DYNAMICS:
Theory and Applications

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PREFACE

Dissatisfaction with available textbooks on the subject of dynamics has been widespread throughout the engineering and physics communities for some years among teachers, students, and employers of university graduates; furthermore, this dissatisfaction is growing at the present time. A major reason for this is that engineering graduates entering industry, when asked to solve dynamics problems arising in fields such as multibody spacecraft attitude control, robotics, and design of complex mechanical devices, find that their education in dynamics, based on the textbooks currently in print, has not equipped them adequately to perform the tasks confronting them. Similarly, physics graduates often discover that; in their education, so much emphasis was placed on preparation for the study of quantum mechanics, and the subject of rigid body dynamics was slighted to such an extent, that they are handicapped, both in industry and in academic research, but their inability to design certain types of experimental equipment, such as a particle detector that is to be mounted on a planetary satellite. In this connection, the ability to analyze the effects of detector scanning motions on the attitude motion of the satellite is just as important as knowledge of the physics of the detection process itself. Moreover, the graduates in question often are totally unaware of the deficiencies in their dynamics education. How did this state of affairs come into being, and is there a remedy?

For the most part, traditional dynamics texts deal with the exposition of eighteenth-century methods and their application to physically simple systems, such as the spinning top with a fixed point, the double pendulum, and so forth. The reason for this is that, prior to the advent of computers, one was justified in demanding no more of students than the ability to formulate equations of motion for such simple systems, for one could not hope to extract useful information from the equations governing the motions of more complex systems. Indeed, considerable ingenuity and a rather extensive knowledge of mathematics were required to analyze even simple systems. Not surprisingly, therefore, even more attention came to be focused on analytical intricacies of the *mathematics* of dynamics, while the process of formulating equations of motion came to be regarded as a rather routine matter. Now that computers enable one to extract highly valuable information from large sets of complicated equations of motion, all this has changed. In fact, the inability to *formulate* equations of motion effectively can be as great a hindrance at present as the inability to *solve* equations was formerly. It follows that the subject of formulation of equations of motion demands careful reconsideration. Or, to say it another way, a major goal of a modern dynamics course must be to produce students who are proficient in the use of the best available methodology for formulating equations of motion. How can this goal be attained?

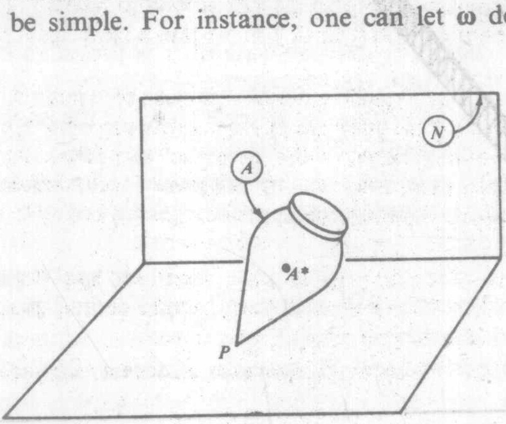
In the 1970s, when extensive dynamical studies of multibody spacecraft, robotic devices, and complex scientific equipment were first undertaken, it became apparent that straightforward use of classical methods, such as those of Newton, Lagrange, and Hamilton, could entail the expenditure of very large, and at times even prohibitive, amounts of analysts' labor, and could lead to equations of motion so unwieldy as to render computer solutions unacceptably slow for technical and/or economic reasons. Now, while it may be impossible to overcome this difficulty entirely, which is to say that it is unlikely that a way will be found to reduce formulating equations of motion for complex systems to a truly simple task, there does exist a method that is superior to the classical ones in that its use leads to major savings in labor, as well as to simpler equations. Moreover, being highly systematic, this method is easy to teach. Focusing attention on motions, rather than on configurations, it affords the analyst maximum physical insight. Not involving variations, such as those encountered in connection with virtual work, it can be presented at a relatively elementary mathematical level. Furthermore, it enables one to deal directly with nonholonomic systems without having to introduce and subsequently eliminate Lagrange multipliers. It follows that the resolution of the dilemma before us is to instruct students in the use of this method (which is often referred to as Kane's method). This book is intended as the basis for such instruction.

Textbooks can differ from each other not only in content but also in organization, and the sequence in which topics are presented can have a significant effect on the relative ease of teaching and learning the subject. The rationale underlying the organization of the present book is the following. We view dynamics as a deductive discipline, knowledge of which enables one to describe in quantitative and qualitative terms how mechanical systems move when acted upon by given forces, or to determine what forces must be applied to a system in order to cause it to move in a specified manner. The solution of a dynamics problem is carried out in two major steps, the first being the formulation of equations of motion, and the second the extraction of information from these equations. Since the second step cannot be taken fruitfully until the first has been completed, it is imperative that the distinction between the two be kept clearly in mind. In this book, the extraction of information from equations of motion is deferred formally to the last chapter, while the preceding chapters deal with the material one needs to master in order to be able to arrive at valid equations of motion.

