

TRIGONOMETRY: A New Approach

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Preface

This text is written for a one-semester or one-quarter course in plane trigonometry. Upon completion, the student will be prepared for many technical courses, including some physics courses and the analytic geometry and calculus sequence. Every student wishing to pursue a study of mathematics must know trigonometry. The prerequisites for a trigonometry course are 2 years of high-school algebra and a study of plane geometry or the equivalent of these courses.

Trigonometry is basically concerned with the six circular (or trigonometric) functions. Because these functions are important in the mathematics courses that follow trigonometry, a functional approach is used throughout this text.

The wrapping function on the unit circle is first introduced, and its properties are studied as a springboard to the sine and cosine as circular functions (see Note to the Instructor below). The remaining four circular functions are presented after an in-depth study is made of the sine and cosine. The six functions are later defined as trigonometric functions on the domain of generated angles. Both the unit circle and the generated angle approaches are used because each has its own merits in the theoretical development and applications of trigonometry.

Other features of the text are as follows:

- It is written for the student. The main topics are illustrated by examples, solved in detail, which guide the reader in the solution of homework exercises. Over 275 solved examples are presented.
- 2. Many applications of trigonometry are included to show its usefulness. Applied problems are found in the appropriate exercise lists.
- Historical references discuss the origin and development of trigonometry. These notes add perspective and present trigonometry as a work of humanity.
- 4. Several sections feature instructions for using an electronic calculator to solve problems, and some exercise lists have problems specially designed to be solved in that manner;

these problems are indicated by the symbol . It is recommended that students should purchase scientific calculators that have keys for the trigonometric and inverse trigonometric functions. An adequate scientific calculator can be purchased for less that the price of this book, and the money will be well spent in terms of the time saved and the accuracy gained when solving the homework exercises.

It is not necessary to have a calculator when using this text because exercises marked for a calculator can be omitted, and a full course in trigonometry still remains. In the absence of a calculator, the tables of Appendix E are sufficient for exercises that are not

marked . Instructions for using these tables are given in the appropriate sections.

An extensive list of exercises found in each section ranges in difficulty from drill-type problems to problems that challenge the student. Other exercises question the student about the definitions, theorems, and theoretical aspects of trigonometry. A comprehensive set of review exercises concludes each chapter. Exercises are also included in Appendix A, Preliminary Algebraic Topics, and Appendix B, Mathematical Induction. Approximately 2700 exercises appear in the text; the answers to most odd-numbered problems are at the back of the book, and answers to the even-numbered problems are found in the instructor's manual.

6. Chapter 0 includes those geometric facts most essential for trigonometry. The necessary algebraic facts are found in Appendix A and in Sections 1.1 and 7.1.

Note to the Instructor

Use of the wrapping function on the unit circle is exploited to its fullest in this text. The advantages of this approach are as follows:

- 1. The wrapping function W is a periodic function. W is used to define the six circular functions, which "inherit" the periodic nature of W. For example, from $W(\theta) = W(\theta + 2\pi k)$ we obtain $\sin \theta = \sin(\theta + 2\pi k)$. The periodic nature of the circular (or trigonometric) functions is one reason for their importance in scientific applications and in higher mathematics.
- 2. W is defined with reference to the unit circle $x^2 + y^2 = 1$. The sine and cosine have the same domain as W and "inherit" their ranges from the fact that on the unit circle $|x| \le 1$ and $|y| \le 1$.
- 3. In Chapter 1, several important properties of W are exploited to quickly yield identities for the circular functions. For example, from the fact that W is defined on the unit circle $x^2 + y^2 = 1$, we obtain $\cos^2\theta + \sin^2\theta = 1$. Also, one of the first theorems states that if $W(\theta) = (x, y)$ then $W(-\theta) = (x, -y)$. This implies that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In a similar fashion, the following identities are obtained in Chapter 1: $\sin(\pi \theta) = \sin\theta$; $\sin(\pi + \theta) = -\sin\theta$; $\cos(\pi \theta) = -\cos\theta$; and $\cos(\pi + \theta) = -\cos\theta$.
- 4. The identities of number 3 (above) are obtained early in the course and are important for the reference numbers (or reference angles) needed in Chapter 1 for solving trigonometric equations and in reading tables.

