Lecture Notes in Mathematics 1536

Man Kam Kwong Anton Zettl

Norm Inequalities for Derivatives and Differences

$\|y'\|^2 \leq K \|y\| \cdot \|y''\|$
$\|\Delta x\|^2 \leq C \|x\| \cdot \|\Delta^2 x\|$

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

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Preface

Edmund Landau's 1913 paper "Einige Ungleichungen für zweimal differenzierbare Funktionen", based on earlier work of Hardy and Littlewood, initiated a vast and fruitful research activity involving the study of the relationship between the norms of (i) a function and its derivatives and (ii) a sequence and its differences. These notes are an attempt to give a connected account of this effort. Detailed elementary proofs of basic inequalities are given. These are accessible to anyone with a background of advanced calculus and a rudimentary knowledge of the L^p and l^p spaces, yet the reader will be brought to the frontier of knowledge regarding several aspects of these problems. Many open questions are raised.

We thank Judy Beumer and Diane Keding for their careful typing of this difficult manuscript. Special thanks are also due Mari-Anne Hartig for never losing patience with our repeated requests for changes and without whose expertise in constructing graphs, tables, getting cross-references right, etc. this final version would not have been possible.

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Introduction

The norm of a function y may not be related to the norm of its derivative y'. One may be large while the other is small. More precisely, given any positive numbers u_0 and u_1, there exists a differentiable function y satisfying

$$||y|| = u_0 \text{ and } ||y'|| = u_1. \tag{0.1}$$

This is true and easy to prove in particular for the classical p-norms:

$$\begin{aligned} ||y||_p^p &= \int_J |\, y(t) \,|^p \, dt, \;\; 1 \leq p < \infty \\ ||y||_\infty &= \text{ess. sup } |\, y(t) \,|, \;\; p = \infty \end{aligned}$$

where J is any bounded or unbounded interval of the real line.

Given another positive number u_2: Does there exist a function y which satisfies, in addition to (0.1), also

$$||y|| = u_2 \ ?$$

The answer is *no*, thus giving rise to another question: How are the norms of a function y and its derivatives related to each other? It is this question we study in this monograph.

Of primary interest is the classical inequality

$$||y^{(k)}||^n \;\leq\; K \, ||y||^{n-k} \, ||y^{(n)}||^k$$

often associated with the names of Landau, Hardy and Littlewood, Kolmogorov, among others, and its discrete analogue

$$||\Delta^k x||^n \;\leq\; C \, ||x||^{n-k} \, ||\Delta^n x||^k.$$

Our goal is to give a basically self-contained exposition requiring only a background of advanced calculus and the basics of Lebesque integration theory, yet we aim to bring the reader to the frontier of knowledge for some aspects of these inequalities. Many results obtained here are from papers less than 15 years old; in the discrete case, less than 10 years. Some are more recent than that. A lot of open problems are mentioned, many of which, we believe, are "accessible". An extensive bibliography is also provided which includes some of the vast Soviet literature on this topic.

Chapter 1

Unit Weight Functions

In this chapter, we discuss the values that the norm of a function and its derivatives can assume. Considered are the classical L^p norms with unit weight. Some basic inequalities are discussed, including those often associated with the names of Landau, Hardy-Littlewood and Kolmogorov.

Although the subject matter treated in this chapter is very classical and the methods used are elementary, there are some results here which do not seem to have been published before.

1.1 The Norms of y and $y^{(n)}$

The classic p-norms are defined by

$$||y||_p = \left(\int_J |y(t)|^p \, dt\right)^{1/p}, \quad 1 \le p < \infty,$$

$$||y||_\infty = \text{ess sup}\,|y(t)|, \quad t \in J, \quad p = \infty.$$

Here J is any nondegenerate interval of the real line, bounded or unbounded.