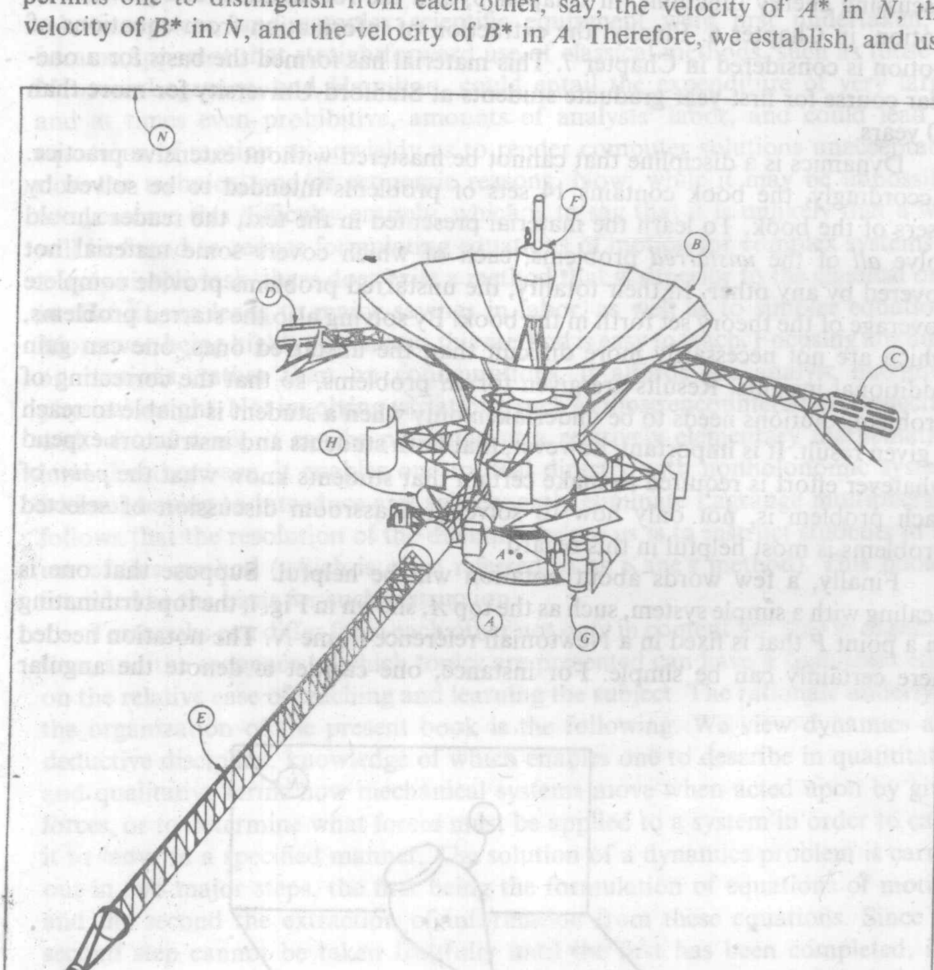
Diverse concepts come into play in the process of constructing equations of motion. Here again it is important to separate ideas from each other distinctly. Major attention must be devoted to kinematics, mass distribution considerations, and force concepts. Accordingly, we treat each of these topics in its own right. First, however, since differentiation of vectors plays a key role in dynamics, we devote the initial chapter of the book to this topic. Here we stress the fact that differentiation of a vector with respect to a scalar variable requires specification of a reference frame, in which connection we dispense with the use of limits because such use tends to confuse rather than clarify matters; but we draw directly on students' knowledge of scalar calculus. Thereafter, we devote one chapter each to the topics of kinematics, mass distribution, and generalized forces, before discussing energy functions, in Chapter 5, and the formulation of equations of motion, in Chapter 6. Finally, the extraction of information from equations of motion is considered in Chapter 7. This material has formed the basis for a one-year course for first-year graduate students at Stanford University for more than 20 years.

Dynamics is a discipline that cannot be mastered without extensive practice. Accordingly, the book contains 14 sets of problems intended to be solved by users of the book. To learn the material presented in the text, the reader should solve *all* of the *unstarred* problems, each of which covers some material not covered by any other. In their totality, the unstarred problems provide complete coverage of the theory set forth in the book. By solving also the starred problems, which are not necessarily more difficult than the unstarred ones, one can gain additional insights. Results are given for *all* problems, so that the correcting of problem solutions needs to be undertaken only when a student is unable to reach a given result. It is important, however, that both students and instructors expend whatever effort is required to make certain that students know what the *point* of each problem is, not only how to solve it. Classroom discussion of selected problems is most helpful in this regard.