The spiral approach is one of the pedagogic methods of this text. After a topic has once been introduced and explained it is presented again at a later time in a different context and/or in greater depth. This procedure is repeated as often as is necessary or appropriate. The reinforcement afforded by this method is definitely advantageous. For example, in Chapter 1 when the sine and cosine functions are studied the topic of identities is first introduced. It is met again in Chapter 2 with the tangent function; in Chapter 3 with the cotangent, secant, and cosecant functions; and finally attains its broadest treatment in Chapter 5. Other topics are presented in a similar fashion.

The words "Include if time permits" appear in parentheses following the title of some subsections. These subsections can be omitted from the course without disturbing its continuity. Interesting sidelights are contained in these subsections, and if they can be included the students will have a better understanding of trigonometry.

In closing I wish to thank Richard Jones of Wadsworth Publishing Company, for his many valuable suggestions that helped shape this text.

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Preliminary Topics

0.1 WHAT IS TRIGONOMETRY?

Have you ever heard of a **trigon?** Perhaps not, but we are acquainted with particular types of polygons such as pentagons, hexagons, and octagons, which have five, six, and eight sides, respectively. How many sides does a trigon have? By now you should have guessed that it is another name (Greek) for a triangle. So trigonometry has something to do with triangles, but what about the suffix **metry?** It, too, is Greek and refers to "the process or science of measuring." Upon joining the two with the letter "o" we see that trigonometry is "the process or science of measuring triangles." In particular, if we have some, but not all, information about the lengths of sides and sizes of angles of a triangle, can we accurately calculate the dimensions of the other parts? Finding the magnitudes of these unknown parts is at the heart of a process known as **indirect measurement.** Indirect measurement is used for many purposes, among which are finding the distances to and sizes of the moon, sun, planets, and other heavenly bodies. This type of problem concerned the ancients who originated the subject of trigonometry, but in our time other applications of trigonometry have assumed greater importance.

There are six different ratios involving the lengths of sides of a right triangle. These ratios are essential to the process of indirect measurement, and over the centuries they evolved into the six trigonometric functions. Basically, trigonometry is the study of these six functions, their inverse functions, and the ramifications of both. The trigonometric functions are important in most higher mathematics courses, and a complete study of analytic geometry, calculus, and differential equations is impossible without them.

Many phenomena in our world are **periodic.** That is, a particular behavior is exhibited over and over again with a high degree of predictability. Examples of these phenomena include sound waves, light waves, alternating-current electricity as used in our homes, the movement of a piston in an automobile engine, and the electromagnetic waves by which radio and television signals are transmitted. The trigonometric functions are periodic functions and provide an excellent means of describing these and other real-life periodic phenomena in a precise mathematical fashion. Many of today's technological advances would not exist if it were not for trigonometry.

0.2 ESSENTIAL GEOMETRIC TOPICS, PART ONE

Before attempting a study of trigonometry it is mandatory that the student have two years of high-school algebra (or its equivalent) and a course in plane geometry. A description of those geometric topics most important to the development of trigonometry is contained in this and the succeeding section. Some of the essential algebraic topics necessary for trigonometry are found in Appendix A and in Sections 1.1 and 7.1.

