INTRODUCTION to MATHEMATICS

Bruce E. Meserve / Max A. Sobel

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BRUCE E. MESERVE
Professor of Mathematics
University of Vermont

MAX A. SOBEL
Professor of Mathematics
Montclair State College

Prentice-Hall, Inc., Englewood Cliffs, N. J.

Library of Congress Catalog Card No. 64-11514
Printed in the United States of America
C-48730

Third printing..... January, 1965

PREFACE

This book has been written with a number of different audiences in mind. The subject matter is suitable for the undergraduate college student that has had moderate secondary school training in mathematics, one who is not a mathematics major, but who wishes to acquire a basic understanding of the nature of mathematics. Many students seeking a knowledge of basic mathematics are so included. Frequently, prospective elementary-school teachers will be among such students. This book is also appropriate for in-service courses for elementary-school teachers. To this end, the emphasis throughout the book is on key concepts and the structure of mathematics, without undue concern over the mechanical procedures.

As the title of the first chapter suggests, mathematics can be fun! The authors have embarked with this idea and have gone ahead to introduce a variety of interesting and timely topics without a major emphasis upon the so-called practical applications. This point of view, it is felt, will leave the reader with a better picture of the true meaning and beauty of mathematics as opposed to a traditional approach with a major emphasis on abstract manipulations.

Very little mathematical background is required of the reader. It is expected that he will have had some secondary school introduction to algebra and geometry, but no working knowledge of any of the skills normally taught in these subjects is presupposed. Maturity, on the other hand, is expected, and interest in the subject is anticipated.

The subject matter presented in this text is sufficient for a three semester hour college course for the above-mentioned type of student. Chapters 1 through 7 may be used for a two-semester hour course. Preliminary editions of these chapters have been used for the past two years at Montclair State College in a two semester hour course for college juniors who were prospective teachers of subjects other than mathematics. Chapters 8, 9, and 10 enable the teacher of a three-semester hour course to include more thorough introductions to algebra, logic, and geometry. The authors recommend the sequence of chapters as they are ordered in the book. However, Chapters 1, 2, 3, 4, and 9 may be studied independently; the dependence of the other chapters is shown in the following array:

```
             5—10
            /
1,  2,  3,  4—6     ,  9.
            \
             7— 8
```

Note that Chapter 4 on "Sets of Elements" provides a basis for most of the remaining chapters of the book.

The authors wish to express their appreciation of the assistance of numerous students who participated actively in college courses at Montclair State College and used the materials that provided a basis for this book. In particular, they appreciate the suggestions of Mr. Jack Ott and Mr. Martin Cohen, who along with the authors taught courses using these materials. The authors also appreciate the cooperation of Prentice-Hall in preparing preliminary paperback editions of much of this material for classroom testing purposes.

The famous French mathematician René Descartes concluded his famous *La Géométrie* with the statement: "I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery." The authors have attempted to provide a great deal of exposition in this text. They have, however, left a great deal for the reader so that he may experience the true beauty of mathematics through discovery.

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CONTENTS

CHAPTER ONE

FUN WITH MATHEMATICS

Mathematics has numerous practical applications ranging from everyday usage to the charting of astronauts through outer space. Mathematics also interests many just by the sheer beauty of its structure. We hope in this text to examine some of the beauty without unduly emphasizing the practical values. Thus, we hope to show that mathematics can be studied just for one's interest in it—that is, just for fun. Accordingly, this chapter contains a smorgasbord variety of items served up to whet the reader's appetite for the main course that follows.

1-1 Mathematical Patterns

Mathematicians love to search for patterns and generalizations in all branches of their subjects—in arithmetic, in algebra, and in geometry. A search for such patterns may not only be interesting but may also help one develop insight into mathematics as a whole.

Multiples of nine

Many patterns that often escape notice may be found in the structure of arithmetic. For example, consider the multiples of 9:

$$\begin{aligned} 1 \times 9 &= 9, \\ 2 \times 9 &= 18, \\ 3 \times 9 &= 27, \\ 4 \times 9 &= 36, \\ 5 \times 9 &= 45, \\ 6 \times 9 &= 54, \\ 7 \times 9 &= 63, \\ 8 \times 9 &= 72, \\ 9 \times 9 &= 81. \end{aligned}$$

What patterns do you notice in the column of multiples on the right? You may note that the sum of the digits in each case is always 9. You should also see that the units digit decreases (9, 8, 7, . . .), whereas the tens digit increases (1, 2, 3, . . .). What lies behind this pattern?