Finally, a few words about notation will be helpful. Suppose that one is dealing with a simple system, such as the top A, shown in Fig. *i*, the top terminating in a point P that is fixed in a Newtonian reference frame N. The notation needed here certainly can be simple. For instance, one can let $\boldsymbol{\omega}$ denote the angular



velocity of A in N, and let $\mathbf{v}$ stand for the velocity in N of point A^*, the mass center of A. Indeed, notations more elaborate than these can be regarded as objectionable because they burden the analyst with unnecessary writing. But suppose that one must undertake the analysis of motions of a complex system, such as the Galileo spacecraft, modeled as consisting of eight rigid bodies $A, B, \ldots, H$, coupled to each other as indicated in Fig. *ii*. Here, unless one employs notations more elaborate than $\boldsymbol{\omega}$ and $\mathbf{v}$, one cannot distinguish from each other such quantities as, say, the angular velocity of A in a Newtonian reference frame N, the angular velocity of B in N, and the angular velocity of B in A, all of which may enter the analysis. Or, if A^* and B^* are points of interest fixed on A and B, perhaps the respective mass centers, one needs a notation that permits one to distinguish from each other, say, the velocity of A^* in N, the velocity of B^* in N, and the velocity of B^* in A. Therefore, we establish, and use



consistently throughout this book, a few notational practices that work well in such situations. In particular, when a vector denoting an angular velocity or an angular acceleration of a rigid body in a certain reference frame has two superscripts, the right superscript stands for the rigid body, whereas the left superscript refers to the reference frame. Incidentally, we use the terms "reference frame" and "rigid body" interchangeably. That is, every rigid body can serve as a reference frame, and every reference frame can be regarded as a massless rigid body. Thus, for example, the three angular velocities mentioned in connection with the system depicted in Fig. *ii*, namely, the angular velocity of A in N, the angular velocity of B in N, and the angular velocity of B in A, are denoted by ${}^N\boldsymbol{\omega}^A$, ${}^N\boldsymbol{\omega}^B$, and ${}^A\boldsymbol{\omega}^B$, respectively. Similarly, the right superscript on a vector denoting a velocity or acceleration of a point in a reference frame is the name of the point, whereas the left superscript identifies the reference frame. Thus, for example, the aforementioned velocity of A^* in N is written ${}^N\mathbf{v}^{A^*}$, and ${}^A\mathbf{v}^{B^*}$ represents the velocity of B^* in A. Similar conventions are established in connection with angular momenta, kinetic energies, and so forth.

While there are distinct differences between our approach to dynamics, on the one hand, and traditional approaches, on the other hand, there is no fundamental conflict between the new and the old. On the contrary, the material in this book is entirely compatible with the classical literature. Thus, it is the purpose of this book not only to equip students with the skills they need to deal effectively with present-day dynamics problems, but also to bring them into position to interact smoothly with those trained more conventionally.

Thomas R. Kane
David A. Levinson

TO THE READER

Each of the seven chapters of this book is divided into sections. A section is identified by two numbers separated by a decimal point, the first number referring to the chapter in which the section appears, and the second identifying the section within the chapter. Thus, the identifier 2.14 refers to the fourteenth section of the second chapter. A section identifier appears at the *top of each page*.

Equations are numbered serially within sections. For example, the equations in Secs. 2.14 and 2.15 are numbered (1)–(31) and (1)–(50), respectively. References to an equation may be made both within the section in which the equation appears and in other sections. In the first case, the equation number is cited as a single number; in the second case, the section number is included as part of a three-number designation. Thus, within Sec. 2.14, Eq. (2) of Sec. 2.14 is referred to as Eq. (2); in Sec. 2.15, the same equation is referred to as Eq. (2.14.2). To locate an equation cited in this manner, one may make use of the section identifiers appearing at the tops of pages.

Figures appearing in the chapters are numbered so as to identify the sections in which the figures appear. For example, the two figures in Sec. 4.8 are designated Fig. 4.8.1 and Fig. 4.8.2. To avoid confusing these figures with those in the problem sets and in Appendix I, the figure number is preceded by the letter P in the case of problem set figures, and by the letter A in the case of Appendix I figures. The double number following the letter P refers to the problem statement in which the figure is introduced. For example, Fig. P12.3 is introduced in Problem 12.3. Similarly, Table 3.4.1 is the designation for a table in Sec. 3.4, and Table P14.6.2 is associated with Problem 14.6.

