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Editor's Preface

For a period of more than a decade some members of the Jena Department of Relativistic Physics have been studying the literature on the exact solutions of Einstein's gravitational field equations. Following this line of work, they also did a considerable amount of research on the classification of these solutions and on new methods of finding such solutions (H. Stephani on the embedding method, D. Kramer and G. Neugebauer on the invariance transformation method, E. Herlt on stationary axisymmetric solutions). After some years, conditions seemed favourable for a systematic treatment of these subjects. As is well known among specialists, many interesting exact solutions have been independently discovered two or more times.

This situation inspired H. Stephani and D. Kramer to write a monograph on exact solutions of Einstein's field equations, presenting the most important methods and solutions in a systematic way, with the aim of accomplishing a kind of catalogue which would give a good survey of the solutions already discovered and help to avoid duplications of discovery. It was originally hoped to offer this monograph to the international relativity community on the occasion of Einstein's 100th birthday, but soon an enormous amount of material accumulated. Therefore the target date was amended to that of the 9th International Conference on General Relativity and Gravitation in July 1980 at Jena, and to achieve even this aim required very substantial support from the Jena Department of Relativistic Physics and the Sektion Physik of the Friedrich Schiller University. We were all very happy that, in addition to E. Herlt, M. MacCallum (Queen Mary College, London) could be enlisted as an author. Furthermore, M. MacCallum agreed to revise the English text. As the head of the department and as the editor of this monograph I would like to thank the authorities of the Friedrich Schiller University, Jena, for their constant help.

Jena, February 1979

Ernst Schmutzer

Friedrich Schiller
University Jena

Authors' Preface

When, in 1975, two of the authors (D. K. and H. S.) proposed to change their field of research back to the subject of exact solutions of Einstein's field equations, they of course felt it necessary to make a careful study of the papers published in the meantime, so as to avoid duplication of known results. A fairly comprehensive review or book on the exact solutions would have been a great help, but no such book was available. This prompted them to ask "Why not use the preparatory work we have to do in any case to write such a book?" After some discussion, they agreed to go ahead with this idea, and then they looked for co-authors. They succeeded in finding two.

The first was E. H., a member of the Jena relativity group, who had been engaged before on the exact solutions and was also inclined to return to them.

The second, M. M., became involved by responding to the existing authors' appeal for information and then (during a visit by H. S. to London) agreeing to look over the English text. Eventually he also agreed to write some parts of the book. He wishes to record that any infelicities remaining in the English arose because the generally good standard of his colleagues' English lulled him into a false sense of security.

Our original optimism somewhat diminished when references to over 2000 papers had been collected and the magnitude of the task became all too clear. How could we extract even the most important information from this mound of literature? How could we avoid constant re-writing to incorporate new information, which would have made the job akin to the proverbial painting of the Forth bridge? How could we decide which topics to include and which to omit? How could we check the calculations, put the results together in a readable form, and still finish in a reasonable time?

Looking back now at the result of three years' work, we cannot really feel that we solved any of these questions in a completely convincing manner. In particular, we feel sure we must have accidentally overlooked many useful results and solutions. However, we did manage to produce an outcome in a finite time, largely because the labour of reading those papers conceivably relevant to each chapter, and then drafting the related manuscript, was divided. (Roughly, D. K. was responsible for most of the introductory Part I., M. M., D. K. and H. S. dealt with groups (Part II.), H. S., D. K. and E. H. with algebraically special solutions, and D. K. and H. S. with Part IV. (special methods) and Part V. (tables).) Each draft was then criticized by the other authors, so that its writer could not be held wholly responsible for any errors or omissions. (Since we hope to maintain up-to-date information, we will be glad to hear from any reader who detects such errors or omissions; we will also be pleased to answer as best we can any requests for further information.)

This book could not have been written, of course, without the efforts of the many scientists whose work is recorded here, and especially the many contemporaries who sent preprints, reprints, references and advice. More immediately, it would not have appeared without the help of Frau Kaschlik and Frau Reichardt in Jena, and Mrs. Smith in London, who did all the secretarial work including typing the illegible and apparently interminable manuscript, of the students in Jena who maintained our reference files, of Prof. Schmutzer, who supported the project from the beginning, and of the Sektion Physik in Jena and the Department of Applied Mathematics at Queen Mary College, London. Last but not least, we thank wives, families and colleagues for tolerating our incessant brooding and discussions.

January 1979

Dietrich Kramer
Hans Stephani
Eduard Herlt
Jena

Malcolm MacCallum
London

Contents

Notation

All symbols are explained in the text. Here we list only some important conventions which are frequently used throughout he book.

Complex conjugation is denoted by a bar over the symbol.

Indices

Small Latin indices run, in an n-dimensional space, from 1 to n, in space-time V_4 from 1 to 4. Indices from the first part of the alphabet ($a, b, \ldots, h$) are tetrad indices, i.e. they refer to a general basis $\{\boldsymbol{e}_a\}$ or its dual $\{\omega^a\}$; $i, j, \ldots$ are reserved for a coordinate basis $\{\partial/\partial x^i\}$ or its dual $\{\mathrm{d}x^i\}$. For a vector $\boldsymbol{v}$ and a 1-form σ we write $\boldsymbol{v} = v^a \boldsymbol{e}_a = v^i\, \partial/\partial x^i$, $\boldsymbol{\sigma} = \sigma_a \omega^a = \sigma_i\, \mathrm{d}x^i$. Small Greek indices run from 1 to 3, if not otherwise stated. Capital Latin indices are either spinor indices ($A, B = 1, 2$) or indices in group space ($A, B = 1 \ldots r$), or they label the coordinates in a Riemannian 2-space V_2 ($M, N = 1, 2$).