The set of equivalence classes (with respect to Lebesgue measure) of functions whose p norms are finite is the classical Banach space $L^p(J)$, $1 \le p \le \infty$. Below, $y^{(n)}$ denotes the n^{th} derivative of y and $y^{(n)} \in L^q(J)$ means that $y^{(n-1)}$ is absolutely continuous on any compact subinterval of J, so that $y^{(n)}$ exists a.e. and is locally integrable, and $||y^{(n)}||_q$ is finite. The symbol $||y||_{p,J}$ will be used when we wish to emphasize the dependence of the norm on the interval J. If p or J is fixed in a result or argument we may merely use the symbol $||y||$. Throughout the book, p and q are assumed to satisfy $1 \le p, q \le \infty$.

We define $W^n_{p,q}(J)$ to be the subspace of $L^p(J)$ consisting of functions $y \in L^p(J)$ such that $y^{(n)} \in L^q(J)$. No integrability conditions are imposed on $y^{(k)}$ for $1 \le k < n$.

Two exponents p and q $(1 \le p, q \le \infty)$ are said to be conjugate to each other if $p^{-1} + q^{-1} = 1$. We follow the usual convention that $p = 1$ when $q = \infty$ and $p = \infty$ when $q = 1$.

The symbol $C^n(J)$ denotes the set of complex-valued functions with a continuous n^{th} derivative on J, $C^\infty(J)$ is the set of infinitely differentiable functions on J, and $C_0^\infty(J)$ is the set of infinitely differentiable functions with compact support in the interior of J.

In this monograph we are concerned with various inequalities among the norms of derivatives of a function. Our first result ensures the existence of a function whose norm and whose n^{th} derivative's norm are respectively equal to two arbitrary given positive numbers. This is not at all surprising and the proof is not difficult.

Theorem 1.1 *Let $1 \le p,\ q \le \infty$, let n be a positive integer, and let J be any interval on the real line, bounded or unbounded. Given any positive numbers u and v there exists a function $y \in C^n(J)$ such that*

$$\|y\|_p = u, \ \|y^{(n)}\|_q = v. \tag{1.1}$$

Proof.

Case 1. $1 \le p,\ q < \infty,\ J = R = (-\infty, \infty)$.

Choose y in $C_0^\infty(R)$ such that $y \ge 0$ but not identically zero and y has compact support. Consider

$$y_{ab}(t) = ay(bt), \quad a > 0, \quad b > 0$$

and note that with $x = bt$

$$\begin{aligned}
\|y_{ab}\|_p^p &= \int_R a^p |y(bt)|^p dt = a^p b^{-1} \int_R |y(x)|^p dx = a^p b^{-1} \|y\|_p^p \\
\|y_{ab}^{(n)}\|_q^q &= \int_R a^q b^{nq} |y^{(n)}(bt)|^q dt = a^q b^{nq-1} \int_R |y^{(n)}(x)|^q dx \\
&= a^q b^{nq-1} \|y^{(n)}\|_q^q.
\end{aligned}$$

Choose b such that

$$b^{nq-1+q/p} = v^q \|y\|_p^q u^{-q} \|y^{(n)}\|_q^{-q} \tag{1.2}$$

(observe that $y^{(n)}$ is not identically zero since y has compact support and is not the zero function), and then choose $a = b^{1/p} u \,/ \|y\|_p$.

The proof for $J = R^+ = (0, \infty)$ is similar. Since translation and reflection preserve norms, all other half-line cases $(-\infty, a)$ or $(a, \infty), -\infty < a < \infty$, reduce to R^+.

Case 2. $1 \le p, q < \infty$, $J = (a, b)$, $-\infty < a < b < \infty$.