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CONTENTS

CHAPTER
ONE
DIFFERENTIATION OF VECTORS

The discipline of dynamics deals with changes of various kinds, such as changes in the position of a particle in a reference frame, changes in the configuration of a mechanical system, and so forth. To characterize the manner in which some of these changes take place, one employs the differential calculus of vectors, a subject that can be regarded as an extension of material usually taught under the heading of the differential calculus of scalar functions. The extension consists primarily of provisions made to accommodate the fact that *reference frames* play a central role in connection with many of the vectors of interest in dynamics. For example, let A and B be reference frames moving relative to each other, but having one point O in common at all times, and let P be a point fixed in A, and thus moving in B. Then the velocity of P in A is equal to zero, whereas the velocity of P in B differs from zero. Now, each of these velocities is a time-derivative of the same vector, $\mathbf{r}^{OP}$, the position vector from O to P. Hence, it is meaningless to speak simply of *the* time-derivative of $\mathbf{r}^{OP}$. Clearly, therefore, the calculus used to differentiate vectors must permit one to distinguish between differentiation with respect to a scalar variable in a reference frame A and differentiation with respect to the same variable in a reference frame B.

When working with elementary principles of dynamics, such as Newton's second law or the angular momentum principle, one needs only the ordinary differential calculus of vectors, that is, a theory involving differentiations of vectors with respect to a single scalar variable, generally the time. Consideration of advanced principles of dynamics, such as those presented in later chapters of this book, necessitates, in addition, *partial* differentiation of vectors with respect to several scalar variables, such as generalized coordinates and generalized speeds. Accordingly, the present chapter is devoted to the exposition of definitions, and consequences of these definitions, needed in the chapters that follow.

1.1 VECTOR FUNCTIONS

When either the magnitude of a vector **v** and/or the direction of **v** in a reference frame A depends on a scalar variable q, **v** is called *a vector function of q in A*. Otherwise, **v** is said to be *independent of q in A*.

Example In Fig. 1.1.1, P represents a point moving on the surface of a rigid sphere S, which, like any rigid body, may be regarded as a reference frame. (Reference frames should not be confused with coordinate systems. Many coordinate systems can be embedded in a given reference frame.) If **p** is the position vector from the center C of S to point P, and if q_1 and q_2 are the angles shown, then **p** is a vector function of q_1 and q_2 in S because the direction of **p** in S depends on q_1 and q_2, but **p** is independent of q_3 in S, where q_3 is the distance from C to a point R situated as shown in Fig. 1.1.1. The position vector **r** from C to R is a vector function of q_3 in S, but is independent of q_1 and q_2 in S, and the position vector **q** from P to R is a vector function of q_1, q_2, and q_3 in S.

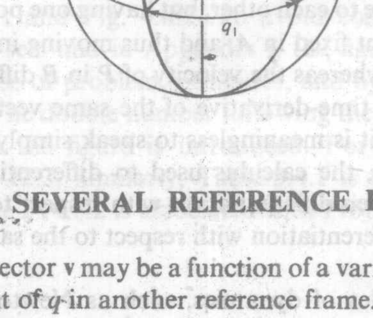

Figure 1.1.1

1.2 SEVERAL REFERENCE FRAMES

A vector **v** may be a function of a variable q in one reference frame, but be independent of q in another reference frame.

Example The outer gimbal ring A, inner gimbal ring B, and rotor C of the gyroscope depicted in Fig. 1.2.1 each can be regarded as a reference frame. If **p** is the position vector from point O to a point P of C, then **p** is a function of

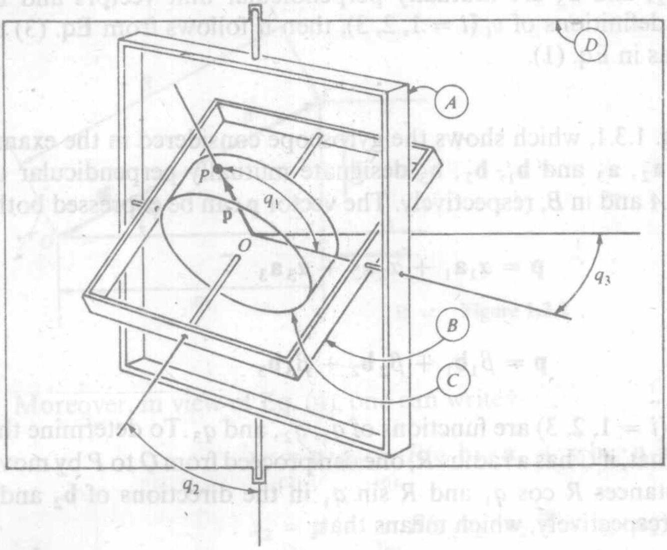

Figure 1.2.1

q_1 both in A and in B, but is independent of q_1 in C; **p** is a function of q_2 in A, but is independent of q_2 both in B and in C; and **p** is independent of q_3 in each of A, B, and C, but is a function of q_3 in reference frame D.

