

# EXACT SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS

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## Editor's Preface

For a period of more than a decade some members of the Jena Department of Relativistic Physics have been studying the literature on the exact solutions of Einstein's gravitational field equations. Following this line of work, they also did a considerable amount of research on the classification of these solutions and on new methods of finding such solutions (H. Stephani on the embedding method, D. Kramer and G. Neugebauer on the invariance transformation method, E. Herlt on stationary axisymmetric solutions). After some years, conditions seemed favourable for a systematic treatment of these subjects. As is well known among specialists, many interesting exact solutions have been independently discovered two or more times.

This situation inspired H. Stephani and D. Kramer to write a monograph on exact solutions of Einstein's field equations, presenting the most important methods and solutions in a systematic way, with the aim of accomplishing a kind of catalogue which would give a good survey of the solutions already discovered and help to avoid duplications of discovery. It was originally hoped to offer this monograph to the international relativity community on the occasion of Einstein's 100th birthday, but soon an enormous amount of material accumulated. Therefore the target date was amended to that of the 9th International Conference on General Relativity and Gravitation in July 1980 at Jena, and to achieve even this aim required very substantial support from the Jena Department of Relativistic Physics and the Sektion Physik of the Friedrich Schiller University. We were all very happy that, in addition to E. Herlt, M. MacCallum (Queen Mary College, London) could be enlisted as an author. Furthermore, M. MacCallum agreed to revise the English text. As the head of the department and as the editor of this monograph I would like to thank the authorities of the Friedrich Schiller University, Jena, for their constant help.

Jena, February 1979

Ernst Schmutzer  
Friedrich Schiller  
University Jena

## Authors' Preface

When, in 1975, two of the authors (D. K. and H. S.) proposed to change their field of research back to the subject of exact solutions of Einstein's field equations, they of course felt it necessary to make a careful study of the papers published in the meantime, so as to avoid duplication of known results. A fairly comprehensive review or book on the exact solutions would have been a great help, but no such book was available. This prompted them to ask "Why not use the preparatory work we have to do in any case to write such a book?" After some discussion, they agreed to go ahead with this idea, and then they looked for co-authors. They succeeded in finding two.

The first was E. H., a member of the Jena relativity group, who had been engaged before on the exact solutions and was also inclined to return to them.

The second, M. M., became involved by responding to the existing authors' appeal for information and then (during a visit by H. S. to London) agreeing to look over the English text. Eventually he also agreed to write some parts of the book. He wishes to record that any infelicities remaining in the English arose because the generally good standard of his colleagues' English lulled him into a false sense of security.

Our original optimism somewhat diminished when references to over 2000 papers had been collected and the magnitude of the task became all too clear. How could we extract even the most important information from this mound of literature? How could we avoid constant re-writing to incorporate new information, which would have made the job akin to the proverbial painting of the Forth bridge? How could we decide which topics to include and which to omit? How could we check the calculations, put the results together in a readable form, and still finish in a reasonable time?

Looking back now at the result of three years' work, we cannot really feel that we solved any of these questions in a completely convincing manner. In particular, we feel sure we must have accidentally overlooked many useful results and solutions. However, we did manage to produce an outcome in a finite time, largely because the labour of reading those papers conceivably relevant to each chapter, and then drafting the related manuscript, was divided. (Roughly, D. K. was responsible for most of the introductory Part I., M. M., D. K. and H. S. dealt with groups (Part II.), H. S., D. K. and E. H. with algebraically special solutions, and D. K. and H. S. with Part IV. (special methods) and Part V. (tables).) Each draft was then criticized by the other authors, so that its writer could not be held wholly responsible for any errors or omissions. (Since we hope to maintain up-to-date information, we will be glad to hear

from any reader who detects such errors or omissions; we will also be pleased to answer as best we can any requests for further information.)

This book could not have been written, of course, without the efforts of the many scientists whose work is recorded here, and especially the many contemporaries who sent preprints, reprints, references and advice. More immediately, it would not have appeared without the help of Frau Kaschlik and Frau Reichardt in Jena, and Mrs. Smith in London, who did all the secretarial work including typing the illegible and apparently interminable manuscript, of the students in Jena who maintained our reference files, of Prof. Schmutzer, who supported the project from the beginning, and of the Sektion Physik in Jena and the Department of Applied Mathematics at Queen Mary College, London. Last but not least, we thank wives, families and colleagues for tolerating our incessant brooding and discussions.

