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ABSTRACT

Let $H^\infty(G)$ denote the algebra of bounded analytic functions on a bounded region G. Let $\pi: H^\infty(G) \to \mathcal{B}(\mathcal{H})$ be a continuous algebra homomorphism with $\pi(1)=1$ and such that $S \equiv \pi(z)$ is a subnormal operator. If S is pure, then π is weak-star, weak-star continuous and unique. Under various hypotheses we describe the spectrum and essential spectrum of $\pi(f)$ for f in $H^\infty(G)$.

Let G be the open unit disc and let S be multiplication by z on $L^2(m)$ where m is Lebesgue measure on the unit circle. In the setting of the paragraph above we may assume that the range of π is contained in $L^\infty(m)$. Our basic structure theorem establishes a one-to-one correspondence between such π's and certain measures on the maximal ideal space of H^∞. The mapping π can be onto. It is an isometry if and only if it is one-to-one and has closed range if and only if $\pi(f)=f$ for each f in H^∞. The mapping π is one-to-one if and only if there exists a measurable set E with positive measure such that $(\pi f)|_E = f|_E$ for each f in H^∞. Theorems for more general regions are also obtained.

In the last chapter these results are generalized to the following setting: let Y and Z be compact spaces. Let μ be a probability measure on Z and p a continuous map of Y onto Z. We obtain a structure theorem similar to the one mentioned above for those representations π of C(Y) into $L^\infty(\mu)$ such that $\pi(h \circ p)=h$ for all $h \in C(Z)$.

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Subnormal Operators and Representations of Algebras of Bounded Analytic Functions and other Uniform Algebras

Thomas L. Miller[1],

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and

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CHAPTER I

INTRODUCTION

Let G be a bounded domain in the plane $\mathbb{C}$ and let $H^\infty(G)$ denote the Banach algebra of bounded analytic functions on G. Let χ denote the function whose value at λ is λ for every $\lambda \in \mathbb{C}$. This paper is concerned with the theory of the continuous algebra homomorphisms from $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ that send 1 to 1 and χ to S where S is a subnormal operator acting on a separable Hilbert space $\mathcal{H}$. The Banach algebra $\mathcal{B}(\mathcal{H})$ consists of the bounded operators on $\mathcal{H}$.

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[1]Some of the results in Sections 2 and 3 appear in the first author's Ph.D thesis written under the supervision of Robert Olin.

[2]The last two authors were partially supported by a grant from the National Science Foundation during the preparation of this paper.

Even the reader whose interest does not reside in the structure of subnormal operators may still find some interesting results in this work (in particular, Chapters 5, 6 and 7). For example, suppose μ is a probability measure whose support, denoted spt μ, is contained in ∂G, the boundary of G. Let S be the normal operator on $L^2(\mu)$ given by multiplication by χ; i.e., $S=M_\chi$ where

$$M_\chi f = \chi f$$

for all $f \in L^2(\mu)$. If π is a continuous algebra homomorphism of $H^\infty(G)$ into $\mathcal{B}(L^2(\mu))$ with $\pi(1)=1$ and $\pi(\chi)=M_\chi$, then the range of π, ran π, is contained in the commutant of M_χ, denoted $\{M_\chi\}'$. Since this last algebra equals $\{M_g\colon g \in L^\infty(\mu)\}$ and, for each $g \in L^\infty(\mu)$, one has $\|M_g\| = \|g\|$, we may view π as a representation into $L^\infty(\mu)$. (For $g \in L^\infty(\mu)$ the definition of the operator M_g on $L^2(\mu)$ is the obvious one; i.e.,

$$M_g f \equiv gf$$

for all $f \in L^2(\mu)$.) Question: Given any such measure μ, are there any representations $\pi\colon H^\infty(G) \to L^\infty(\mu)$ that send χ to χ? The answer is yes. There are many. What are they? That is, describe how they arise. We will give a classification theorem that answers this last question (consult Theorem 59).