1.3 SCALAR FUNCTIONS

Given a reference frame A and a vector function **v** of n scalar variables $q_1, \ldots, q_n$ in A, let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be a set of nonparallel, noncoplanar (but not necessarily mutually perpendicular) unit vectors fixed in A. Then there exist three unique scalar functions v_1, v_2, v_3 of $q_1, \ldots, q_n$ such that

$$\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3 \tag{1}$$

This equation may be regarded as a bridge connecting scalar to vector analysis; it provides a convenient means for extending to vector analysis various important concepts familiar from scalar analysis, such as continuity, differentiability, and so forth. The vector $v_i\mathbf{a}_i$ is called the $\mathbf{a}_i$ *component* of **v**, and v_i is known as the $\mathbf{a}_i$ *measure number* of **v** ($i = 1, 2, 3$).

When $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are *mutually perpendicular* unit vectors, then it follows from Eq. (1) that the $\mathbf{a}_i$ measure number of **v** is given by

$$v_i = \mathbf{v} \cdot \mathbf{a}_i \qquad (i = 1, 2, 3) \tag{2}$$

and that Eq. (1) may, therefore, be rewritten as

$$\mathbf{v} = \mathbf{v} \cdot \mathbf{a}_1\mathbf{a}_1 + \mathbf{v} \cdot \mathbf{a}_2\mathbf{a}_2 + \mathbf{v} \cdot \mathbf{a}_3\mathbf{a}_3 \tag{3}$$

Conversely, if $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are mutually perpendicular unit vectors and Eqs. (2) are regarded as definitions of v_i $(i = 1, 2, 3)$, then it follows from Eq. (3) that $\mathbf{v}$ can be expressed as in Eq. (1).

Example In Fig. 1.3.1, which shows the gyroscope considered in the example in Sec. 1.2, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ designate mutually perpendicular unit vectors fixed in A and in B, respectively. The vector $\mathbf{p}$ can be expressed both as

$$\mathbf{p} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 \tag{4}$$

and as

$$\mathbf{p} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 \tag{5}$$

where α_i and β_i $(i = 1, 2, 3)$ are functions of q_1, q_2, and q_3. To determine these functions, note that, if C has a radius R, one can proceed from O to P by moving through the distances $R \cos q_1$ and $R \sin q_1$ in the directions of $\mathbf{b}_2$ and $\mathbf{b}_3$ (see Fig. 1.3.2), respectively, which means that

$$\mathbf{p} = R(\mathrm{c}_1 \mathbf{b}_2 + \mathrm{s}_1 \mathbf{b}_3) \tag{6}$$

where c_1 and s_1 are abbreviations for $\cos q_1$ and $\sin q_1$, respectively. Comparing Eqs. (5) and (6), one thus finds that

$$\beta_1 = 0 \qquad \beta_2 = R\mathrm{c}_1 \qquad \beta_3 = R\mathrm{s}_1 \tag{7}$$

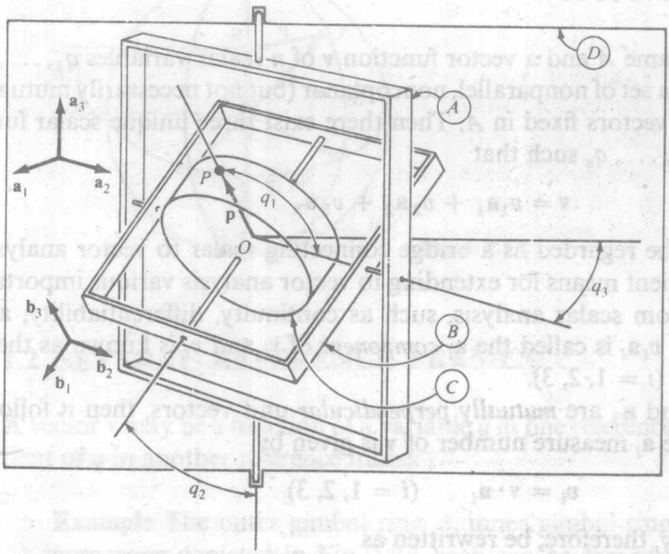

Figure 1.3.1