January 1979

Dietrich Kramer  
Hans Stephani  
Eduard Herlt  
Jena

Malcolm MacCallum  
London





# Contents

<b>Notation</b>	<b>15</b>
<b>Chapter 1. Introduction</b>	<b>19</b>
1.1. What are exact solutions, and why study them?	19
1.2. The development of the subject	21
1.3. The contents and arrangement of this book	22
1.4. Using this book as a catalogue.	23
<b>Part I. GENERAL METHODS</b>	<b>25</b>
<b>Chapter 2. Differential geometry without a metric</b>	<b>27</b>
2.1. Introduction	27
2.2. Differentiable manifolds	27
2.3. Tangent vectors	29
2.4. One-forms	30
2.5. The exterior product	31
2.6. Tensors	32
2.7. The exterior derivative	34
2.8. The Lie derivative	37
2.9. The covariant derivative	39
2.10. The curvature tensor	41
<b>Chapter 3. Some topics in Riemannian geometry</b>	<b>43</b>
3.1. Introduction	43
3.2. The metric tensor and null tetrads	43
3.3. Calculation of curvature from the metric	46
3.4. Bivectors	47
3.5. Decomposition of the curvature tensor	49
3.6. Spinors	52
3.7. Conformal transformations	55
<b>Chapter 4. The Petrov classification</b>	<b>57</b>
4.1. The eigenvalue problem	57
4.2. The Petrov types	58
4.3. Principal null directions	61
4.4. Determination of the Petrov type	63

<b>Chapter 5. Classification of the Ricci tensor and the energy-momentum tensor</b>	66
5.1. The algebraic types of the Ricci tensor	66
5.2. The energy-momentum tensor	69
5.3. The energy conditions	71
5.4. The Rainich conditions	72
5.5. Perfect fluids	74
<b>Chapter 6. Vector fields</b>	76
6.1. Vector fields and their invariant classification	76
6.2. Vector fields and the curvature tensor	79
<b>Chapter 7. The Newman-Penrose formalism</b>	82
7.1. The spin coefficients and the field equations	82
7.2. The commutators and Bianchi identities	85
7.3. The modified calculus	87
7.4. Geodesic null congruences	89
7.5. The Goldberg-Sachs theorem and its generalizations	90
<b>Chapter 8. Continuous groups of transformations. Groups of motions</b>	93
8.1. Introduction: Lie groups and Lie algebras	93
8.2. Enumeration of distinct group structures	96
8.3. Transformation groups	98
8.4. Groups of motions	99
8.5. Spaces of constant curvature	101
8.6. Orbits of isometry groups	104
<b>Part II. SOLUTIONS WITH GROUPS OF MOTIONS</b>	111
<b>Chapter 9. Classification of solutions with isometries</b>	113
9.1. The cases to be considered	113
9.2. Isotropy and the curvature tensor	114
<b>Chapter 10. Homogeneous space-times</b>	117
10.1. The possible metrics	117
10.2. Homogeneous vacuum and null Einstein-Maxwell space-times	119
10.3. Homogeneous non-null electromagnetic fields	120
10.4. Homogeneous perfect fluid solutions	122
10.5. Other homogeneous solutions	124
10.6. Summary	125
<b>Chapter 11. Hypersurface-homogeneous space-times</b>	126
11.1. The possible metrics	126
11.2. Formulation of the field equations	130
11.3. Vacuum, $\Lambda$ -term and Einstein-Maxwell solutions	133
11.4. Perfect fluid solutions homogeneous on $T_3$	139
11.5. Summary of all metrics with $G_T$ on $V_3$	140

