one) on the space of all Riemannian C^∞-metrics following A.E. Fischer and J. Marsden

In this section we complete the formalism of Einstein's evolution equation presented in the introduction to the extend that we state the precise form of the notions. The review follows the presentation in [16]. The reader is asked to consult [1],[15] and [6] as well.

Let M be an oriented compact C^∞-manifold of dimension three. Consider $\mathcal{M}(M)$, the collection of all C^∞-Riemannian metrics defined on M. This collection forms an open set of the Frechet space $S_2(M)$ of all symmetric two-tensors of class C^∞ which carries Whitney's C^∞-topology. Thus $\mathcal{M}(M)$ is a Fréchet manifold. Its tangent bundle is identified with $\mathcal{M}(M) \times S_2(M)$.

We now proceed to define De Witt's metric G_{DW} on $\mathcal{M}(M)$. Given $G \in \mathcal{M}(M)$, any tangent vector $H \in S_2(M)$ can be represented by a unique C^∞-bundle map $\tilde{H} : TM \longrightarrow TM$ which satisfies

$$G(\tilde{H}X,Y) = H(X,Y) \tag{1}$$

for any couple X,Y of C^∞-vector fields. Denote the collection of all C^∞-maps from M into the C^∞-manifold Q by $C^\infty(M,Q)$.

We have a $C^\infty(M,\mathbb{R})$-valued scalar product on $S_2(M)$ which is given by

(2) $$H\cdot K := \operatorname{tr} \tilde{H}\cdot\tilde{K}$$

for any two $H,K \in S_2(M)$. The dot on the right hand side means the fibrewise composition. Clearly $H\cdot G = \operatorname{tr} \tilde{H}$. Instead of $\operatorname{tr} \tilde{H}$ write $\operatorname{tr}_G H$, which is called the trace of H with respect to G.

$\mathcal{G}_{DW}\, G \in (\mathcal{M})$ evaluated on any pair $H,K \in S_2(M)$ is then defined by

(3) $$\mathcal{G}_{DW}(G)(H,K) = \int (H\cdot K - \operatorname{tr}_G K\cdot \operatorname{tr}_G K)\, d\mu(G) \quad ,$$

where $\mu(G)$ denotes the Riemannian volume associated with $G \in \mathcal{M}(M)$.

To define the Lagrangian L mentioned in the introduction, consider $G \in \mathcal{M}(M)$ and its Ricci-tensor $\operatorname{Ric} G \in S_2(M)$. The trace of $\operatorname{Ric} G$ with respect to G is called the scalar curvature and is denoted by $\lambda(G)$. The Lagrangian $L : \mathcal{M}(M) \times S_2(M) \longrightarrow \mathbb{R}$ is then defined by

(4) $$L(G,H) = \mathcal{G}_{DW}(G)(H,H) + \int \lambda(G)\, d\mu(G) \quad ,$$

which is derived out of Hilberts variational principle for the vacuum equation as mentioned in [7].

An extremal γ (defined on an open interval $I \subset \mathbb{R}$ centered around zero) of L satisfies the Euler-Lagrange equation which reads in this case:

(5) $$\begin{aligned}\ddot{\gamma}(t) &= \dot{\gamma}(t) \times \dot{\gamma}(t) - \tfrac{1}{2}(\operatorname{tr}_{\gamma(t)}\dot{\gamma}(t))\cdot\dot{\gamma}(t)\\ &\quad - \tfrac{1}{8}(\dot{\gamma}(t)\cdot\dot{\gamma}(t) - (\operatorname{tr}_{\gamma(t)}\dot{\gamma}(t))^2)\cdot\gamma(t)\\ &\quad + 2\operatorname{Ric}\gamma(t) - \tfrac{1}{2}\,\lambda(\gamma(t))\cdot\gamma(t)\end{aligned}$$

for each $t \in I$. Here x denotes the cross product in $S_2(M)$. It is defined as follows: for any $H,K \in S_2(M)$

(6) $$H \times K = G(\tilde{H}\cdot\tilde{K},\ldots) \quad .$$

More precisely:

(7) $$(H \times K)(X,Y) = G(\tilde{H}\cdot\tilde{K}\, X,Y)$$

for all C^∞-vector fields X,Y on M. The first three terms of the right hand side of (5) form the spray of $\mathcal{G}_{\mathcal{D}\omega}$, the second two the gradient of the potential $\int \lambda(G)\, d\mu(G)$ formed with respect to $\mathcal{G}_{\mathcal{D}\omega}$.