Axiomatic Systems

All branches of mathematics are axiomatic systems. Certain words called undefined terms are used to write definitions of other technical terms, which, in turn, can be used to define more terms and symbols. For example, the words point, line, and plane are undefined terms in geometry and are used to write definitions of other words such as angle and transversal. Then the undefined and defined terms are used to write implicative statements. Implicative statements consist of two parts: the first is called the **hypothesis** and the second, the **conclusion**. We shall be concerned with those implicative statements in which the truth of the conclusion depends upon the truth of the hypothesis. For example, consider the implicative statement "If two points are equidistant from the ends of a line segment, then they determine the perpendicular bisector of that segment." The hypothesis is "Two points are equidistant from the ends of a line segment," and the conclusion is "They determine the perpendicular bisector of that segment." This particular implicative statement is a theorem. If we start with the truth of the hypothesis, by using the rules of logic we can deduce that the conclusion is true. Hence a theorem is an implicative statement that must be proven true. On the other hand, there must be some initial "truths" upon which we base the proofs of theorems. That is, a minimum number of implicative statements must be accepted as true without being proven. These are known as axioms and are the fundamental truths upon which the axiomatic system is based. Trigonometry as an axiomatic system depends upon algebra and geometry in the sense that the undefined terms, definitions, axioms, and theorems of those two subjects form the foundation for the definitions and theorems of trigonometry. No new, undefined terms or axioms will be necessary for our studies. We will develop trigonometry through new definitions and theorems only.

Angles

Line and point are two undefined terms in geometry. Two distinct points determine exactly one line. A particular line can be named by a single letter or by using any two points on the line. In Figure 0.01 the line has the names \overrightarrow{l} , \overrightarrow{BA} , and \overrightarrow{AB} , where the double-headed arrow indicates the infinite extent of the line in both directions. Note the contrasting ray or half-line \overrightarrow{CD} ; its end point is C and it extends indefinitely from C only in the direction of D. (Note that $\overrightarrow{CD} \neq \overrightarrow{DC}$.) Unlike a line or ray, a line segment has a finite length and is named by the points at its extremities. Both \overrightarrow{EF} and \overrightarrow{FE} name the line segment in Figure 0.01. Associated with each line segment is a unique real number that is the measure of its length, depending upon the unit of measure used. \overrightarrow{FE} and \overrightarrow{EF} both designate the length of \overrightarrow{EF} .

此为试读, 需要完整的原见,FE and EF both designate the length of EF.

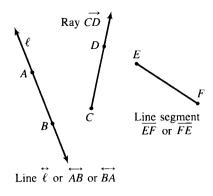


Figure 0.01

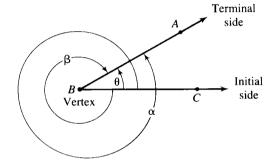


Figure 0.02

An angle is the set of points formed by two rays having a common end point. Rays \overrightarrow{BA} and \overrightarrow{BC} form the angle shown in Figure 0.02, and the rays are the **sides** of the angle. One name for this angle is $\angle ABC$, where the symbol \angle is interpreted as "angle." The middle letter of the three used to name the angle is the common end point of the two rays. This is B, which is the **vertex** of the angle. The vertex does not lie on either side of the angle. The other two letters, A and C, come from the two different rays that form the angle. $\angle CBA$ is also a correct name for the angle. Since there is only one angle having point B as its vertex, then $\angle B$ is another name for $\angle CBA$.

The concept of angle that is needed in trigonometry goes be-

yond the present description. We need to define a **generated angle**, which adds the concept of rotation. We arbitrarily designate \overrightarrow{BC} as the **initial side** and \overrightarrow{BA} as the **terminal side** of $\angle CBA$ in Figure 0.02. The angle shown can be "generated," or formed, by a counterclockwise rotation of the initial side \overrightarrow{BC} to the position of the terminal side \overrightarrow{BA} . This direction of rotation is defined to be **positive**, and the angle so formed, θ , is shown with a curved arrow in the counterclockwise direction. (θ is the Greek letter "theta.") A **negative** generated angle is formed by a clockwise rotation from the initial to the ter-