<b>Chapter 12. Spatially-homogeneous perfect fluid cosmologies . . . . .</b>	<b>143</b>
12.1. Introduction . . . . .	143
12.2. Robertson-Walker cosmologies . . . . .	144
12.3. Cosmologies with a $G_4$ on $S_3$ . . . . .	145
12.4. Cosmologies with a $G_3$ on $S_3$ . . . . .	148
<b>Chapter 13. Groups <math>G_3</math> on non-null orbits <math>V_2</math>. Spherical and plane symmetry . . . . .</b>	<b>152</b>
13.1. Metric, Killing vectors, and Ricci tensor . . . . .	152
13.2. Some implications of the existence of an isotropy group $H_1$ . . . . .	154
13.3. Spherical and plane symmetry . . . . .	154
13.4. Vacuum, Einstein-Maxwell and pure radiation fields. . . . .	155
13.5. Dust solutions . . . . .	159
13.6. Plane symmetric perfect fluids . . . . .	161
<b>Chapter 14. Spherically symmetric perfect fluid solutions . . . . .</b>	<b>163</b>
14.1. Static solutions . . . . .	163
14.2. Non-static solutions . . . . .	165
<b>Chapter 15. Groups <math>G_2</math> and <math>G_1</math> on non-null orbits . . . . .</b>	<b>174</b>
15.1. Group structures $G_2$ and group orbits $V_2$ . . . . .	174
15.2. Colliding plane waves . . . . .	176
15.3. Closed universes built from gravitational waves . . . . .	177
15.4. Group $G_1$ on non-null orbits . . . . .	178
<b>Chapter 16. Stationary gravitational fields . . . . .</b>	<b>180</b>
16.1. The projection formalism . . . . .	180
16.2. The Ricci tensor on $\Sigma_3$ . . . . .	181
16.3. Conformal transformation of $\Sigma_3$ and the field equations . . . . .	183
16.4. Vacuum and Einstein-Maxwell equations for stationary fields. . . . .	184
16.5. Geodesic eigenrays . . . . .	186
16.6. Static fields . . . . .	187
16.7. The conformastationary class of Einstein-Maxwell fields . . . . .	191
<b>Chapter 17. Stationary axisymmetric fields: basic concepts and field equations . . . . .</b>	<b>192</b>
17.1. The Killing vectors . . . . .	192
17.2. Orthogonal surfaces . . . . .	192
17.3. The metric and the projection formalism . . . . .	194
17.4. The field equations for stationary axisymmetric Einstein-Maxwell fields . . . . .	196
17.5. Various forms of the field equations for stationary axisymmetric vacuum fields . . . . .	197
<b>Chapter 18. Stationary axisymmetric vacuum solutions . . . . .</b>	<b>200</b>
18.1. Static axisymmetric vacuum solutions (Weyl's class) . . . . .	200
18.2. Fields of uniformly accelerated particles . . . . .	202
18.3. The class of solutions $U = U(\omega)$ (Papapetrou's class) . . . . .	203
18.4. The class of solutions $S = S(A)$ . . . . .	204
18.5. The Kerr solution and the Tomimatsu-Sato class . . . . .	205
18.6. Other solutions . . . . .	206
<b>Chapter 19. Non-empty stationary axisymmetric solutions. . . . .</b>	<b>209</b>
19.1. Einstein-Maxwell fields . . . . .	209
19.2. Perfect fluid solutions . . . . .	215