Abbreviate

(7^+) $$\frac{1}{2}\,(\dot\gamma(t)\cdot\dot\gamma(t) - (tr_{\gamma(t)}\dot\gamma(t))^2) +$$
$$+\, 2\cdot\lambda(\gamma(t))\cdot\gamma(t) \quad \text{by} \quad \mathcal{H}(t) \ .$$

The group Diff M of all C^∞-diffeomorphisms operates on $\mathcal{M}(M)$ by pull back:

Given $g \in$ Diff M and $G \in \mathcal{M}(M)$, define $g\cdot G$ by $(g^{-1})^*G$. The latter symbol denotes the pull back of G by g^{-1}. For any pair X,Y of C^∞-vector fields on M we have

(8) $$(g\cdot G)\ (X,Y) = G(Tg^{-1}X, Tg^{-1}Y) \quad .$$

This operation from the left of Diff M on $\mathcal{M}(M)$ yields for each C^∞-vector field X, regarded as a tangent vector at $id \in$ Diff M, a first integral for the extremals of L. According to [16], this first integral

$$F_X : \mathcal{M}(M) \times S_2(M) \longrightarrow \mathbb{R}$$

is given by $F_X(G,H) = \mathcal{G}_{\mathcal{D}\omega}(G)(-L_X G,H)$, where L_X denotes the Lie derivative in the direction of X. This collection of first integrals has the following effect on extremals of L : An extremal $\gamma : L \longrightarrow \mathcal{M}(M)$ of L, a.e. a curve satisfying (5), yields a Ricci-flat Lorentz metric ${}^4G$ of type 3+1 (described in the introduction) iff the following two additional constraint equations are satisfied:

(9) $\delta(\dot{\gamma}(t) - (tr_{\gamma(t)}\dot{\gamma}(t))\cdot\gamma(t)) = 0$

(δ denotes the covariant divergence)

and

(10) $\mathcal{H}(\gamma(t)) = 0$ for all $t \in I$.

For the proof see [16] again.

We start our reformulation of the above formalism to the space of differentials of immersion by first investigating the latter space more closely in order to derive the necessary techniques.

2) Differentials of immersions

Let M be a three-dimensional compact oriented C^∞-manifold. On $\mathbb{R}^n$ we fix a scalar product $\langle,\rangle$. A C^∞-map $i : M \longrightarrow \mathbb{R}^n$ is called an immersion if the tangent map has maximal rank. The collection $I(M,\mathbb{R}^n)$ of all C^∞-immersions of M into $\mathbb{R}^n$ is an open subset of the Fréchet space $C^\infty(M,\mathbb{R}^n)$ consisting of all C^∞ maps from M into $\mathbb{R}^n$, endowed with Whitney's C^∞-topology [10].

Thus $I(M,\mathbb{R}^n)$ is a Fréchet manifold. Using Gutknecht's calculus [9] on Fréchet spaces, the tangent space at each immersion is (analogous to the case of a finite dimensional manifold) canonically identified with $C^\infty(M,\mathbb{R}^n)$.

The path components of $I(M,\mathbb{R}^n)$ consist evidently of all immersions which are isotopic, a.e. are deformable (in the sense of a homotopy) within the space of immersions. We refer to [12] for a detailed study of the deformations of immersions.

In case $n \geqslant 7$, any two immersions in $I(M,\mathbb{R}^n)$ are connected by a C^∞-path [12]. Given $i \in I(M,\mathbb{R}^n)$, denote by 0_i its path-component or (which amounts to the same) the connected component. Our first goal in this section is to describe the nature of the differentials of immersions in 0_i. Let dj be the principal part of the tangent map of $j \in 0_i$, called the differential of j. Locally, e.g. in a chart

$U \subset M$, the $\mathbb{R}^n$-valued one-form dj at $p \in M$ is the Fréchet derivative $Dj(p)$ of $j : U \longrightarrow \mathbb{R}^n$, mapping any $v \in \mathbb{R}^3$ into $Dj(p)(v)$. Hence the tangent map Ti is of the form (j, dj). The nature of di is resolved by looking at the tangential representation of j :
The tangential representation of M into the Grassmanian $G(3,n)$ of all 3-planes in $\mathbb{R}^n$ is given by