First consider the case $J = (0, 1)$. Define

$$Q(y) = \|y^{(n)}\|_q / \|y\|_p, \quad y \in W^n_{p,q}(J) = X.$$

Since the norm is a continuous function from X into $[0, \infty)$ it follows that Q is a continuous function from $X - \{0\}$ into $[0, \infty)$. We show that Q is onto. Let $Q(y) = \alpha$, $Q(z) = \beta$, $0 \le \alpha < \beta$. Then y and z are linearly independent. Thus $S = \text{span}\{y, z\} - \{0\}$ is a two-dimensional connected subset of X. Since the continuous image of a connected set is connected, it follows that $[\alpha, \beta] \subset$ range Q. By letting $y(t) = 1$ we see that we can choose $\alpha = 0$. To show that β can be chosen arbitrarily large, consider

$$y(t) = t^d, \quad 0 < t < 1.$$

Then

$$\|y\|_p^p = \int_0^1 t^{pd} dt = 1/(pd + 1)$$

$$\|y^{(n)}\|_q^q = h(d) \int_0^1 t^{(d-n)q} dt = h(d)/(q(d - n) + 1),$$

where

$$h(d) = d(d - 1) \ldots [d - (n - 1)].$$

Thus

$$Q(y) = (h(d))^{1/q} (pd + 1)^{1/p} (q(d - n) + 1)^{-1/q} \to \infty \text{ as } d \to \infty.$$

We conclude that the range of Q is $[0, \infty)$. Let $r = v/u$. ¿From the above argument we know there is a $y \in W^n_{p,q}(0, 1)$ such that $Q(y) = r$. Choose the constant c so that $\|cy\|_p = u$, then $\|cy^{(n)}\|_q = v$. This completes the proof for the case $J = (0, 1)$. The general case of a bounded interval J follows from this case and a transformation of the form $t \to ct + d = x$.

The proofs of the remaining cases ($p = \infty$ or $q = \infty$, J bounded or unbounded) are left to the reader as exercises. □

The proof of Theorem 1.1 is presented as above to point out the difference between the cases of unbounded and bounded J. In the first case, the technique of "horizontal scaling" (change of the independent variable) is a very useful tool, which is not available in the second case.

1.2 The norms of y, $y^{(k)}$, and $y^{(n)}$

The special case $p = q$, $n = 1$ of Theorem 1.1 says that the norm of a function and its derivative on any interval, bounded or unbounded, can assume arbitrary positive values. Can the norms of y, y', and y'' assume arbitrary positive values? Below we will see that the answer is no. But first we discuss some preliminary results.

Lemma 1.1 *Let $1 \leq p \leq \infty$. Assume $J = [a, b]$ is a compact interval of length $L = b - a$. If $y \in L^p(J)$ and $y^{(k)}$ exists on J and*

$$\alpha_k = \alpha_k(J) = \inf |y^{(k)}(t)|, \quad t \in J; \tag{1.3}$$

then

$$\alpha_k(J) \leq A \, \|y\|_{p,J} \quad k = 1, 2, 3, \ldots, \tag{1.4}$$

where A is a constant independent of y given by

$$A = A(k, p, L) = 2^k \cdot 3^{x(k)} L^{-k-1/p}, \tag{1.5}$$

where $x(k)$ is defined recursively by

$$x(1) = 1/p, \quad x(k+1) = x(k) + k + 1/p, \quad k = 1, 2, 3, \ldots. \tag{1.6}$$

Proof. The proof uses a "triple interval" argument and induction on k.

Case 1. $p = \infty$.

Divide J into three equal subintervals: $J_1 = [a, a + L/3]$, $J_2 = [a + L/3, \quad a + 2L/3]$, and $J_3 = [a + 2L/3, b]$. By the mean value theorem, for any t_1 in J_1 and t_3 in J_3 there exists $t^* \in (t_1, t_3)$ such that

$$\begin{aligned}\alpha_1 &\leq |y'(t^*)| = |(y(t_1) - y(t_3))/(t_1 - t_3)| \\ &\leq 3L^{-1}(|y(t_1)| + |y(t_3)|) \leq 6L^{-1}\|y\|_{\infty,J}.\end{aligned}$$