<b>Chapter 20. Cylindrical symmetry</b> . . . . .	220
20.1. General remarks . . . . .	220
20.2. Stationary cylindrically symmetric fields . . . . .	221
20.3. Vacuum fields . . . . .	223
20.4. Einstein-Maxwell fields and pure radiation fields . . . . .	224
20.5. Perfect fluid solutions . . . . .	227
<b>Chapter 21. Groups on null orbits. Plane waves</b> . . . . .	228
21.1. Introduction . . . . .	228
21.2. Groups $G_3$ on $N_3$ . . . . .	228
21.3. Groups $G_2$ on $N_2$ . . . . .	230
21.4. Null Killing vectors ( $G_1$ on $N_1$ ) . . . . .	231
21.5. The plane-fronted gravitational waves with parallel rays ( <i>pp</i> waves) . . . . .	233
<b>PART III. ALGEBRAICALLY SPECIAL SOLUTIONS</b> . . . . .	237
<b>Chapter 22. The various classes of algebraically special solutions.</b>	
Newman-Tamburino solutions . . . . .	239
22.1. Solutions of Petrov type <i>II</i> , <i>D</i> , <i>III</i> , or <i>N</i> . . . . .	239
22.2. Conformally flat solutions . . . . .	243
22.3. Newman-Tamburino solutions . . . . .	243
<b>Chapter 23. The line element for metrics with <math>\kappa = \sigma = 0 = R_{11} = R_{14} = R_{44}</math>,</b>	
$\Theta + i\omega \neq 0$ . . . . .	245
23.1. The line element in the case with twisting rays ( $\omega \neq 0$ ) . . . . .	245
23.2. The line element in the case with non-twisting rays ( $\omega = 0$ ) . . . . .	249
<b>Chapter 24. Robinson-Trautman solutions</b> . . . . .	250
24.1. Robinson-Trautman vacuum solutions . . . . .	250
24.2. Robinson-Trautman Einstein-Maxwell fields . . . . .	254
24.3. Robinson-Trautman pure radiation solutions . . . . .	260
24.4. Robinson-Trautman solutions with a cosmological constant $\Lambda$ . . . . .	261
<b>Chapter 25. Twisting vacuum solutions</b> . . . . .	262
25.1. Twisting vacuum solutions — the field equations . . . . .	262
25.2. Some general classes of solutions . . . . .	267
25.3. Solutions of type <i>N</i> ( $\Psi_2 = \Psi_3 = 0$ ) . . . . .	273
25.4. Solutions of type <i>III</i> ( $\Psi_2 = 0, \Psi_3 \neq 0$ ) . . . . .	274
25.5. Solutions of type <i>D</i> ( $3\Psi_2\Psi_4 = 2\Psi_3^2, \Psi_2 \neq 0$ ) . . . . .	274
25.6. Solutions of type <i>II</i> . . . . .	276
<b>Chapter 26. Twisting Einstein-Maxwell and pure radiation field solutions</b> . . . . .	277
26.1. The structure of the Einstein-Maxwell field equations . . . . .	277
26.2. Determination of the radial dependence of the metric and the Maxwell field . . . . .	278
26.3. The remaining field equations . . . . .	279

26.4. Charged vacuum metrics . . . . .	280
26.5. Remarks concerning solutions of the different Petrov types . . . . .	281
26.6. Pure radiation fields . . . . .	282
<b>Chapter 27. Non-diverging solutions (Kundt's class) . . . . .</b>	<b>286</b>
27.1. Introduction . . . . .	287
27.2. The line element for metrics with $\Theta + i\omega = 0$ . . . . .	287
27.3. The Ricci tensor components . . . . .	288
27.4. The structure of the vacuum and Einstein-Maxwell equations . . . . .	289
27.5. Vacuum solutions . . . . .	292
27.6. Einstein-Maxwell null fields and pure radiation fields . . . . .	294
27.7. Einstein-Maxwell non-null fields . . . . .	295
<b>Chapter 28. Kerr-Schild metrics . . . . .</b>	<b>298</b>
28.1. General properties of Kerr-Schild metrics . . . . .	298
28.2. Application of the Kerr-Schild ansatz to Einstein's vacuum field equations . . . . .	304
28.3. Application of the Kerr-Schild ansatz to the Einstein-Maxwell equations . . . . .	305
28.4. Application of the Kerr-Schild ansatz to pure radiation fields . . . . .	308
28.5. Generalizations of the Kerr-Schild ansatz . . . . .	312
<b>Chapter 29. Algebraically special perfect fluid solutions . . . . .</b>	<b>313</b>
29.1. Generalized Robinson-Trautman solutions . . . . .	313
29.2. Solutions with a geodesic, shearfree, non-expanding multiple null eigenvector . . . . .	315
29.3. Type <i>D</i> solutions . . . . .	316
29.4. Type <i>III</i> and type <i>N</i> solutions . . . . .	319
<b>Part IV. SPECIAL METHODS . . . . .</b>	<b>321</b>
<b>Chapter 30. Generation techniques . . . . .</b>	<b>323</b>
30.1. Introduction . . . . .	323
30.2. The potential space . . . . .	323
30.3. The invariance transformations for Einstein-Maxwell fields . . . . .	325
30.4. The generation theorems for Einstein-Maxwell fields admitting an Abelian group $G_2$ . . . . .	333
30.5. Applications . . . . .	336
30.6. Other generation methods . . . . .	341
<b>Chapter 31. Special vector and tensor fields . . . . .</b>	<b>343</b>
31.1. Space-times that admit constant vector and tensor fields . . . . .	343
31.2. Complex recurrent, conformally recurrent, recurrent and symmetric space-times . . . . .	345
31.3. Killing tensors of order two . . . . .	348
31.4. Some remarks concerning space-times with other special properties . . . . .	352
<b>Chapter 32. Local isometric embedding of four-dimensional Riemannian manifolds . . . . .</b>	<b>354</b>
32.1. The why of embedding . . . . .	354
32.2. The basic formulae governing embedding . . . . .	355
32.3. Some theorems on local isometric embedding . . . . .	356

