Algebraic

K-Theory

ALGEBRAIC K-THEORY

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W. A. BENJAMIN, INC.

1968 Amsterdam

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Library of Congress Catalog Card Number 68-24366
Manufactured in the United States of America
1234K21098

The manuscript was put into production December 13, 1967;
this volume was published on December 1, 1968

W. A. BENJAMIN, INC.
NEW YORK, NEW YORK 10016

PREFACE

This book is based on a course I gave at Columbia University in 1966-67. Its writing was greatly facilitated by the notes for that course which were taken by Tsit-Yuen Lam, M. Pavaman Murthy, and Charles Small. I am extremely grateful to them for their assistance and criticism.

I had originally hoped to make the exposition here more or less self-contained, modulo a first year algebra course. Because of the variety of techniques employed, however, this ambition threatened to lead to an infinite regress. Thus, Part 1 on preliminaries still contains, despite its length, a few results which are merely quoted without proof.

Time prevented me from including here a treatment of the "K-theory of symplectic modules," which I hope to publish in the near future. For the theory of "quadratic modules" there is so far only a discussion of the formalism (construction of the classical invariants) in my Tata lectures [4], and only partial results are known at present in the way of general stability theorems. It is worth noting, however, that the discussion in Chapter VII has been deliberately arranged so that it can be applied directly to a variety of contexts. Thus, for example, one has Mayer–Vietoris sequences and excision isomorphisms for the theories of symplectic, quadratic, and Hermitian forms, for the Brauer group, and for various other theories (roughly speaking, for those based on projective modules supplied with some type of tensor).

An important feature of algebraic K-theory, and one which has led to genuinely new insights in pure algebra, is its ability to exploit the techniques of a highly developed branch of topology—the homotopy theory of vector bundles. In turn, and for entirely different reasons, which go back to J.H.C. Whitehead's theory of simple homotopy types, the topologists are active patrons of the subject, providing an abundant supply of interesting and difficult questions with which the theory can be tested and expanded.

Under these circumstances it seemed worthwhile to make available a reasonably comprehensive and systematic treatment of the main ideas of the subject, as so far developed. I have written these notes with that intention. I hope they may be useful, as a reference to topologists, and as an invitation to an area of new techniques and problems to algebraists. Finally, I have tried to organize the notes so that they might serve as the basis for a second-year graduate algebra course, such as the one from which they originated.

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New York, New York
October 1967

INTRODUCTION

The "algebraic K-theory" presented here is, essentially, a part of general linear algebra. It is concerned with the structure theory of projective modules, and of their automorphism groups. Thus, it is a generalization, in the most naive sense, of the theorem asserting the existence and uniqueness of bases for vector spaces, and of the group theory of the general linear group over a field. One witnesses here the evolution of these theorems as the base ring becomes more general than a field. There is a satisfactory "stable form" in which the above theorems survive (Part 2). In a stricter sense these theorems fail in the general case, and the Grothendieck groups (K_0) and Whitehead groups (K_1) which we study can be viewed as providing a measure of their failure.

A topologist can similarly seek such a generalization of the structure theorems of linear algebra. He views a vector space as a special case of a vector bundle. The homotopy theory of vector bundles, and topological K-theory, then provide a completely satisfactory framework within which to treat such questions. It is remarkable that there exists, in algebra, nothing of remotely comparable depth or generality, even though many of these questions are algebraic in character.

The techniques used here are, therefore, topologically inspired. They are based on the philosophy, supported by theorems of Swan (Chapter XIV) and Serre (cf. Chapter IV), that a projective module should be thought of as the module of sections of a vector bundle. This dictates the choice of projective modules (rather than some wider class of modules) as the objects of the theory. This point of view further exhibits the stability theorems (Part 2) as direct imitations of their topological precursors (cf. Chapter XIV). It was Serre [1] who originated the techniques for proving such stability theorems in a purely algebraic setting.

