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Jacob Kogan

# Bifurcation of Extremals in Optimal Control

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**To my uncle Boris**

## Preview

The study concerns bifurcations of extremals in optimal control problems. The roots of this topic descends to the classical theory of the calculus of variations. The question of intersection of extremals has been considered in the calculus of variations under a sufficiency criterion, which enables to derive the famous Jacobi necessary condition. This condition guarantees the absence of conjugate points, namely the points of intersection of neighboring extremals.

We show in this work that in optimal control conjugate points exist even under a natural generalization of the sufficiency criterion of the calculus of variations. However, we discover that the set of the conjugate points has a simple and elegant structure.

The set of the conjugate points is described in the study for three different types of optimal control problems: an optimal control system with a scalar cost, an optimal control system with a vector cost functional, and an optimal control problem with constraints. In the case of a linear control equation we find out that the conjugate points are the points where the dimension of the attainable set increases; in particular these points do not depend on a cost functional. In a nonlinear case the conjugate points of an extremal $x(t)$ are determined by the attainable set of the system linearized about the extremal $x(t)$. The Jacobi necessary condition is recovered as a special particular case.

The first chapter of the study is an overview of the concepts, definitions, methods and results, the last, however, without proofs. The remainder of the work contains full proofs of the results.

In conclusion I wish to thank the numerous people for their assistance in this work. I am indebted especially to Zvi Artstein with whom I had many stimulating conversations. My thanks go to good people at Physics Department, University of Toronto for their cooperation and help with TEX typesetting system. Last, but not least I wish to acknowledge the much appreciated support of the ALD Ltd.

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# Contents

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*Chapter 1*

# Overview

## §1.1 The problem

In this study we analyze extremal trajectories of optimal control problems. The purpose of this work is to investigate under what conditions an extremal trajectory branches or bifurcates in time. By an extremal ( or an extremal trajectory ) of an optimal control problem we mean a trajectory which satisfies a first order necessary condition. In this study we examine trajectories of a control problem that satisfy the Pontryagin Maximum Principle as the necessary condition. We also examine extremals generated by the Euler–Lagrange equations. Connection between extremals generated by the different necessary conditions will be discussed.

The modern optimal control theory has deep roots originating from the classical theory of the calculus of variations. Numerous problems of the calculus of variations were treated under sufficient conditions which guarantee the nonexistence of branching points. In this work we adopt a natural generalization of these conditions and clarify reasons for the absence of branching points in the calculus of variations and, on the other hand, their appearance in problems of optimal control. We discover that when these branching points do occur, they have a nice detectable structure.

The study is organized as follows: This overview takes a general look at the results obtained in the study, indicating links to the fundamental ideas of the classical calculus of variations. The main concepts, ideas and methods are presented in this chapter. The detailed examples here will illustrate what is meant by words like branching and branching points. For the sake of convenience almost all examples are assembled in the last section of the chapter. Only those examples that help to explain new concepts and relations are displayed in the corresponding sections. The remainder of the study is devoted to full proofs of the results along with all the respective details.

## §1.2 Background from the calculus of variations

We introduce first, for the sake of convenience, the following convention: In order to distinguish between functions of a real variable and elements of the real Euclidean space $R^n$, the functions will be denoted throughout by boldface letters, in contrast with points in $R^n$. Namely $\boldsymbol{x}$, as well as $x(t)$, denotes a function, and $x$ is an element of $R^n$. The norm of $x \in R^n$ is denoted by $|x|$, $\langle x, y\rangle$ denotes the scalar product in $R^n$ and $\dot{\boldsymbol{x}}$ denotes differentiation with respect to time, i.e. $\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt}$. The norm of $n \times n$ matrix $M$ is denoted by $|M|$.