minal side, and in Figure 0.02 β is such an angle. (β is the Greek letter "beta.") The amount of rotation is another factor to be considered. In Figure 0.02, if \overrightarrow{BC} makes one additional complete rotation before coming to rest on \overrightarrow{BA} the generated angle $\angle \alpha$ is formed. (α is the Greek letter "alpha.") The curved arrow again identifies the amount and direction of rotation. This process could continue indefinitely, making additional clockwise or counterclockwise rotations to form an infinite number of generated angles, all of which share \overrightarrow{BC} as the initial side and \overrightarrow{BA} as the terminal side. α , β , θ , and all angles so formed in Figure 0.02 are known as **coterminal** angles. Each generated angle has a unique real number, called its measure, associated with it. The most common unit of measure is the degree. A comprehensive discussion on the measurement of angles is found in Section 4.1. An acute angle has a measure between 0° and 90°, a **right angle** has a measure of 90°, the measure of an obtuse angle is between 90° and 180°, and the measure of a straight angle is 180°. If the initial side of a generated angle makes one complete counterclockwise rotation before reaching the terminal side its measure is 360°. Two angles are complementary if the sum of their measures is 90°. If the sum of the measures of two angles is 180° they are supplementary. Two or more angles are congruent if they have the same measure. For example, all right angles are congruent, since their common measure is 90°. The symbol \cong means "is congruent to." Hence if the measure of $\angle OMP =$ 12° = the measure of $\angle RNS$, we may write $\angle OMP \cong \angle RNS$. The \cong symbol also applies to line segments that have the same length. If two line segments, \overline{RS} and \overline{TU} , have the same length, this can be expressed as $\overline{RS} \cong \overline{TU}$.

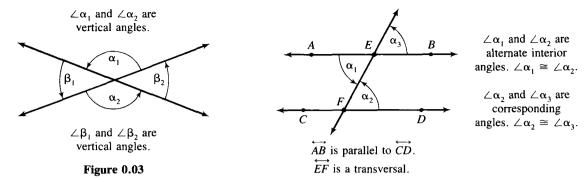


Figure 0.04

Vertical angles, which are always congruent, are formed when two lines intersect. In Figure 0.03 there are two pairs of vertical angles: $\angle \alpha_1 \cong \angle \alpha_2$ and $\angle \beta_1 \cong \angle \beta_2$. If two lines or rays intersect so that a right angle is formed, then the lines or rays are **perpendicular**. Two intersecting line segments can also be perpendicular if the lines determined by the segments are perpendicular. If two lines are the same distance apart at all points they are **parallel lines**, and any line that intersects them is known as a **transversal**. Refer to Figure 0.04. The **alternate interior angles**, $\angle \alpha_1$ and $\angle \alpha_2$, are congruent, as are the **corresponding angles**, $\angle \alpha_2$ and $\angle \alpha_3$.

Triangles

When three noncollinear points of the plane are connected by three line segments, a **triangle** is formed. Each point is a **vertex** of the triangle, and the letters at the vertices are used to name the triangle. Hence, in Figure 0.05, we have triangles ABC, DEF, HIJ, and KLM. The symbol $\triangle ABC$ means 'triangle ABC.' The line segments forming the triangle are its **sides**. Triangles are classified according to the relative sizes of their angles and sides, as shown in the following chart.

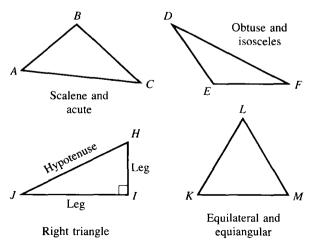


Figure 0.05

 $\triangle DEF$ with $\overline{DE} \cong \overline{EF}$

 $\triangle KLM$

Sides	Example in Figure 0.05
All sides of unequal lengths	$\triangle ABC$

Triangle Classification by Sides

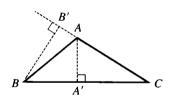
Trionale	Classification	hw.	Angles	

Two congruent sides

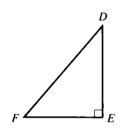
All sides congruent

Name	Angles	Example in Figure 0.05
Acute	All angles acute	$\triangle ABC$
Obtuse	One obtuse angle	$\triangle DEF$ with obtuse $\angle DEF$
Right	One right angle	$\triangle HIJ$ with right $\angle HIJ$
Equiangular	All angles congruent	$\triangle \mathit{KLM}$

A right angle will sometimes be identified by the symbol \Box placed within the angle at its vertex, as in $\angle HIJ$ of Figure 0.05. In a right triangle the side opposite the right angle is called the **hypotenuse**, and the other two sides are known as **legs**.