The formalism of K-theory originated with Grothendieck's proof of the generalized Riemann–Roch theorem. The ideas were then quickly developed in topology by Atiyah and Hirzebruch, who made the Grothendieck groups, K(X), part of a generalized cohomology theory, using the suspension functor. While our point of view leads to an obvious translation of K(X), there is no clear algebraic counterpart for suspension. As a result our algebraic K-theory in Part 3 is far from complete, and the treatment here should be regarded as a provisional one, albeit sufficient for a number of applications in later chapters.

The development in Part 3 is axiomatic so that the results can be usefully applied to many categories other than those of projective modules. The exposition there is substantially influenced by ideas of Milnor. It was he who first called attention to the existence and importance of the Mayer–Vietoris sequence of a Cartesian square, and this has become a cornerstone of the whole theory. In particular, it leads to a very general analog of the excision isomorphisms. Otherwise the results of Part 3 are taken largely from a paper of Heller [1]. The latter contains another major tool of the theory, the exact sequence of a localizing functor, which does not seem to have any familiar topological counterpart. Chapter VIII also contains a striking new theorem of Leslie Roberts, with which he has computed K_1 for nonsingular projective algebraic varieties.

There has been some recent progress in finding satisfactory definitions of higher algebraic K's. For example, Milnor has defined a K_2, on which some work has been done by Gersten [2]. From a quite different point of view, A. Nóbile and O. Villamayor [1] have constructed an algebraic K-theory with functors K_n for all $n \geqq 0$. Other (unpublished) definitions have been proposed as well. However, in none of these cases are the new functors yet very well understood. It therefore seemed premature to attempt an excursion in that direction in these notes.

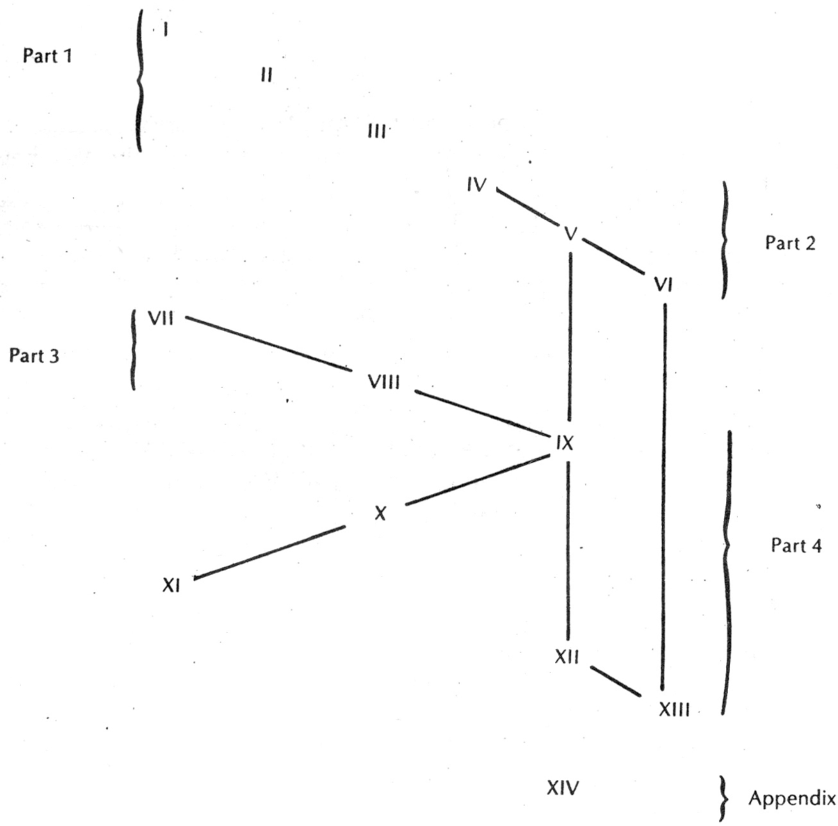
In Part 4 the general results of Parts 2 and 3 are assembled and applied to the computation of Grothendieck groups $K_0(A)$ and Whitehead groups $K_1(A)$ for a variety of rings A. Special emphasis is given to the case of group rings $A = \mathbb{Z}\pi$ because of the interest of the groups $K_1(\mathbb{Z}\pi)$ to topologists. In particular, the long Chapter XI is devoted to a new exposition of techniques, developed by Swan and Lam, which are based on the theory of induced representations for finite groups.