In the classical theory of the calculus of variations the question of intersection of extremals is considered. The so-called simplest variational problem consists of finding an extremum of a functional of the following form

$$c(\boldsymbol{x}) = \int_{t_1}^{t_2} f(t, x(t), \dot{x}(t))dt \quad \text{subject to } x(t_1) = x_1,\ x(t_2) = x_2 \tag{1.2.1}$$

where $f(t, x, y)$ is a function with continuous first and second partial derivatives with respect to all its arguments, and the class of admissible curves consists of all smooth curves $x(t)$ joining two points $x_1$, $x_2$. There are two significant normed linear spaces which are considered in connection with this variational problem:

1. The space $\mathrm{C}[t_1, t_2]$, consisting of all continuous functions $x(t)$ defined on a (closed) interval $[t_1, t_2]$, while the norm is defined as the maximum of the absolute value, i.e.

$$|\boldsymbol{x}|_{\mathrm{C}[t_1,t_2]} = \max_{t_1 \le t \le t_2} |x(t)|.$$

2. The space $D_1[t_1, t_2]$, consisting of all functions $x(t)$ defined on an interval $[t_1, t_2]$ which are continuous and have continuous first derivatives. The norm is defined by the formula

$$|\boldsymbol{x}|_{D_1[t_1,t_2]} = \max_{t_1 \le t \le t_2} |x(t)| + \max_{t_1 \le t \le t_2} |\dot{x}(t)|.$$

( We follow here the notions being used by Gelfand and Fomin in [6]. ) These linear spaces are important in our study. We will emphasize throughout resemblance between the problems of the calculus of variations, which are introduced next, on one hand, and the problems which are considered in our work on the other hand.

A curve $x(t)$ is a weak extremum of (1.2.1) if there exists an $\epsilon > 0$ such that the difference $c(\boldsymbol{x}) - c(\boldsymbol{y})$ has the same sign for all $y(t)$ in the domain of definition of the functional with $|\boldsymbol{y} - \boldsymbol{x}|_{D_1[t_1,t_2]} < \epsilon$. On the other hand, we say that $x(t)$ is a strong extremum of (1.2.1) if there exists an $\epsilon > 0$ such that $c(\boldsymbol{x}) - c(\boldsymbol{y})$ has the same sign for all $y(t)$ in the domain of definition of the functional with $|\boldsymbol{y} - \boldsymbol{x}|_{\mathrm{C}[t_1,t_2]} < \epsilon$.

An extremal trajectory is usually a good candidate for the extremum. This is the main motivation behind beginning a search for the extremum from investigation the extremal trajectories. In the classical theory of the calculus of variations the integral curves of Euler's equation are called extremals. Namely, $x(t)$ is an extremal of the simplest variational problem if $x(t)$ is a solution of the Euler second order differential equation

$$\frac{d}{dt} f_{\dot{x}}(t, x(t), \dot{x}(t)) = f_x(t, x(t), \dot{x}(t)) \tag{1.2.2}$$

The concept of conjugate point which is recalled next, plays an important role in the derivation of sufficient conditions for a functional to have a weak extremum ( see Gelfand and Fomin [6], p. 125, Theorem 6 ). Consider an extremal curve $x(t)$ defined over the time interval $[t_1, t_2]$. An extremal $y(t)$ is a neighboring extremal if the distance $|\boldsymbol{y} - \boldsymbol{x}|$ is small in the appropriate norm. If the norm is chosen to be $D_1[t_1, t_2]$ then the notion of a conjugate point is the following: The point $r$ is said to be conjugate to the point $t_1$ if $x(t)$ has a sequence of neighboring extremals drawn from the same initial point $x(t_1)$, such that this sequence tends to $x(t)$, each neighboring extremal intersects $x(t)$ and the points of intersection $(r_i, x(r_i))$ have $(r, x(r))$ as their limit. Namely, there exists a sequence $\{ \ y_i(t) \ \}$ of extremals such that

$$\lim_{i \to \infty} |\boldsymbol{y}_i - \boldsymbol{x}|_{D_1[t_1, t_2]} = 0, \ y_i(t_1) = x(t_1), \ y_i(r_i) = x(r_i) \text{ and } \lim_{i \to \infty} (r_i, x(r_i)) = (r, x(r)).$$

The Jacobi necessary condition is the following:

*If the extremal $x(t)$ corresponds to a minimum of the functional (1.2.1) and if the matrix $f_{\dot{x}\dot{x}}(t, x(t), \dot{x}(t))$ is positive definite along this extremal, then the open interval $(t_1, t_2)$ contains no points conjugate to $t_1$.* ( see [6], p. 124, Theorem 5. )

