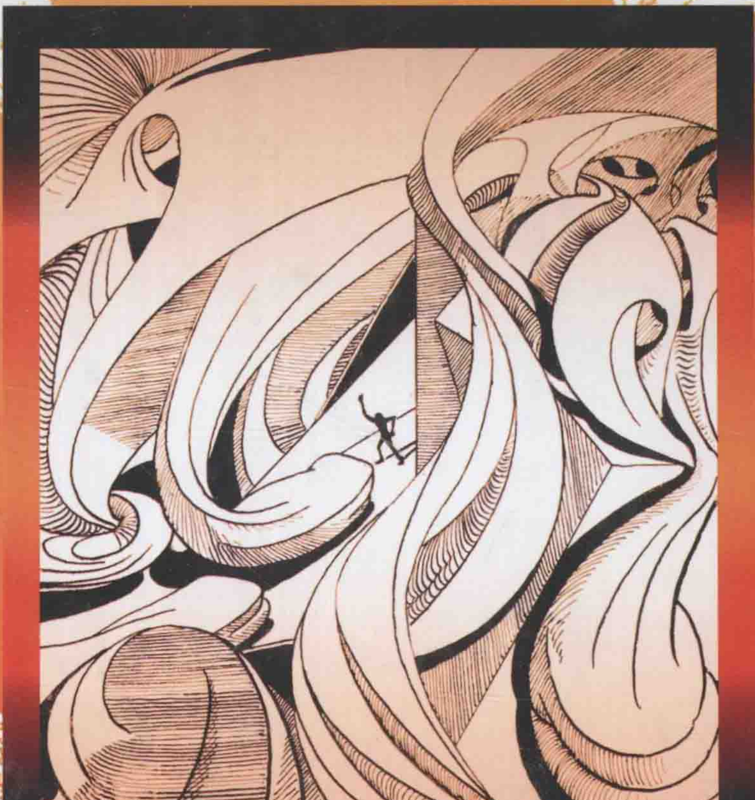
Understanding Game Theory

Introduction to the Analysis of Many Agent Systems with Competition and Cooperation

Vassili N Kolokoltsov • Oleg A Malafeyev



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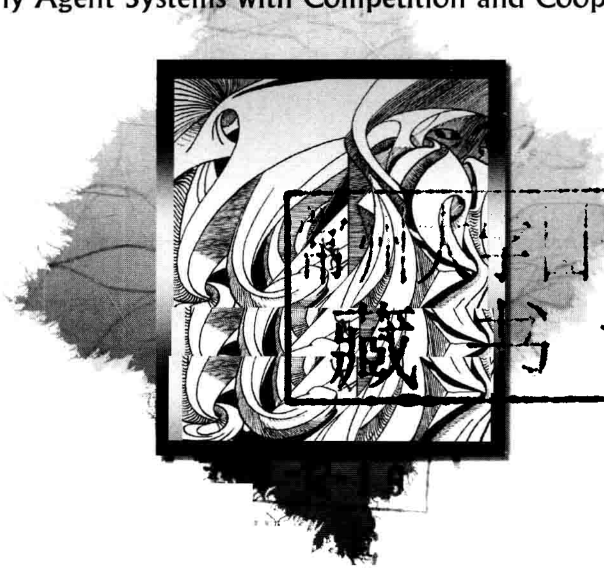

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Dedicated to our daughters

ANYA, DASCHA, MARGARITA

Preface

This text is devoted to game theory and its various applications. With the aim of being accessible to the widest audience the text is clearly separated into two parts with different levels of exposition.

1. The first part (Chapters 1-8) give an elementary but systematic exposition of the main ideas of modern game theory without any special prerequisites in mathematics (secondary school level should be quite sufficient). It requires from a reader only an inclination towards logical thinking, equations, some calculations, and an acceptance of the Greek letters ϵ (epsilon), δ (delta), σ (sigma), η (eta) and Π or π (pi). Nevertheless, not following the tradition of popular science books, our presentation is concise, with rigorous definitions, quantitative as well as qualitative results. On the other hand, due to the elementary nature of this presentation, it often stops just where really serious analysis begins, giving (as a compensation) lots of references for further developments.

2. The second part is devoted mostly to the mathematical methods of the theory. Having carefully separated from the first part of this exposition all higher mathematics, we give in Chapter 9 a concise presentation of mathematical aspects and techniques of game theory. This is supplemented by examples and exercises needed to get used to these techniques and to learn to apply them. The level of mathematics required in Chapter 9 is higher than in earlier chapters, but assumes an acquaintance with only basic notions of calculus, differential equations, linear algebra and probability. Chapter 10 presents several concrete (mostly original) game theoretic models and their analysis including game theoretic treatment of rainbow options in financial mathematics, advanced models of inspection, price war games, investment competitions, etc. The sections of this chapter are supplied with problems that can be used as a starting point for individual projects