$$\tilde{d}i : M \longrightarrow G(3,n) \tag{11}$$

mapping any $p \in M$ into $di(T_pM) \in G(3,n)$. It is a C^∞-map. Denote the canonical 3-plane bundle of $G(3,n)$ by ξ and its normal bundle by η. Hence $G(3,n) \times \mathbb{R}^n = \xi \oplus \eta$. Given any immersion $j \in 0_i$, its tangential representations dj is homotopic to di. Form now the pullbacks $i^*\xi$, $j^*\xi$, $i^*\eta$ and $j^*\eta$ with respect to $\tilde{d}j$ and $\tilde{d}i$ of ξ and η respectively. The bundle $j^*\xi$ serves as the "tangent bundle" of $j(M) \subset \mathbb{R}^n$. The latter only exists if j is an embedding, i.e. if j is one-to-one on M. Clearly, by construction, $i^*\xi \cong j^*\xi \cong TM$. If again i and j are embeddings we verify that (due to the construction of η) both $i^*\eta$ and $j^*\eta$ are isomorphic to the normal bundles of $i(M)$ and $j(M)$ in $\mathbb{R}^n$ respectively. Summarizing we have a C^∞-bundle isomorphism

$$\bar{F} : i^*\xi \oplus i^*\eta \longrightarrow j^*\xi \oplus j^*\eta \quad . \tag{12}$$

Denote by $j^*(\xi)_p$ and $j^*(\eta)_p$ the fibres of $j^*(\xi)$ and $j^*(\eta)$ respectively. Observe that $\bar{F}$ restricted to a fibre $(i^*\xi)_p$ is given by $d_pj \circ (d_pi)^{-1}$. Here d_pj and d_pi denote the restrictions of the differentials of i and j to the tangent space T_pM of $p \in M$.

Since domain and range of $\bar{F}$ are canonically isomorphic with $M \times \mathbb{R}^n$, we identify $\bar{F}$ with a C^∞-map

$$F : M \longrightarrow GL(n) \; .$$

Thus the differentials di and dj are related by the equation

$$dj(v_p) = F(p)(d_pi(v_p))$$

for each $v_p \in T_pM$ and each $p \in M$. Let $C^\infty(M, GL(n))$ be the collection of all C^∞-maps from M into $GL(n)$. Given $F \in C^\infty(M, GL(n))$, consider

$$F\cdot di : TM \longrightarrow \mathbb{R}^n \quad ,$$

mapping any $v_p \in T_pM$ into $F(p)(d_p i(v_p))$ for each $p \in M$. Denote by δ the exterior differential defined by

$$\delta(F\cdot di)(X,Y)(p) = d(F\cdot diY)(X(p)) - d(F\cdot diX)Y(p) - F\cdot di[X,Y](p)$$

for any $p \in M$ and any two members X,Y of the collection ΓTM of all C^∞-vector fields on M. Observe

$$\delta(F\cdot di)(X,Y) = dF(X)\cdot diY - dF(Y)\cdot diX \quad , \tag{13}$$

where $dF(Y)(p) = dF(p)(Y(p))$ is an element of $\mathrm{End}\mathbb{R}^n$ for each $p \in M$. $F\cdot di$ is locally a differential of a C^∞-map iff

$$\delta(F\cdot di) = 0 \quad .$$

This is an immediate consequence of deRham's theorem and the fact that $H^1(M,\mathbb{R}^n) \cong H^1(M,\mathbb{R}) \otimes \mathbb{R}^n$. Since $F \in C^\infty(M,GL(n))$, the $\mathbb{R}^n$-valued form $F\cdot di$ has maximal rank everywhere. Hence if $\delta(F\cdot di) = 0$, then it is locally a differential of an immersion. The observations made in this section can be summarized as follows [2]

<u>THEOREM 1:</u> Any two immersions i,j in the same connected component can be joined by a C^∞-curve within that component. Their differentials are related via a map $F \in C^\infty(M,GL(n))$ by

$$dj = F\cdot di \quad . \tag{14}$$

If M satisfies $H^1(M,Z) = 0$, then for any $F \in C^\infty(M,GL(n))$, $F\cdot di$ is the differential of an immersion iff $\delta(F\cdot di) = 0$.

To investigate more closely the type of $F \in C^\infty(M,GL(n))$ introduced above, we introduce the Riemannian metrics $G(j) = j^*\langle,\rangle$ and $G^0 = i^*\langle,\rangle$, which applied to any $X,Y \in \Gamma TM$ yield the functions

$$G(j)(X,Y) = \langle djX,djY\rangle \quad \text{and} \quad G^0(X,Y) = \langle diX,diY\rangle \quad .$$

By the theorem of Fischer-Riesz [14] applied pointwise, we find a C^∞-bundle map

$$A : TM \longrightarrow TM$$