The theory of the calculus of variations presents also sufficient conditions for the **non–existence** of conjugate points on the interval $[t_1, t_2]$ as follows:

Hypothesis 1.2.1

1. $f_{\dot{x}\dot{x}}(t, x(t), \dot{x}(t))$ is positive definite along the chosen extremal $x(t)$,
2. $\displaystyle\int_{t_1}^{t_2} \begin{pmatrix} h(t) \\ \dot{h}(t) \end{pmatrix}^* \begin{pmatrix} f_{xx}(t, x(t), \dot{x}(t)) & f_{\dot{x}x}(t, x(t), \dot{x}(t)) \\ f_{x\dot{x}}(t, x(t), \dot{x}(t)) & f_{\dot{x}\dot{x}}(t, x(t), x(t)) \end{pmatrix} \begin{pmatrix} h(t) \\ \dot{h}(t) \end{pmatrix} dt$
   is positive definite for each $h(t)$ such that $h(t_1) = h(t_2) = 0$, where $*$ indicates the transpose.

Namely, if the conditions $1 - 2$ hold for the extremal $x(t)$, then the interval $[t_1, t_2]$ contains no point conjugate to $t_1$ ( see [6], p. 123, Theorem 4 ).

Remark 1.2.1 The nonexistence of conjugate points allows, as it is known from the calculus of variations ( see [6], pp. 145–149 ), the following:

To construct a field of extremals of the functional (1.2.1), namely to derive an ordinary differential equation

$$\dot{x}(t) = \psi(t, x(t)) \tag{1.2.3}$$

such that solutions of (1.2.3) are the extremals of (1.2.1). To define the Hilbert invariant integral. To derive sufficient conditions for a strong extremum. To obtain the Weierstrass necessary conditions for a strong extremum.

This remark emphasizes the importance of the absence of conjugate points in the problems of the calculus of variations. This work is devoted to the investigation of bifurcation of extremals in optimal control. We do not discuss ways to extend the presented in Remark 1.2.1 important constructions. However, in section 1.4 will be shown how to do it in some particular case. In the next section the question of existence of conjugate points in the modern control theory will be discussed.

## §1.3 Conjugate points in control theory

Consider an optimal control system

$$\dot{x}(t) = F(t, x(t), u(t)) \tag{1.3.1}$$

where $\boldsymbol{u} : [t_1, t_2] \mapsto R^m$ is measurable and $\boldsymbol{x} : [t_1, t_2] \mapsto R^n$ is absolutely continuous, where measurability is understood to be in the Lebesgue sense, and equalities are always "almost everywhere". Following Berkovitz ( see [1], p. 22 ) we define an admissible pair as follows:

<u>Definition 1.3.1</u> Let $x(t)$ be an absolutely continuous function from $[t_1, t_2]$ to $R^n$ and $u(t)$ be a measurable function from $[t_1, t_2]$ to $R^m$. The pair $(x(t), u(t))$ is admissible if it satisfies (1.3.1).

Consider the cost functional defined on the set of admissible pairs $(x(t), u(t))$ as follows:

$$c(\boldsymbol{x}, \boldsymbol{u}) = \int_{t_1}^{t_2} f(t, x(t), u(t))dt. \tag{1.3.2}$$

We assume throughout that the functions $F(t, x, u)$ and $f(t, x, u)$ have continuous first and second order partial derivatives with respect to $(x, u)$ and measurable in $t$. That is (for example) $\frac{\partial^2 F}{\partial x_i \partial u_j}(t, x, u)$ is a continuous function in $(x, u)$ for each fixed $t$, and is a measurable function of $t$ for each fixed $(x, u)$.