students. The material from Chapters 1-9 fertilized by chosen models from Chapter 10 can serve as a basis for an introductory undergraduate course in game theory for both mathematics related degrees and university degrees with a minimal mathematics background (business, economics, biology, etc.). The last Chapter 11 is meant mostly for mathematics graduate and postgraduate students (and could be of interest to researchers), as it requires in places some mathematical culture, for instance some knowledge of functional analysis and stochastic processes. This chapter is devoted to selected topics of more advanced analysis (partially reflecting the authors' interests and research) including differential geometry approach (transversality and catastrophe) to stability, abstract dynamic system approach to the analysis of differential games, Bellman type equations for multi-criteria optimization, turnpikes for stochastic games, connections with recently becoming popular tropical (or idempotent) mathematics, as well as with statistical physics (interacting particles). Chapter 9 with chosen parts of the last two chapters can be used for various advanced courses on the mathematical methods of game theory.

The text is aimed primarily at undergraduate students and instructors in game theory as well as at postgraduates in mathematics, system biology and social sciences. We have tried to provide an entertaining and easy read for those wishing to get acquainted quickly with (or to refresh their knowledge of) the beauty of the basic ideas of the game theory, its wide range of applicability and some recent developments. We aimed at helping to make teaching and learning a more interesting and exciting process, supplementing a course by a variety of motivating examples, historical, cultural and general science excursions.

The authors believe that due to the elementary character of the first part, the book can be used also (i) by laymen (e.g. businessmen, politicians and everyone interested in scientific problems and methods), as it introduces rigorous quantitative methods that can be helpful for an assessment of a wide range of human interactions and provides some glimpses of relevant problems from biology, economics and psychology; and (ii) by teenagers, as it is aimed to interest them in the problems of science in general, to involve them in the process of logical thinking and to stimulate their interest in mathematics.

With game theory becoming popular, there is a variety of good textbooks on the theory. As examples of accessible and extensive general introductions one can recommend books [25] and [58]. For nonspecialists, introductions to particular areas of theory and applications (well reflected in their titles) one can refer to [2], [12], [21], [24], [29], [30], [31], [32], [33], [39], [40], [42], [47], [51], [53], [56], [57], [135], [139], [143], [144], [145], [160], [180], [181] and [191]. More mathematically oriented introductions can be found in [14], [36], [43], [142], [179], [185] [156], [195] and [129]. The special character of the present text is the clear separation of elementary and more advanced material, the wide covering of the theme in combination with a concise and rigorous exposition, which at the same time is generously spiced with relevant glimpses from literature and history, and finally the discussion of some advanced topics that have not yet found their place in textbooks but have a potential to become a part of the scientific culture of the future.

Almost all chapters of the first part can be read independently (only some general notions introduced in Chapter 1 are used repeatedly). So, according to their wishes and tastes, readers can start from thinking about choosing the best president or prime minister in Chapter 6, or looking through Chapter 1 and then going to explore the biological context in Chapter 4, or touching the curious quantum world in Chapter 7, or going directly to the party games of Chapter 8. On the other hand, the material in each chapter is carefully organized in a logical order and it is advisable to read each chapter from the beginning (the exception being Chapters 10, 11, whose sections are devoted to various topics and perspectives).

The first part of this book is largely based on the Lecture Notes [88] prepared for the students of the Nottingham Trent University. Further work of the authors resulted in the Lecture Notes [99] aimed primarily at the students of St. Petersburg University. The present book is based on the best parts of [88] and [99], fully revised and updated.

To stimulate mathematical and scientific imagination and to add charm to the book, we illustrate it by carefully selecting artistic graphics of a world renowned mathematician and mathematics imaging artist A.T. Fomenko (to whom the authors express their deepest gratitude for allowing the use his works in this text). Though these works were originally designed to illustrate the geometric structure of the Universe, they fit nicely in the circle of ideas dealt with here. Readers who would like to see more of these graphics are referred to the album [55].

It is our pleasure to thank Fredrick Marcowitz, Geert Jan Olsder and James Webb, who read carefully large parts of the book and made comments that helped to improve it immeasurably. The first-named author expresses his gratitude to M. Akian, St. Gaubert, J. Binner, L. Fletcher, B. McEneaney, L. Khodarinova, R. MacKay and V. Maslov for useful discussions and the joy of collaboration on the related topics. The second-named author thanks the participants of the research seminars at the Central Economics and Mathematics Institute of the Russian Academy of Science and the Institute of Mechanics and Mathematics of Ekaterinburg, at the Institute of Mathematics of the Belarus Academy of Science and the Institute of Cybernetics of the Ukrainian Academy of Science, at the Stockholm, Sankt-Petersburg, and Moscow State Universities, where he presented his research in game theory on various occasions. He also expresses his deep gratitude to Professor L.A. Petrosyan for very fruitful discussions.

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Contents

PART 1

Basic ideas

Chapter 1

Around the prisoner's dilemma

1.1 What is a two-player game?

- What's our life? - A game!
- Who is right or happy here, mate?
- Today it's you, tomorrow me!