32.4. Exact solutions of embedding class one . . . . .	360
32.5. Exact solutions of embedding class two . . . . .	367
32.6. Exact solutions of embedding class $p > 2$ . . . . .	373
32.7. Remarks on global embedding . . . . .	374
 <b>Part V. TABLES</b> . . . . .	 375
 <b>Chapter 33. The interconnections between the main classification schemes</b> . . . . .	 377
33.1. Introduction . . . . .	377
33.2. The connection between Petrov types and groups of motion . . . . .	378
33.3. Tables . . . . .	382
 <b>Bibliography</b> . . . . .	 386
 <b>Index</b> . . . . .	 419

## Notation

All symbols are explained in the text. Here we list only some important conventions which are frequently used throughout the book.

Complex conjugation is denoted by a bar over the symbol.

### Indices

Small Latin indices run, in an  $n$ -dimensional space, from 1 to  $n$ , in space-time  $V_4$  from 1 to 4. Indices from the first part of the alphabet ( $a, b, \dots, h$ ) are tetrad indices, i.e. they refer to a general basis  $\{e_a\}$  or its dual  $\{\omega^a\}$ ;  $i, j, \dots$  are reserved for a coordinate basis  $\{\partial/\partial x^i\}$  or its dual  $\{dx^i\}$ . For a vector  $v$  and a 1-form  $\sigma$  we write  $v = v^a e_a = v^i \partial/\partial x^i$ ,  $\sigma = \sigma_a \omega^a = \sigma_i dx^i$ . Small Greek indices run from 1 to 3, if not otherwise stated. Capital Latin indices are either spinor indices ( $A, B = 1, 2$ ) or indices in group space ( $A, B = 1 \dots r$ ), or they label the coordinates in a Riemannian 2-space  $V_2$  ( $M, N = 1, 2$ ).

Symmetrization and antisymmetrization of index pairs:

$$v_{(ab)} \equiv \frac{1}{2} (v_{ab} + v_{ba}), \quad v_{[ab]} \equiv \frac{1}{2} (v_{ab} - v_{ba}).$$

### Metric and tetrads

Line element in terms of dual basis  $\{\omega^a\}$ :  $ds^2 = g_{ab} \omega^a \omega^b$ .

Signature of space-time metric:  $(+ + + -)$ .

Commutation coefficients:  $D_{ab}^c$ ;  $[e_a, e_b] = D_{ab}^c e_c$ .

(Complex) null tetrad:  $\{e_a\} = (\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k})$ ,  $g_{ab} = 2m_{(a}\bar{m}_{b)} - 2k_{(a}l_{b)}$ ,  
 $ds^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4$ .

Orthonormal basis:  $\{E_a\}$ .

Projection tensor:  $h_{ab} \equiv g_{ab} + u_a u_b$ ,  $u_a u^a = -1$ .

### Bivectors

Levi-Civita tensor:  $\varepsilon_{abcd}$ ;  $\varepsilon_{abcd} m^a \bar{m}^b l^c k^d = i$ .