One of the problems which is considered in the control theory is the problem of finding an optimum of (1.3.2) among all admissible pairs $(x(t), u(t))$ satisfying the boundary condition $x(t_1) = x_1$, $x(t_2) = x_2$. An extremal trajectory $x(t)$ is a weak optimum of (1.3.2) if there exists a positive scalar $\epsilon$ such that $c(\boldsymbol{x}, \boldsymbol{u}) - c(\boldsymbol{y}, \boldsymbol{w}) \leq 0$ for each admissible pair $(y(t), w(t))$ with $x(t_1) = y(t_1)$, $x(t_2) = y(t_2)$ and $|(\boldsymbol{x}, \boldsymbol{u}) - (\boldsymbol{y}, \boldsymbol{w})| < \epsilon$ in an appropriate norm. In [11], for example, the norm is chosen to be $C[t_1, t_2]$. On the other hand we say that $x(t)$ is a strong optimum of (1.3.2) if there exists a positive $\epsilon$ such that $c(\boldsymbol{x}, \boldsymbol{u}) - c(\boldsymbol{y}, \boldsymbol{w}) \leq 0$ for each admissible pair $(y(t), w(t))$ with $x(t_1) = y(t_1)$, $x(t_2) = y(t_2)$ and $|\boldsymbol{x} - \boldsymbol{y}|_{C[t_1, t_2]} < \epsilon$. Hence the problem of finding an optimum of (1.3.2) disintegrates naturally into the problem of finding a weak optimum of (1.3.2), and the problem of finding a strong optimum of (1.3.2).

In order to emphasize the connection with the simplest variational problem we wish to mention, that in the special particular case of the control system $\dot{x}(t) = u(t)$ the two above listed problems coincide with the corresponding problems of the calculus of variations which have been presented in section 1.2.

Sufficient conditions for a weak local optimum in optimal control problems have been discussed, e.g., by Lee and Markus in [11], ( see Chapter 5, section 5.2 ). Sufficient conditions for a strong local optimum in optimal control were derived recently by Zeidan in [22] for Hamiltonians with locally Lipschitz gradients. There has been significant research in this area during the last years (see e.g. [2], [9]). However a connection of an extremal trajectory with conjugate points in optimal control problems was not investigated yet. It is shown in this study that extremals in an optimal control problem can intersect each other even under a very natural generalization of Hypothesis 1.2.1 ( see Hypothesis 1.3.1 ). On the other hand, and this is the main contribution of the study, we demonstrate in this work that the branching of extremals in optimal control problems possesses a simple elegant structure. In particular the Jacobi necessary condition is recovered as a special case.

In order to give the formal definition of branching in optimal control problems we need to recall some auxiliary notions.

Definition 1.3.2 A trajectory $x(t)$ is an extremal trajectory of the optimal control problem (1.3.1) – (1.3.2) if $(x(t), u(t))$ is an admissible pair and there exist a scalar $\eta_0 \leq 0$ and an absolutely continuous vector valued function $\eta(t)$ which is defined on $[t_1, t_2]$ and is a solution of the following ordinary differential equation:

$$\frac{d}{dt}\eta(t) = -\eta_0 \frac{\partial f}{\partial x}(t, x(t), u(t)) - \eta(t)\frac{\partial F}{\partial x}(t, x(t), u(t)), \tag{1.3.3}$$

and such that the triple $(x(t), u(t), \eta(t))$ satisfies the Pontryagin Maximum Principle, i.e. the following additional condition holds on $[t_1, t_2]$:

$$\eta_0 f(t, x(t), u(t)) + \eta(t)F(t, x(t), u(t)) = \max_u \{\eta_0 f(t, x(t), u) + \eta(t)F(t, x(t), u)\}. \tag{1.3.4}$$

( See Berkovitz [1], p. 186. ) The last definitions show that an extremal trajectory, which is the main object of the study, usually appears together with an admissible control and a corresponding solution of the adjoint equation (1.3.3). For the sake of the technical convenience we introduce now the notion of an extremal triple as follows:

Definition 1.3.3 A triple $(x(t), u(t), \eta(t))$ is an extremal triple if $(x(t), u(t))$ is an admissible pair, $\eta(t)$ is a solution of (1.3.3) and the triple $(x(t), u(t), \eta(t))$ satisfies condition (1.3.4).

Definition 1.3.4 The Hamiltonian $H$ of the optimal control problem (1.3.1) – (1.3.2) is defined as follows:

$$H(t, x, u, \eta) = \eta_0 f(t, x, u) + \eta F(t, x, u).$$

We present now the definition of a branching point of the extremal trajectory $x(t)$. Suppose that there exists a neighboring extremal $y(t)$ such that $x(t)$ and $y(t)$ coincide over an initial subinterval $[t_1, r]$ of the time interval $[t_1, t_2]$ and differ on the rest of it. Namely, $x(t) = y(t)$ for each $t \in [t_1, r]$ and $x(t) \neq y(t)$ for each $t \in (r, t_2]$. In this case we say that $r$ is a branching point of the extremal trajectory $x(t)$, and $y(t)$ branches out of $x(t)$ at $r$.