The Queen of Spades (opera)

During its life any being attempts to achieve goals which are important to it. However, all beings are in permanent contact with others who are trying to achieve their own goals. Thus any being should always take into consideration the interests and possible actions of other beings: sometimes by attempting to outwit an opponent and sometimes by forming coalitions with partners that have similar interests. This is precisely what is meant by a game: an attempt to achieve one's goal in an environment where there are other beings that may have opposite or similar, but almost never identical, goals.

Science usually tries to decompose complicated interlaced interactions into simple parts (elementary bricks or links), to analyze their workings separately, and at last to reconstruct the whole chain from these simple parts. We shall start with such an elementary brick of an interaction with conflict interests, namely with a two-player game. It is convenient and customary to give the players some names. We shall often denote our players R and C, say, Ruth and Charlie. These letters are not randomly chosen. In a geometrical, or table representation (see below) they correspond to the Row and Column players. Other common names for the players are Alice and Bob with initials being the first two letters of the Latin alphabet.

The two most natural ways to describe a game are the so-called normal form (tables) and extensive form (game trees). Though the latter usually yields the most adequate description, the former is simpler and is well suited to the analysis of static games. In this Chapter we shall use exclusively the normal (or tabular) form.

A game of two players, Ruth and Charlie, *in the normal form* is specified by the set S_R of possible strategies of R, the set S_C of possible strategies of C, and by two payoff functions Π_R and Π_C. The notion of the strategy is capacious. A strategy can be described by some action, or by a sequence of actions, or more generally by a type of behavior. In any case, a strategy must define the action of a player in any situation that can arise in the process of the interaction of the players according to the rules of the game. The payoff functions specifies two real numbers $\Pi_R(s_R, s_C)$ and $\Pi_C(s_R, s_C)$ for any pairs of strategies (s_R, s_C), where s_R is from S_R and s_C from S_C. These two numbers describe the payoffs to Ruth and Charlie when they apply their strategies s_R and s_C. The payoffs can be negative, which means, of course, that in this case the player rather loses than wins.

When the number of possible strategies is not large, one can conveniently describe such a game by a table, in which rows and columns correspond respectively to the strategies of R (the Row player) and C (the Column player), and where two payoffs $\Pi_R(s_R, s_C)$ and $\Pi_C(s_R, s_C)$ are placed in the cell positioned on the intersection of the row s_R and the column s_C.

Example. *Head and Tail (or Matching Pennies) game.* Ruth and Charlie simultaneously put two coins on the table. If two coins are put in the same way (two heads or two tails), C pays to R one dollar. Otherwise, R pays to C one dollar (the procedure is similar to the game where R announces Tail or Head, and C throws the coin). This game can be represented by the following table

		C	
		head	tail
R	head	1,-1	-1,1
	tail	-1,1	1,-1

Table 1.1

Let us stress here that the first number in a cell shows the winning of R (the Row player), and the second number shows the wining of C (the Column player).

Recall another well known children game: C conceals a penny in one of his fists and R has to guess Left or Right. It is described by the same table with the strategies Left and Right instead of the strategies Head and Tail.

Yet another realization of this game is the Penalty-Shooting game: football player C shoots Left or Right, goalkeeper R dives Left or Right.

Example. *Rock-Paper-Scissors game.* This is a well known children game when R and C have both three strategies. They simultaneously display their hands in one of three shapes denoting schematically a rock, a paper, or scissors. The rock wins over the scissors as it can shatter them, the scissors win over the paper as they can cut it, and the paper wins over the rock as it can be wrapped around the latter. A winner takes a penny from the opponent. If both players displays the same, then the game is drawn. The game can be tabulated as

		C		
		R	S	P
R	R	0,0	1,-1	-1,1
	S	-1,1	0,0	1,-1
	P	1,-1	-1,1	0,0

Table 1.2

A quite different example of a game is supplied by chess (or go or draughts). In the normal form of this game a strategy is a rule which prescribes the next move for a player for any possible position of the chessmen. The number of such strategies is so immense that it is impossible to represent this game in a table even using the memory of the most perfect modern computers. So, using normal form is not an adequate tool for such games. Here the extensive form is more preferable that we will look at later.

Two classes of games are often considered separately, as they have some nice, but special, properties. These are strictly competitive and symmetric games.

Strictly competitive games (or *games with opposite interests*) are the games where the gain of one player always equals the lose of another one, i.e. $\Pi_R(s_R, s_C) = -\Pi_C(s_R, s_C)$ for all strategies of R and C. This equation can be rewritten in the form $\Pi_R(s_R, s_C) + \Pi_C(s_R, s_C) = 0$, which means that the joint payoff of two players always vanishes. Hence these games are also called *zero sum games*. Both examples on Tables 1.1 and 1.2 above are zero sum games. In case of strictly competitive games one can leave only the first number in each cell of the table of the game, as the second