There are two unanticipated, and mathematically interesting, high points in the theory. The first is the fact that when A is a Dedekind ring, the group theory of $SL_n(A)$, as formulated in terms of K_1, is intimately connected with certain "reciprocity laws" in A. The latter include the classical power reciprocity laws in totally imaginary number fields as well as certain geometric reciprocity laws on algebraic curves. This phenomenon was first witnessed in the recent papers of C. Moore [1] and of Bass–Milnor–Serre [1]. The discussion of this in Chapter VI is an axiomatization, based the latter reference. I am further indebted here to T.-Y. Lam for a number of suggestions. The upshot of this theory is that known reciprocity laws can be used to compute K_1. Conversely, using the machinery developed in later chapters, we can sometimes compute K_1 directly, and in turn use these calculations to exhibit new reciprocity laws. Examples of both of these procedures occur in the text (cf. Chapters VI and XII).

The other surprise is the "Fundamental Theorem" in Chapter XII, §7, which computes $K_1(A[t, t^{-1}])$. Its principal feature is that $K_0(A)$ appears as a natural direct summand of $K_1(A[t, t^{-1}])$. This is surprising because, at least algebraically, K_0 and K_1 look like rather different kinds of animals. The surprise disappears, however, if one interprets the theorem topologically, whereupon it is seen to be an algebraic analog of Bott's complex periodicity theorem (cf. Chapter XIV, §6). This theorem first appeared (in a less precise form) in the paper of Bass–Heller–Swan [1]. A new feature, which emerged only at the end of the writing of these notes, is that the fundamental theorem has a built-in iteration procedure, which can be used to manufacture a whole sequence of functors K_{-n} $(n \geqq 0)$ with which to extend the (K_1, K_0)—exact sequence to the right. They help to clarify some calculations made in Bass–Murthy [1], but their significance is otherwise still unclear (to me).

LOGICAL DEPENDENCE OF CHAPTERS

The following diagram is a rough indication of the logical interdependence of the chapters. If Chapter B depends logically on Chapter A then A is placed above B; the converse is not necessarily true. In some cases this dependence is rather peripheral, so a line joining A and B appears only when the contents of A are an essential prerequisite for the reading of B.



SOME GENERAL NOTATION

Let A be a ring. We write

$$\text{mod-}A \quad \text{and} \quad A\text{-mod}$$

for the categories of right and left A-modules, respectively. We have the full subcategories

$$\underline{\underline{P}}(A) \subset \underline{\underline{H}}(A) \subset \underline{\underline{M}}(A) \subset \text{mod-}A$$

defined as follows: $M \in \underline{\underline{M}}(A) \Leftrightarrow M$ is a finitely generated A-module, and $M \in \underline{\underline{P}}(A) \Leftrightarrow M$ is also projective. Finally, $M \in \underline{\underline{H}}(A) \Leftrightarrow M$ has a finite resolution by objects of $\underline{\underline{P}}(A)$ (see Chapter III, §6).

Let R be a commutative ring and suppose A is an R-algebra. Let S be a multiplicative set in R and let $\underline{\underline{C}}$ be a subcategory of mod-A. Then $\underline{\underline{C}}_S$ denotes the full subcategory of all $M \in \underline{\underline{C}}$ such that $S^{-1} M = 0$.

The ring of n by n matrices over A is denoted $M_n(A)$, and its invertible elements constitute the group $GL_n(A)$. We often identify $M_n(A)$ with the A-endomorphisms of the right A-module A^n. When $n = 1$ we write

$$U(A) = GL_1(A)$$

so that $GL_n(A) = U(M_n(A))$.