We wish to emphasize that our interest concentrates on the branching points of the extremal trajectory $x(t)$ formed by neighboring extremals initiating at the same point $x(t_1)$, where by a neighboring extremal we mean an extremal $y(t)$ such that the norm $|(\boldsymbol{x}, \boldsymbol{\eta}) - (\boldsymbol{y}, \boldsymbol{\mu})|_{C[t_1,t_2]}$ is small ( how small will be specified later ). Here $\eta(t)$, $\mu(t)$ are the corresponding solutions of the adjoint equation (1.3.3). Note, that in the particular case of the control system $\dot{x}(t) = u(t)$ this definition of a neighboring extremal leads to the old definition of a neighboring extremal defined by $D_1[t_1, t_2]$ norm in the calculus of variations.

Let $y(t)$ be an extremal trajectory of the control problem (1.3.1), (1.3.2) with $y(t_1) = x(t_1)$. It is shown in the study ( see Chapter 4, section 4.3 ) that there exists a positive $\epsilon$ such that, if $|(\boldsymbol{x}, \boldsymbol{\eta}) - (\boldsymbol{y}, \boldsymbol{\mu})|_{C[t_1,t_2]} < \epsilon$ and $x(r) \neq y(r)$ for some $r \in [t_1, t_2]$, then $x(t) \neq y(t)$ for each $t \in [r, t_2]$. On the other hand we show by an example ( see Example 1.10.1 ) that, if the condition $|(\boldsymbol{x}, \boldsymbol{\eta}) - (\boldsymbol{y}, \boldsymbol{\mu})|_{C[t_1,t_2]} < \epsilon$ is not satisfied, the extremal $y(t)$ can be different from $x(t)$ on a subinterval $[r, r+\delta)$ and intersect $x(t)$ once again on $[r+\delta, t_2]$.

In order to present the formal definition of a branching point we need to define rigorously what is meant by a neighboring extremal in this study. We say, first, that an extremal $y(t)$ is an $\epsilon$-neighboring extremal if $|(\boldsymbol{x}, \boldsymbol{\eta}) - (\boldsymbol{y}, \boldsymbol{\mu})|_{C[t_1,t_2]} < \epsilon$. A point $r$ is a $\epsilon$-branching point of the extremal trajectory $x(t)$ if there exists an $\epsilon$-neighboring extremal $y(t)$ such that

$$x(t) = y(t) \text{ on } [t_1, r], \text{ and } x(t) \neq y(t) \text{ on } (r, t_2].$$

It is clear, that if $\epsilon_1 < \epsilon_2$, then the set of $\epsilon_1$-branching points is a subset of that of $\epsilon_2$-branching points. It is shown in the study ( see Chapter 4, section 4.3, Definition 4.3.1 ), that there exists an $\epsilon^* > 0$ which depends only on the matrix $H_{uu}(t, x(t), u(t), \eta(t))$, such that for each positive $\delta$ less than $\epsilon^*$, the set of $\epsilon^*$-branching points coincides with that of $\delta$-branching points. From here on this $\epsilon^*$ will define the set of neighboring extremal trajectories as follows:

Definition 1.3.5 An extremal trajectory $y(t)$ is a neighboring extremal if the inequality $|(\boldsymbol{x}, \boldsymbol{\eta}) - (\boldsymbol{y}, \boldsymbol{\mu})|_{C[t_1,t_2]} < \epsilon^*$ holds.

This definition enables us to define a branching point as follows:

Definition 1.3.6 A point $r$ is a branching point of the extremal trajectory $x(t)$ if there exists a neighboring extremal trajectory $y(t)$ such that

$$x(t) = y(t) \text{ on } [t_1, r], \text{ and } x(t) \neq y(t) \text{ on } (r, t_2].$$

Our main purpose in this work is: to determine the conditions under which the branching points of an extremal $x(t)$ exist or do not exist and to characterize the set of branching points of