Symmetrization and antisymmetrization of index pairs:

$$v_{(ab)} \equiv \frac{1}{2}\,(v_{ab} + v_{ba}), \qquad v_{[ab]} \equiv \frac{1}{2}\,(v_{ab} - v_{ba}).$$

Metric and tetrads

Line element in terms of dual basis $\{\omega^a\}$: $\mathrm{d}s^2 = g_{ab}\omega^a\omega^b$.
Signature of space-time metric: $(+ + + -)$.
Commutation coefficients: $D^c{}_{ab}$; $[\boldsymbol{e}_a, \boldsymbol{e}_b] = D^c{}_{ab}\boldsymbol{e}_c$.
(Complex) null tetrad: $\{\boldsymbol{e}_a\} = (\boldsymbol{m}, \overline{\boldsymbol{m}}, \boldsymbol{l}, \boldsymbol{k})$, $g_{ab} = 2m_{(a}\overline{m}_{b)} - 2k_{(a}l_{b)}$,
$\mathrm{d}s^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4$.
Orthonormal basis: $\{\boldsymbol{E}_a\}$.
Projection tensor: $h_{ab} \equiv g_{ab} + u_a u_b$, $u_a u^a = -1$.

Bivectors

Levi-Civita tensor: ε_{abcd}; $\varepsilon_{abcd}m^a\overline{m}^b l^c k^d = i$.

Dual bivector: $\tilde{X}_{ab} \equiv \frac{1}{2}\,\varepsilon_{abcd}X^{cd}$.

(Complex) self-dual bivector: $X^*_{ab} \equiv X_{ab} + \mathrm{i}\tilde{X}_{ab}$.
Basis of self-dual bivectors: $U_{ab} \equiv 2\overline{m}_{[a}l_{b]}$, $V_{ab} \equiv 2k_{[a}m_{b]}$, $W_{ab} \equiv 2m_{[a}\overline{m}_{b]} - 2k_{[a}l_{b]}$.

Derivatives

Partial derivative: comma in front of index or coordinate, e.g.

$$f_{,i} \equiv \partial f/\partial x^i \equiv \partial_i f, \qquad f_{,\zeta} \equiv \partial f/\partial \zeta.$$

Directional derivative: denoted by stroke or comma, $f_{|a} \equiv f_{,a} \equiv \boldsymbol{e}_a(f)$, if followed by a numerical (tetrad) index, we prefer the stroke, e.g. $f_{|4} = f_{,i}k^i$. Directional derivatives with respect to the null tetrad $(\boldsymbol{m}, \overline{\boldsymbol{m}}, \boldsymbol{l}, \boldsymbol{k})$ are symbolized by $\delta f = f_{|1}$, $\bar{\delta} f = f_{|2}$, $\Delta f = f_{|3}$, $Df = f_{|4}$.

Covariant derivative: ∇; in component calculus, semicolon. (Sometimes other symbols are used to indicate that in V_4 a metric different from g_{ab} is used, e.g. $h_{ab\|c} = 0$, $\gamma_{ab=c} = 0$.)

Lie derivative of a tensor $\boldsymbol{T}$ with respect to a vector $\boldsymbol{v}$: $\pounds_{\boldsymbol{v}}\boldsymbol{T}$.

Exterior derivative: d.

Connection and curvature

Connection coefficients: $\Gamma^a{}_{bc}$, $v^a{}_{;c} = v^a{}_{,c} + \Gamma^a{}_{bc}v^b$.

Connection 1-forms: $\boldsymbol{\Gamma}^a{}_b \equiv \Gamma^a{}_{bc}\boldsymbol{\omega}^c$, $\mathrm{d}\boldsymbol{\omega}^a = -\boldsymbol{\Gamma}^a{}_b \wedge \boldsymbol{\omega}^b$.

Riemann tensor: $R^d{}_{abc}$, $2v_{a;[bc]} = v_d R^d{}_{abc}$.

Curvature 2-forms: $\boldsymbol{\Theta}^a{}_b \equiv \frac{1}{2} R^a{}_{bcd}\boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^d = \mathrm{d}\boldsymbol{\Gamma}^a{}_b + \boldsymbol{\Gamma}^a{}_c \wedge \boldsymbol{\Gamma}^c{}_b$.

Ricci tensor, Einstein tensor, and scalar curvature:

$$R_{ab} \equiv R^c{}_{acb}, \quad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}, \quad R \equiv R_a{}^a.$$

Weyl tensor in V_4:

$$C_{abcd} \equiv R_{abcd} + \frac{R}{3} g_{a[c}g_{d]b} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a}.$$

Null tetrad components of the Weyl tensor:

$$\Psi_0 \equiv C_{abcd}k^a m^b k^c m^d, \qquad \Psi_1 \equiv C_{abcd}k^a l^b k^c m^d,$$

$$\Psi_2 \equiv \frac{1}{2} C_{abcd}k^a l^b (k^c l^d - m^c \overline{m}^d),$$

$$\Psi_3 \equiv C_{abcd}l^a k^b l^c \overline{m}^d, \qquad \Psi_4 \equiv C_{abcd}l^a \overline{m}^b l^c \overline{m}^d.$$

Metric of a 2-space of constant curvature:

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 \pm \Sigma^2(x, \varepsilon)\,\mathrm{d}y^2,$$

$$\Sigma(x, \varepsilon) = \sin x,\ x,\ \sinh x \text{ resp. when } \varepsilon = 1,\ \ 0 \text{ or } -1.$$

Gaussian curvature: K.

Physical fields

Energy-momentum tensor: T_{ab}, $T_{ab}u^a u^b \geqq 0$ if $u_a u^a = -1$.

Electromagnetic field: Maxwell tensor F_{ab}, $T_{ab} = \frac{1}{2} F^{*c}_{a}\overline{F^{*}_{bc}}$.