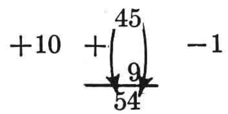
Consider the product

$$5 \times 9 = 45.$$

To find 6×9 we need to add 9 to 45. Instead of adding 9, we may add 10 and subtract 1.

$$\begin{array}{r} 45 \\ +10 \\ \hline 55 \end{array} \qquad \begin{array}{r} 55 \\ -1 \\ \hline 54 \end{array} = 6 \times 9$$

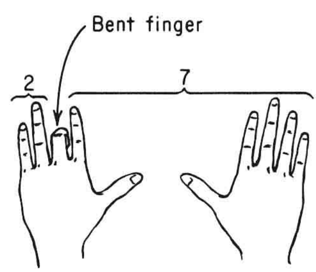
That is, by adding 1 to the tens digit, 4, of 45, we are really adding 10 to 45. We then subtract 1 from the units digit, 5, of 45 to obtain 54 as our product.



A similar explanation can be given for each of the other multiples in the table.

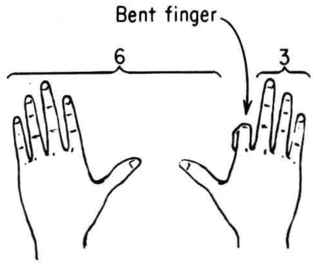
Finger multiplication

The number 9, incidentally, has other fascinating properties. Of special interest is a procedure for multiplying by 9 on one's fingers. For example, to multiply 9 by 3, place both hands together as in the figure, and bend the third finger from the left. The result is read as 27.



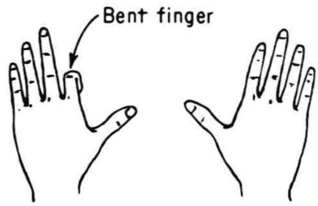
3 x 9 = 27

The next figure shows the procedure for finding the product 7 × 9. Note that the seventh finger from the left is bent, and the result is read in terms of the tens digit, to the left, and the units digit to the right of the bent finger.



7 x 9 = 63

What number fact is shown in the next figure?



Patterns of numbers

Here is one more pattern related to the number 9. You may, if you wish, verify that each of the following is correct:

$$\begin{aligned} 1 \times 9 + 2 &= 11, \\ 12 \times 9 + 3 &= 111, \\ 123 \times 9 + 4 &= 1{,}111, \\ 1{,}234 \times 9 + 5 &= 11{,}111, \\ 12{,}345 \times 9 + 6 &= 111{,}111. \end{aligned}$$

Try to find a correspondence of the number of 1's in the number symbol on the right with one of the numbers used on the left. Now, see if you can supply the answers, without computation, to the following:

$$\begin{aligned} 123{,}456 \times 9 + 7 &= ? \\ 1{,}234{,}567 \times 9 + 8 &= ? \end{aligned}$$

Now, let's see *why* this pattern works. To do so we shall examine just one of the statements. A similar explanation can be offered for each of the other statements. Consider the statement:

$$12{,}345 \times 9 + 6 = 111{,}111.$$

We can express 12,345 as a sum of five numbers as follows:

$$\begin{array}{r} 11{,}111 \\ 1{,}111 \\ 111 \\ 11 \\ 1 \\ \hline 12{,}345 \end{array}$$

Next we multiply each of the five numbers by 9:

$$\begin{aligned} 11{,}111 \times 9 &= 99{,}999, \\ 1{,}111 \times 9 &= 9{,}999, \\ 111 \times 9 &= 999, \\ 11 \times 9 &= 99, \\ 1 \times 9 &= 9. \end{aligned}$$

Finally, we add 6 by adding six ones as in the following array, and find the total sum:

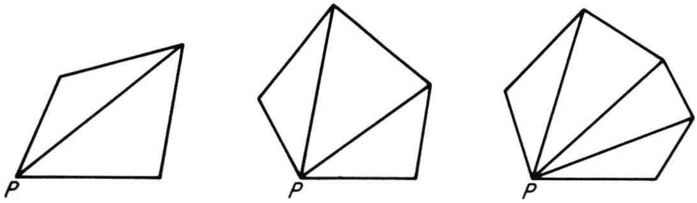
$$\begin{array}{rcr} 99{,}999 + 1 &=& 100{,}000 \\ 9{,}999 + 1 &=& 10{,}000 \\ 999 + 1 &=& 1{,}000 \\ 99 + 1 &=& 100 \\ 9 + 1 &=& 10 \\ 1 &=& 1 \\ \hline & & 111{,}111 \end{array}$$

Here is another interesting pattern. After studying the pattern, see if you can add the next three lines to the table.

$$\begin{array}{ccc} 1 \times 1 &=& 1 \\ 11 \times 11 &=& 121 \\ 111 \times 111 &=& 12{,}321 \\ 1{,}111 \times 1{,}111 &=& 1{,}234{,}321 \\ 11{,}111 \times 11{,}111 &=& 123{,}454{,}321 \end{array}$$

Geometric patterns

In the study of geometry we frequently jump to conclusions on the basis of a small number of examples, together with an exhibited pattern. Consider, for example, the problem of determining the number of triangles that can be formed from a given polygon by drawing diagonals from a given vertex, *P*. First we draw several figures and consider the results in tabular form as follows.