 $\overline{AA'}$ and $\overline{BB'}$ are altitudes. \overline{AC} is the base for $\overline{BB'}$. \overline{BC} is the base for $\overline{AA'}$.



Legs \overline{DE} and \overline{FE} are also altitudes.

Figure 0.06

Name

Scalene

Isosceles

Equilateral

If a line segment extends from any vertex of a triangle perpendicular to the line containing the opposite side, then its length is an **altitude** of the triangle. The line segment is sometimes called the altitude. In Figure 0.06 \overline{AA}' and \overline{BB}' represent two of the three altitudes of $\triangle ABC$. The legs of a right triangle are also altitudes. \overline{DE} and \overline{FE} are altitudes of right $\triangle DEF$. A **base** is the length of the side to which an altitude is perpendicular. The name also refers to the line segment itself. In Figure 0.06, \overline{BC} is the base for altitude \overline{BB}' . Likewise, \overline{DE} is the base for altitude \overline{FE} . The numerical

meanings of "base" and "altitude" are employed in the formula for the area of a triangle.

Area of a triangle =
$$\frac{1}{2}$$
 (base) (altitude)

The following theorem is of paramount importance for the study of trigonometry:

0.2.1 Theorem: Sum of Angles of a Triangle

In any triangle, the sum of the measures of its angles equals 180°.

From this theorem we can deduce that a triangle can have no more than one right angle or one obtuse angle, and the acute angles of a right triangle are complementary.

Cartesian Coordinates

In trigonometry it is necessary to locate points in the plane by means of a coordinate system. The **Cartesian** or **rectangular coordinate system** is based upon two perpendicular number lines that intersect at a point called the **origin**. See Figure 0.07. The

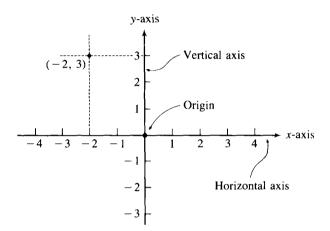


Figure 0.07

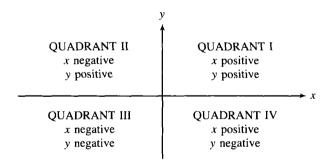


Figure 0.08

horizontal line is called the x-axis, while the vertical line is known as the y-axis. Each point on the x-axis (or y-axis) corresponds to exactly one real number and vice versa. Ordered pairs of numbers are formed where the first number in each ordered pair is the abscissa or x-coordinate, and the second is the ordinate or y-coordinate. The Cartesian coordinate system associates these ordered pairs with points in the plane. The system depends upon the axiom that says each point in the plane corresponds to exactly one ordered pair. and each ordered pair corresponds to exactly one point of the plane. To locate the point corresponding to a given ordered pair we proceed as follows, using (-2, 3) as an example. Locate the point on the x-axis corresponding to the abscissa, -2, and draw a line through this point perpendicular to the xaxis. See Figure 0.07. Next draw a line perpendicular to the y-axis where y = 3. The intersection of these two lines locates the unique point whose Cartesian (or rectangular) coordinates are (-2, 3). The point which corresponds to a given ordered pair is called the graph of the ordered pair, and the abscissa and ordinate of the ordered pair are the coordinates of the point.

The coordinate axes divide the plane into four nonintersecting regions called quad-

rants. Refer to Figure 0.08. The coordinate axes themselves are not part of any quadrant but serve to separate one from another. All points on the x-axis have y-coordinates that are zero, and all points on the y-axis have x-coordinates that are zero. The origin has coordinates (0, 0). It is important to know in which quadrants the x-and y-coordinates are positive and negative, as shown in Figure 0.08.