This establishes the case $k = 1$. To establish the inductive step it is convenient to use the notation $\alpha_k(J)$ to denote the dependence of α_k on the interval J. For the sake of clarity we consider first the case $k = 2$. Using the above notation, choose t_i in J_i so that $\alpha_1(J_i) = |y'(t_i)|$, $i = 1, 3$. By the mean value theorem we have, for some $t^* \in (t_1, t_3)$

$$\begin{aligned}\alpha_2 &\leq |y''(t^*)| = |(y'(t_1) - y'(t_3))/(t_1 - t_3)| \\ &\leq 3L^{-1}(\alpha_1(J_1) + \alpha_1(J_3)) \\ &\leq 3L^{-1}(6(L/3)^{-1}\|y\|_{\infty,J_1} + 6(L/3)^{-1}\|y\|_{\infty,J_3}) \\ &\leq 3 \cdot 6^2 \cdot L^{-2}\|y\|_{\infty,J}.\end{aligned}$$

Assume (1.4), (1.5) hold in the k^{th} stage of our induction process. Let $J = J_1 \cup J_2 \cup J_3$ as above in step $k = 1$. Then using the mean value theorem again with $t_1 \in J_1$, $t_3 \in J_3$ chosen so that $\alpha_k(J_i) = |y^{(k)}(t_i)|$, $i = 1, 3$ we get

$$\begin{aligned}
\alpha_{k+1}(J) &\leq |y^{(k+1)}(t^*)| = |(y^{(k)}(t_1) - y^{(k)}(t_3))/(t_1 - t_3)| \\
&\leq L^{-1}3(\alpha_k(J_1) + \alpha_k(J_3)) \\
&\leq 3L^{-1}2^k 3^{x(k)}(L/3)^{-k}(\|y\|_{\infty,J_1} + \|y\|_{\infty,J_3}) \\
&\leq L^{-k-1}2^{k+1}3^{1+x(k)+k}\|y\|_{\infty,J}.
\end{aligned}$$

This completes the proof of (1.4), (1.5) for $p = \infty$.

Case 2. $1 \leq p < \infty$.

Let $p^{-1} + q^{-1} = 1$. Let $J = J_1 \cup J_2 \cup J_3$ be as in Case 1 above. Choose $t_i \in J_i$ such that $|y(t_i)| = \min |y(t)|$, $t \in J_i$, $i = 1, 3$. ¿From the mean value theorem and Hölder's inequality we have for some t^* between t_1 and t_3

$$\alpha_1 \leq |y(t^*)| \leq L^{-1}3(|y(t_1)| + |y(t_3)|)$$

and

$$\begin{aligned}
L|y(t_i)|/3 &= \int_{J_1} |y(t_i)|dt \leq \int_{J_i} |y(t)|dt \\
&\leq (L/3)^{1/q}\|y\|_{p,J_i}, \quad i = 1, 3.
\end{aligned}$$

¿From these inequalities

$$\alpha_1 \leq 3^{1-1/q}L^{1/q-2}2\|y\|_{p,J}.$$

This is (1.4), (1.5) for $k = 1$. Assume (1.4), (1.5) hold for k. Decompose J as above and choose $t_i \in J_i$ such that $|y^{(k)}(t_i)| = \inf |y^{(k)}(t)|$ for $t \in J_i$, $i = 1, 3$. Then, as above,

$$\begin{aligned}
\alpha_{k+1}(J) &\leq |y^{(k+1)}(t^*)| \leq 3L^{-1}(|y^{(k)}(t_1)| + |y^{(k)}(t_3)|) \\
&= 3L^{-1}(\alpha_k(J_1) + \alpha_k(J_3)) \\
&\leq 3L^{-1}2(2^k 3^{x(k)} L^{1/q-k})\|y\|_{p,J},
\end{aligned}$$

and the proof of Lemma 1.1 is complete. □

Lemma 1.2 *Let* $1 \leq p, q, r \leq \infty$, $l(J) = L < \infty$. *If* $y \in L^p(J)$ *and* $y'' \in L^r(J)$ *then* $y' \in L^q(J)$ *and*