If $\underline{\underline{C}}$ is any category we write

$$\Sigma \underline{\underline{C}}$$

for the category of pairs (M, α) $(M \in \underline{\underline{C}}, \alpha \in \text{Aut}_{\underline{\underline{C}}}(M))$ (see Chapter VII, §1), i.e., the category of automorphisms of objects of $\underline{\underline{C}}$.

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PART 3. *ALGEBRAIC K-THEORY*

PART 4. *K-THEORY OF PROJECTIVE MODULES*

Part 1

PRELIMINARIES

Chapter I

SOME CATEGORICAL ALGEBRA

This chapter introduces some of the basic language of categories and functors. It should be used mainly for reference, rather than being read outright. The first sections lead up to the notion of an Abelian category, in §4. In §§5-6 we assemble some basic facts about homology and projective resolutions which will be used extensively in the following sections. In §8 we prepare some less standard results on direct limits, which are needed in Chapter VII.

Essentially all of the material of this chapter can be found in the books of MacLane [1] and Mitchell [1].

§1. CATEGORIES AND FUNCTORS

Recall that a category $\underline{\underline{A}}$ consists of <u>objects</u>, ob $\underline{\underline{A}}$, a set of morphisms, $\underline{\underline{A}}(A, B)$, for each $A, B \in$ ob $\underline{\underline{A}}$, and a composition

$$\underline{\underline{A}}(B, C) \times \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{A}}(A, C), \quad (a, b) \longrightarrow ab$$

The latter is associative, and there are identities $1_A \in \underline{\underline{A}}(A, A)$ with the usual properties. The <u>dual category</u> $\underline{\underline{A}}^0$ has the same objects, $\underline{\underline{A}}^0(A, B) = \underline{\underline{A}}(B, A)$, and composition is reversed. The dual of a statement about categories is the same statement but interpreted in $\underline{\underline{A}}^0$. In this sense, general theorems about categories have duals, and the latter are also theorems.

The notion of subcategory is obvious. Similarly, we can form the Cartesian product of categories, in a naive way, to obtain new categories.

We shall often confuse $\underline{\underline{A}}$ with ob $\underline{\underline{A}}$, and write $A \in \underline{\underline{A}}$ in place of $A \in$ ob $\underline{\underline{A}}$. The class of all morphisms in $\underline{\underline{A}}$ is denoted mor $\underline{\underline{A}}$.

$$a: \; A \longrightarrow B \quad \text{means} \quad a \in \underline{\underline{A}}(A, B)$$

as usual. We call a an <u>isomorphism</u> if there exists $b \in \underline{\underline{A}}(B, A)$ such that $ab = 1_B$ and $ba = 1_A$, i.e. if a is invertible. We call a a monomorphism (resp., epimorphism) if $ab = ac \Rightarrow b = c$ (resp., $ba = ca \Rightarrow b = c$), whenever the indicated compositions are defined. Note that an isomorphism is both a monomorphism and an epimorphism. The converse fails in general. For example, in the category of topological groups and continuous homomorphisms, an inclusion of a dense subgroup is an epimorphism and a monomorphism.

We shall commonly use the following alternative notations:

$$\mathrm{Hom}_{\underline{\underline{A}}}(A, B) = \underline{\underline{A}}(A, B)$$

$$\mathrm{End}_{\underline{\underline{A}}}(A) = \underline{\underline{A}}(A, A)$$

$$\mathrm{Aut}_{\underline{\underline{A}}}(A) = \text{the group of automorphisms of } A \text{ (in } \underline{\underline{A}}).$$

A <u>functor</u> T: $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ consists of a map on objects, $A \longmapsto TA$, and maps on morphisms

$$T(=T_{A,\,B}): \; \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{B}}(TA, TB)$$

which preserve composition and identities. T is called <u>faithful</u> (resp., <u>full</u>) if $T_{A,\,B}$ is injective (resp., surjective) for all $A, B \in \underline{\underline{A}}$. Note that a faithful functor might carry nonismorphic objects to ismorphic ones (e.g., the functor (topological groups) $\xrightarrow[\text{topology}]{\text{ignore the}}$ (groups)) but this cannot happen if it is also full. A <u>contravariant functor</u> $\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is a functor $\underline{\underline{A}}^0 \longrightarrow \underline{\underline{B}}$. <u>Functors of several variables</u> are just functors on product categories.