Number of sides of polygon	4	5	6
Number of triangles formed	2	3	4

From the pattern of entries in the table it appears that the number of triangles formed is two less than the number of sides of the polygon. Thus we expect that we can form 10 triangles from a dodecagon, a polygon with 12 sides, from a given vertex. In general then, for a polygon with n sides, called an n-gon, we can form $n - 2$ triangles.

This is reasoning by induction. We formed a generalization on the basis of several specific examples and an obvious pattern. It does not however, constitute a proof. To *prove* that $n - 2$ triangles can be formed we must observe that two of the n sides of the polygon intersect at the common point of the diagonals and each of the other $n - 2$ sides is used to form a different triangle.

Exercises

1. Verify that the process for finger multiplication shown in this section will work for each of the multiples of nine from 1×9 through 9×9.

2. Follow the procedure outlined in this section and show that $1{,}234 \times 9 + 5 = 11{,}111$.

3. An addition problem can be checked by a process called "casting out nines." To do this, you first find the sum of the digits of each of the addends (that is, numbers that are added), divide by 9, and record the remainder. The sum of these remainders is then divided by 9 to find a final remainder. This should be equal to the remainder found by considering the sum of the addends (that is, the answer), adding its digits, dividing the sum of these digits by 9, and finding the remainder. Here is an example:

Addends	*Sum of digits*	*Remainders*
4,378	22	4
2,160	9	0
3,872	20	2
1,085	14	5
11,495		11

When the sum of the remainders is divided by 9, the final remainder is 2. This corresponds to the remainder obtained by dividing the sum of the digits in the answer ($1 + 1 + 4 + 9 + 5 = 20$) by 9.

Try this procedure for several other examples and verify that it works in each case.

4. Try to discover a procedure for checking multiplication by casting out nines. Verify that this procedure works for several cases.

5. Study the following pattern and use it to express the squares of 6, 7, 8, and 9 in the same manner.

$$1^2 = 1$$
$$2^2 = 1 + 2 + 1$$
$$3^2 = 1 + 2 + 3 + 2 + 1$$
$$4^2 = 1 + 2 + 3 + 4 + 3 + 2 + 1$$
$$5^2 = 1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$$

6. Study the entries that follow and use the pattern that is exhibited to complete the last three rows.

$$1 + 3 = 4 \text{ or } 2^2$$
$$1 + 3 + 5 = 9 \text{ or } 3^2$$
$$1 + 3 + 5 + 7 = 16 \text{ or } 4^2$$
$$1 + 3 + 5 + 7 + 9 = ?$$
$$1 + 3 + 5 + 7 + 9 + 11 = ?$$
$$1 + 3 + 5 + 7 + 9 + 11 + 13 = ?$$

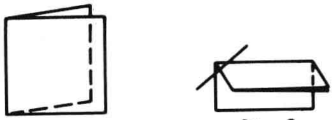
7. There is a procedure for multiplying a two-digit number by 9 on one's fingers provided that the tens digit is smaller than the ones digit. The accompanying diagram shows how to multiply 28 by 9. Reading from the left, put a space after the second finger and bend the eighth finger. Read the product in groups of fingers as 252.

Use this procedure to find:

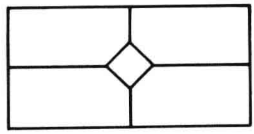
(a) 9×47; **(b)** 9×36; **(c)** 9×18; **(d)** 9×29.

Check each of the answers you have obtained.

8. Take a piece of notebook paper and fold it in half. Then fold it in half again and cut off a corner that does not involve an edge of the original piece of paper.



Step 1 Step 2



Your paper, when unfolded, should look like the accompanying sketch.

That is, with two folds we produced one hole. Repeat the same process but this time make three folds before cutting off an edge. Try to predict the number of holes that will be produced. How many holes will be produced with four folds? With n folds?

9. A famous mathematician named Gauss is said to have found the sum of the first 100 counting numbers at a very early age by the following procedure:

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

He reasoned that there would be 50 pairs of numbers, each with a sum of 101 (100 + 1, 99 + 2, 98 + 3, etc.). Thus, the sum would be 50 × 101 or 5050. Use this procedure to find:

(a) the sum of the first 50 counting numbers;
(b) the sum of the first 200 counting numbers;
(c) the sum of all the odd numbers from 1 through 49;
(d) the sum of all the odd numbers from 1 through 99;
(e) the sum of all the even numbers from 2 through 200.

10. Use the results obtained in Exercise 9 and try to find a formula for the sum of:

(a) the first n counting numbers [that is, $1 + 2 + 3 + \cdots + (n - 1) + n$];
(b) the first n odd numbers [that is, $1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)$].