$$\|y'\|_q \leq AL^{1/r'+1/q}\|y''\|_r + BL^{-1-1/p+1/q}\|y\|_p, \tag{1.7}$$

where $1/r' + 1/r = 1$ *and*

$$A = 2^{1-1/q}, \quad B = 2^{2-1/q} \cdot 3^{1/p}. \tag{1.8}$$

Proof. We may assume, without loss of generality, that J is compact. With the notation of Lemma 1.1 we have

$$|y'(t)| \leq \alpha_1 + \left|\int_{t_1}^{t} y''\right| \leq 2 \cdot 3^{1/p} L^{-1-1/p} \|y\|_p + L^{1/r'} \|y''\|_r. \tag{1.9}$$

If $q = \infty$, (1.7), (1.8) follow from (1.9) since $1 < 2$, $2 \cdot 3^{1/p} < B$. If $q < \infty$, we obtain from (1.9)

$$\begin{aligned} \int_J |y'(t)|^q dt &\leq 2^{q-1}(2^q \cdot 3^{q/p} L^{-q-q/p+1} \|y\|_p^q + L^{q/r'+1} \|y''\|_r^q) \\ &= A^q L^{q/r'+q} \|y''\|_r^q + B^q L^{-q-q/p+1} \|y\|_p^q \end{aligned} \tag{1.10}$$

using

$$a^s + b^s \leq (a+b)^s \leq 2^{s-1}(a^s + b^s), \quad 1 \leq s, \quad a > 0, \quad b > 0. \tag{1.11}$$

Now (1.7) follows from (1.10) and the second half of the elementary inequality

$$2^{s-1}(a^s + b^s) \leq (a+b)^s \leq a^s + b^s, \quad 0 \leq s \leq 1, \quad a > 0, \quad b > 0. \quad \square \tag{1.12}$$

Lemma 1.3 *Let $1 \leq p \leq \infty$, $l(J) = L \leq \infty$. Given $\epsilon > 0$ there exists a $K(\epsilon) > 0$ such that if $y \in L^p(J)$, y' is locally absolutely continuous on J, $y'' \in L^p(J)$ then $y' \in L^p(J)$ and*

$$\|y'\|_p \leq \epsilon \, \|y''\|_p + K(\epsilon) \, \|y\|_p. \tag{1.13}$$

Furthermore, for fixed ϵ, $K(\epsilon)$ can be chosen to be a nonincreasing function of the length of J.

Proof. We consider $p < \infty$ first.

Case 1. Assume $L < \infty$.

Let $\epsilon > 0$. If $L \leq \epsilon/A$ then (1.13) follows from (1.7) with

$$K(\epsilon) = BL^{-1}, \quad B = 2^{2-1/p}. \tag{1.14}$$

If $\epsilon_1 = \epsilon/A < L < \infty$ let $J = \cup_{i=1}^n J_i$, where J_i are nonoverlapping, $l(J_1) = \epsilon_1/2$, $i = 1, \ldots, n-1$, and $\epsilon_1/2 \leq l(J_n) \leq \epsilon_1$. Apply (1.10) to J_i, $i = 1, \ldots, n-1$ with L replaced by $\epsilon_1/2 = \epsilon/2A$, and q, r by p, we get

$$\int_I |y'|^p \leq \epsilon^p 2^{-p} \int_I |y''|^p + (2AB)^p \epsilon^{-p} \int_I |y|^p \leq \epsilon^p \int_I |y''|^p + (2AB)^p \epsilon^{-p} \int_I |y|^p \tag{1.15}$$

holding on each interval $I = J_i$, $i = 1, \ldots, n-1$. On $I = J_n$ we get from (1.10)

$$\begin{aligned} \int_I |y'|^p &\leq \epsilon^p \int_I |y''|^p + B^p L^{-p} \int_I |y|^p \\ &\leq \epsilon^p \int_I |y''|^p + (2AB)^p \epsilon^{-p} \int_I |y|^p. \end{aligned} \tag{1.16}$$