Dual bivector:  $\tilde{X}_{ab} \equiv \frac{1}{2} \varepsilon_{abcd} X^{cd}$ .

(Complex) self-dual bivector:  $X_{ab}^* \equiv X_{ab} + i\tilde{X}_{ab}$ .

Basis of self-dual bivectors:  $U_{ab} \equiv 2\bar{m}_{[a}l_{b]}$ ,  $V_{ab} \equiv 2k_{[a}m_{b]}$ ,  $W_{ab} \equiv 2m_{[a}\bar{m}_{b]} - 2k_{[a}l_{b]}$ .



## Derivatives

Partial derivative: comma in front of index or coordinate, e.g.

$$f_{,i} \equiv \partial f / \partial x^i \equiv \partial_i f, \quad f_{,t} \equiv \partial f / \partial \xi.$$

Directional derivative: denoted by stroke or comma,  $f_{|a} \equiv f_{,a} \equiv e_a(f)$ , if followed by a numerical (tetrad) index, we prefer the stroke, e.g.  $f_{|4} = f_{,i} k^i$ . Directional derivatives with respect to the null tetrad  $(m, \bar{m}, l, k)$  are symbolized by  $\delta f = f_{|1}$ ,  $\bar{\delta} f = f_{|2}$ ,  $\Delta f = f_{|3}$ ,  $Df = f_{|4}$ .

Covariant derivative:  $\nabla$ ; in component calculus, semicolon. (Sometimes other symbols are used to indicate that in  $V_4$  a metric different from  $g_{ab}$  is used, e.g.  $h_{ab|c} = 0$ ,  $\gamma_{ab=c} = 0$ .)

Lie derivative of a tensor  $T$  with respect to a vector  $v$ :  $\mathcal{L}_v T$ .

Exterior derivative:  $d$ .

## Connection and curvature

Connection coefficients:  $\Gamma^a_{bc}$ ,  $v^a{}_{;c} = v^a{}_{,c} + \Gamma^a_{bc} v^b$ .

Connection 1-forms:  $\Gamma^a_b \equiv \Gamma^a_{bc} \omega^c$ ,  $d\omega^a = -\Gamma^a_b \wedge \omega^b$ .

Riemann tensor:  $R^d_{abc}$ ,  $2v_{a;[bc]} = v_d R^d_{abc}$ .

Curvature 2-forms:  $\Theta^a_b = \frac{1}{2} R^a_{bcd} \omega^c \wedge \omega^d = d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b$ .

Ricci tensor, Einstein tensor, and scalar curvature:

$$R_{ab} \equiv R^c_{acb}, \quad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}, \quad R \equiv R^a_a.$$

Weyl tensor in  $V_4$ :

$$C_{abcd} \equiv R_{abcd} + \frac{R}{3} g_{a[c} g_{d]b} - g_{a[c} R_{d]b} + g_{b[c} R_{d]a}.$$

Null tetrad components of the Weyl tensor:

$$\Psi_0 \equiv C_{abcd} k^a m^b k^c m^d, \quad \Psi_1 \equiv C_{abcd} k^a l^b k^c m^d,$$

$$\Psi_2 \equiv \frac{1}{2} C_{abcd} k^a l^b (k^c l^d - m^c \bar{m}^d),$$

$$\Psi_3 \equiv C_{abcd} l^a k^b l^c \bar{m}^d, \quad \Psi_4 \equiv C_{abcd} l^a \bar{m}^b l^c \bar{m}^d.$$

Metric of a 2-space of constant curvature:

$$d\sigma^2 = dx^2 \pm \Sigma^2(x, \varepsilon) dy^2,$$

$$\Sigma(x, \varepsilon) = \sin x, \quad x, \sinh x \text{ resp. when } \varepsilon = 1, \quad 0 \text{ or } -1.$$

Gaussian curvature:  $K$ .

## Physical fields

Energy-momentum tensor:  $T_{ab}$ ,  $T_{ab} u^a u^b \geq 0$  if  $u_a u^a = -1$ .

Electromagnetic field: Maxwell tensor  $F_{ab}$ ,  $T_{ab} = \frac{1}{2} F^{*c}_a \bar{F}^*_{bc}$ .