1-2 Finite and Infinite

In modern times all of us have become familiar with very large numbers merely by observing the national budget and the national debt. The number "one million" no longer seems exceptionally large, but have you ever stopped to consider just how big this number really is?

Just for fun, can you estimate how long it would take you to count to a million? Assume that you count at the rate of one number per second with no time off to eat, rest, or sleep. (Don't do any computation yet; merely estimate.) Would you guess that it would take you less than an hour? A few hours? A day? A few days? More than a week? A month?

Let's find out how long it would take to count to a million. At the rate we suggested it would take a million seconds. There are 3,600 seconds in an hour, so it would take 1,000,000 ÷ 3,600 or approximately 278 hours. This is equivalent to about $11\frac{1}{2}$ days, counting night and day without rest, to reach one million!

Yes, one million is a large number, but it is **finite**; that is, it is countable. In this sense we also consider the number "one trillion" as finite even though we would not want to actually count to it.

Is there a largest number? The answer is clearly "no," for if someone claimed to be in possession of the world's largest number, you would merely have to add one to this number and then you would be in possession of a larger number.

One of the largest numbers ever named is a **googol**. This has been defined as 1 followed by one hundred zeros:

10000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000.

This number is larger than what is considered to be the total number of protons or electrons in the universe! A googol can be expressed, using exponents, as

$$10^{100}.$$

Even larger than a googol is a **googolplex**, defined as 1 followed by a googol of zeros. One famous mathematician claimed that there would not even be room between the earth and the moon to write all the zeros in a googolplex!

Although large beyond human comprehension, these numbers are still finite. The question one might well ask is whether or not there are numbers that are **infinite**—that is, beyond the finite. To answer this we turn to the procedure used by man in ancient times.

When ancient man wished to count the number of animals he owned he did so by placing a pebble on the ground for each animal. Then the number of pebbles was the same as the number of animals. He had established a **one-to-one correspondence** between the set of animals that he owned and a set of pebbles; for each pebble there was an animal, and for each animal there was a pebble.

When we wish to count today we make use of the set of numbers 1, 2, 3, 4, 5, and so on. We place the things which we count in a one-to-one correspondence with members of this set of numbers. The last number used tells the size of the collection. For example, to count the numbers of letters in the word "Thursday" we proceed as follows:

$$\begin{array}{cccccccc} T & h & u & r & s & d & a & y \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

We say that the **cardinality** of the set of letters in the word is 8. A number used to denote the size of a collection is called a **cardinal number**. (Actually when we write "8" we are writing a **numeral** and not a number. Numbers are abstract concepts which cannot be written on paper. However, here, and in later work, we shall distinguish between a number and a numeral only when it proves helpful to do so.)

Whenever two sets are such that their members may be placed in a one-to-one correspondence with each other we say that the two sets are **equivalent**. Thus the set of letters in the word "Thursday" is equivalent to the set of the first eight counting numbers. In a similar manner, the following two sets are equivalent.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 \end{array}$$

Also, the set of even numbers greater than zero is equivalent to the set of counting numbers.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 \ \ldots & n & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 \ \ldots & 2n & \ldots \end{array}$$

Each counting number, n, can be matched with an even counting number, $2n$.

It appears strange to be able to say that these two sets are equivalent. We are really saying that there are just as many even counting numbers as there are counting numbers altogether! This puzzled mathematicians for centuries until, at the turn of the twentieth century, a German mathematician named Georg Cantor developed an entire theory of infinite sets of numbers. This is essentially what he did.

He assigned a cardinal number to the set of counting numbers, namely, $\aleph_0$. This is read **"aleph-null"** and is really a **transfinite** (beyond the finite) **cardinal number**. It is correct then to say that there are $\aleph_0$ counting numbers, just as you might say that there are 7 days of the week or 10 fingers on your hands. Furthermore, any set that can be matched in a one-to-one correspondence with the set of counting numbers is also of size $\aleph_0$.

The discussion of transfinite numbers gives rise to some very interesting apparent paradoxes. One of the most famous of these is the story of the infinite house. This is a house that contains an infinite number of rooms, numbered 1, 2, 3, 4, 5, and so on. Each room is occupied by a single tenant. That is, there is a one-to-one correspondence between rooms and occupants. There are $\aleph_0$ rooms, and $\aleph_0$ occupants. One day a stranger arrived at the house and asked to be admitted. The caretaker was an amateur mathematician and was able to accommodate this visitor in the following manner. He asked the occupant of room 1 to move to room 2, the occupant of room 2 to move to room 3, the occupant of room 3 to move to room 4, and in general the occupant of room n to move to room $n + 1$. Now everyone had a room and room number 1 was available for the visitor! In other words we have demonstrated the interesting fact