Summing inequalities (1.15) and (1.16) over all the intervals J_i, $i = 1, \ldots, n$ and then taking the p^{th} root we obtain (1.13) with

$$K(\epsilon) = 2AB\epsilon^{-1}. \tag{1.17}$$

Case 2. $L = \infty$. Let $J = \cup_{i=1}^{\infty} J_i$, where J_i are nonoverlapping intervals each of length ϵ_1. Proceeding as above we get inequality (1.15) on each interval $I = J_i$, $i = 1, 2, 3, \ldots$. Summing over the intervals J_i and taking the p^{th} root yields (1.13) with $K(\epsilon)$ given by (1.17).

For fixed ϵ it is clear that $K(\epsilon)$, chosen according to (1.14) and (1.17) is a nonincreasing function of the length of the interval J. This completes the proof for $p < \infty$. The modifications needed with $p = \infty$ are straightforward and hence omitted. □

Theorem 1.2 *Let $1 \le p \le \infty$, let n, k be integers with $1 \le k < n$, and let J be any interval of the real line, bounded or unbounded. Given any $\epsilon > 0$ there exists a positive $K(\epsilon)$ such that if $y \in L^p(J)$, $y^{(n-1)}$ is locally absolutely continuous and $y^{(n)} \in L^p(J)$ (i.e. $y \in W^n_{p,p}(K)$) then $y^{(k)} \in L^p(J)$ and*

$$\|y^{(k)}\|_p \le \epsilon\, \|y^{(n)}\|_p + K(\epsilon)\, \|y\|_p. \tag{1.18}$$

Furthermore, for a given $\epsilon > 0$ the constant $K(\epsilon)$ can be chosen to be a non-increasing function of the length of the interval J.

Proof. The proof is by induction on n. Since p is fixed throughout the proof we will suppress the subscript p on the norm symbol. The case $n = 2$ is Lemma 1.3. Assume Theorem 1.2 is true for $n = N$ (and $k = 1, 2, \ldots, N-1$) and suppose that $y \in W^{N+1}_{p,p}(J)$. We need to show that $y^{(k)} \in L^p(J)$ for $1 \le k < N+1$ and (1.18) holds with $n = N+1, k \le N$. Note that it does not follow immediately from the induction hypothesis that $y^{(k)} \in L^p(J)$ for $k = 1, \ldots, N$. But $y^{(k)} \in L^p(I)$ for any compact subinterval I of J since $y^{(k)}$ is absolutely continuous on I, $k = 1, \ldots, N$. Hence by Lemma 1.3 given $\epsilon_1 > 0$ there exists a $K(\epsilon_1) > 0$ such that

$$\begin{aligned}\|y^{(N)}\|_I &\le \epsilon_1 \|y^{(N+1)}\|_I + K(\epsilon_1)\|y^{(N+1)}\|_I \\ &\le \epsilon_1 \|y^{(N+1)}\|_J + K(\epsilon_1)(\epsilon_2 \|y^{(N)}\|_I + K(\epsilon_2)\|y\|_I).\end{aligned}$$

Here we used the inductive hypothesis in the last step. Rearranging terms we get

$$(1 - K(\epsilon_1)\epsilon_2 \|y^{(N)}\|_I \le \epsilon_1 \|y^{(N+1)}\|_J + K(\epsilon_1)K(\epsilon_2)\|y\|_I.$$

Choose $\epsilon_1 < \epsilon/2$ and ϵ_2 such that $1/2 \le 1 - K(\epsilon_1)\epsilon_2 < 1$. Then dividing by $1 - K(\epsilon_1)\epsilon_2$ and using $\|y\|_I \le \|y\|_J$ we get

$$\|y^{(N)}\|_I \le \epsilon\, \|y^{(N+1)}\|_J + 2K(\epsilon_1)K(\epsilon_2)\|y\|_J. \tag{1.19}$$