In practice a category will often be specified by naming only its objects. Such license will be allowed when either the morphisms and composition are clear from the context, or, if there is some ambiguity, it is of no consequence for the discussion at hand. Similarly, we shall often define functors by specifying their effect on objects when their effect on morphisms is then clear from the context.

The functors from $\underline{\underline{A}}$ to $\underline{\underline{B}}$ are themselves the objects of a category, denoted $\underline{\underline{B}}^{\underline{\underline{A}}}$. The morphisms are sometimes called <u>natural transformations</u>, so we write

$$\text{Nat. Tran.}(T, S) = \underline{\underline{B}}^{\underline{\underline{A}}}(T, S)$$

A natural transformation $\alpha: T \longrightarrow S$ is a family, $\alpha = (\alpha_A)_{A \in \underline{\underline{A}}}$, of $\underline{\underline{B}}$ - morphisms $\alpha_A : TA \longrightarrow SA$ such that $Sf\,\alpha_A = \alpha_B\,Tf$ whenever $f : A \longrightarrow B$ in $\underline{\underline{A}}$. (Rather innocent assumptions on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ will guarantee that $\underline{\underline{B}}^{\underline{\underline{A}}}(T, S)$ is a set; this will always be so in the examples we treat.) Composition is defined in the obvious way. Suppose we are given functors

$$\underline{\underline{A}} \xrightarrow{S} \underline{\underline{B}} \underset{T_2}{\overset{T_1}{\rightrightarrows}} \underline{\underline{C}} \xrightarrow{U} \underline{\underline{D}}$$

and a morphism $\alpha: T_1 \longrightarrow T_2$. Then we have the composite functors, T_iS, UT_i, etc., and we also have morphisms

$$\alpha S : T_1S \longrightarrow T_2S \qquad (\alpha S)_A = \alpha_{SA} \quad (A \in \underline{\underline{A}})$$

and

$$U\alpha : UT_1 \longrightarrow UT_2 \qquad (U\alpha)_B = U(\alpha_B) \quad (B \in \underline{\underline{B}})$$

If $S^1: \underline{\underline{A}}^1 \longrightarrow \underline{\underline{A}}$ and $U^1: \underline{\underline{D}} \longrightarrow \underline{\underline{D}}^1$ are functors, and if $\alpha^1: T_2 \longrightarrow T_3$ is a morphism of functors then we have the following easily verified rules:

$$\alpha(SS^1) = (\alpha S)S^1 \quad , \qquad (U^1U)\alpha = U^1(U\alpha)$$
$$1_{T_i}S = 1_{T_iS} \quad , \qquad U\,1_{T_i} = 1_{UTi}$$
$$(\alpha^1\alpha)S = (\alpha^1S)(\alpha S) \quad , \qquad U(\alpha^1\alpha) = (U\alpha^1)(U\alpha)$$

The latter show that composition with S and U defines functors $\cdot S : \underline{\underline{C}}^{\underline{\underline{B}}} \longrightarrow \underline{\underline{C}}^{\underline{\underline{A}}}$ and $U\cdot : \underline{\underline{C}}^{\underline{\underline{B}}} \longrightarrow \underline{\underline{D}}^{\underline{\underline{B}}}$, respectively.

A functor $T : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is an <u>isomorphism</u> if there is a functor $S : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ such that $TS = 1_{\underline{\underline{B}}}$ and $ST = 1_{\underline{\underline{A}}}$.