Pythagorean Theorem and Distance Formula

If two points lie on the same vertical or horizontal line, it is a simple matter to find the distance between them. For example, points A and B, having coordinates (3, 5)

and (3, -7), respectively, both lie on the vertical line x = 3, and the distance between them is equal to the absolute value of the difference in the y-coordinates. Hence AB = |5 - (-7)| = 12. How do we find the distance between points which do not lie on the same vertical or horizontal line? The method depends upon the Pythagorean theorem, which is stated here for reference.

0.2.2 Pythagorean Theorem

If the lengths of the legs and hypotenuse of a right triangle are a, b, and c, respectively, then $a^2 + b^2 = c^2$.

Example 1 Find the length of the indicated side in each of the given right triangles.

Solution:
$$c^2 = a^2 + b^2 = 24^2 + 7^2 = 576 + 49 = 625.$$

 $c = \sqrt{625} = 25.$
 $b = 7$

$$a = 24$$

Solution:
$$c^2 = a^2 + b^2$$
 implies that $b = \sqrt{c^2 - a^2}$.
 $b = \sqrt{12^2 - 6^2} = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}$. $a = 6$

The formula for calculating the distance between any two points in the plane is found in the next theorem, whose proof depends upon the Pythagorean theorem.

0.2.3 Theorem: Distance Formula

For any two points in the Cartesian plane having coordinates (x_1, y_1) and (x_2, y_2) , the distance d between the points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

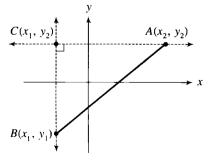


Figure 0.09

Proof: Refer to Figure 0.09, where d = AB is the distance to be found. Although points A and B are shown in quadrants I and III, they can be in any quadrant or on any axis, and the generality of the proof will not be affected. Through A and B, construct horizontal and vertical lines that intersect at C, forming right triangle ABC whose hypotenuse is \overline{AB} . The coordinates (x_1, y_1) of B and (x_2, y_2) of A are known. The coordinates of C must be (x_1, y_2) , since \overline{AC} is parallel to the x-axis and \overline{BC} is parallel to the y-axis. In the following calculations we use the facts that

$$AC = |x_2 - x_1|, BC = |y_2 - y_1|, |x_2 - x_1|^2 = (x_2 - x_1)^2$$
, and $|y_2 - y_1|^2 = (y_2 - y_1)^2$. By the Pythagorean theorem $(AB)^2 = (AC)^2 + (BC)^2$

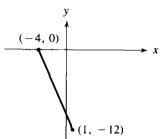
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$
Therefore $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, the final step in the proof.

The application of the distance formula is explained in Example 2.

Example 2 Find the distance between two points whose coordinates are (1, -12) and (-4, 0).

Solution: Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, it makes no difference whether $(1, -12) = (x_1, y_1)$ or (x_2, y_2) . We arbitrarily choose $(1, -12) = (x_2, y_2)$ and $(-4, 0) = (x_1, y_1)$.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{[1 - (-4)]^2 + [-12 - 0]^2} = \sqrt{5^2 + (-12)^2}$$

$$= \sqrt{25 + 144} = \sqrt{169} = 13 = d.$$

Circles

Our introduction to trigonometry in Chapter 1 will include both the wrapping function and two of the circular functions. Circles play an important role in both types of functions. A **circle** is a set of all points in the plane that are at a fixed distance r from a fixed point having coordinates (a, b). The **radius** of the circle is r and its **center** is (a, b). (Note: We sometimes use the coordinates of a point in place of the name of the point. Hence, point P, whose coordinates are (a, b), can be written as point (a, b).) If the radius and center of a circle are known, the circle has an equation of a particular type. This is the subject of the following theorem.

0.2.4 Theorem: Equation of a Circle

If (x, y) is any point on a circle C, having radius r and center (a, b), then

$$(x - a)^2 + (y - b)^2 = r^2$$

is an equation of C.

The proof of 0.2.4 depends upon the distance formula and is left for the list of exercises. The following examples utilize 0.2.4.