In the last chapter we show how the problems mentioned in the last paragraph are special cases of the following problem. Let Y and Z be compact spaces. Suppose μ is a probability measure on Z and p is a continuous map of Y onto Z. Describe those representations π of C(Y) into $L^\infty(\mu)$ such $\pi(h \circ p) = h$ for all $h \in C(Z)$. Our answer to this problem describes a very natural one-to-one correspondence between the extreme points of the set of measures ν on Y such that $p(\nu)=\mu$ and the set of representations π mentioned. (We would like to thank C. Foias for suggesting that our methods in Chapter Five might be general enough to carry out this latter description. Those readers who like abstraction first and examples second should read Chapter Eight before they read Chapter

Five.)

From now on, we shall refer to a continuous algebra homomorphism π from $H^{\infty}(G)$ into $\mathcal{B}(\mathcal{H})$ that sends 1 to 1 and χ to S (where S is a subnormal operator) as a <u>unital representation</u>.

Some remarks seem in order as to why the theory of unital representations has some importance to that of subnormal operators. One of the tools that has been used successfully to discover the structure of the lattice of invariant subspaces of a subnormal operator S is the algebra homomorphism from $P^{\infty}(\mu)$ to $\mathcal{A}(S)$, described in [11]. If S acts on the Hilbert space $\mathcal{H}$ and its minimal normal extension N is defined on $\mathcal{K}$, then the scalar spectral measure for N is denoted by μ. (An excellent account for the general theory of subnormal operators is contained in [10].) Define $P^{\infty}(\mu)$ as the weak-star closure of the polynomials in $L^{\infty}(\mu)$ (the dual of $L^{1}(\mu)$), and $\mathcal{A}(S)$ as the weak-star closure of the polynomials (in the variable S) in $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{B}(\mathcal{H})$ is the dual of the trace class operators.) It turns out that $\mathcal{A}(S)$ is weakly closed and the weak operator topology and the weak-star topology agree on this algebra [33].

If $P^{\infty}(\mu)$ is antisymmetric; i.e., every real-valued function in it is constant, then $P^{\infty}(\mu)$ is isometrically isomorphic and weak-star homeomorphic to the algebra $H^{\infty}(G)$ for a suitably chosen domain G. (The regions G that arise in this fashion are characterized in [32].) The space $H^{\infty}(G)$, for any region G, is the dual of a separable Banach space [36]; a sequence $\{f_n\}$ in $H^{\infty}(G)$ converges weak-star if and only if it is bounded and converges pointwise everywhere on G. Therefore, the algebra homomorphism referred to earlier is an example of a unital representation that is an isometry and a weak-star homeomorphism.

Looking at the other techniques used in [5,33] to study subnormal operators (or those techniques used in [2,6,8] to study other classes of operators), one realizes the existence of a unital representation π in the $P^{\infty}(\mu)$ case) implies some important information about the lattice of invariant subspaces for S. (For a specific result, consult Theorem 3.2 in [8].)

One is naturally led into a vast array of problems; this paper should be viewed as an initial assault on this mound. We have taken the liberty of asking many questions which we were unable to resolve. They are scattered throughout this work. There are many questions left that we did not ask that are begging to be answered.

Our probing has been directed in two ways. On one hand, we have let the problems of existence and uniqueness dictate the course of investigation. On the other hand, we have let the problems related to the functional calculus and continuity steer our inquiries. An example of our journey along the first course has already been sketched in the second paragraph. If π is a unital representation of $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$, then in a natural way, π describes a functional calculus. Our objective in this light (the second course) has been twofold; describe the spectral mapping theorems associated with π, and discuss the weak-star continuity of π (does it hold?).

Clearly the latter investigation is motivated from the results in [11], as we mentioned earlier. There is another motivating source for this inquiry.

Example 1. Let G be a bounded domain in $\mathbb{C}$ and let μ be planar Lebesgue measure restricted to G. Let $\mathcal{H}$ be the space of analytic functions on G that belong to $L^2(\mu)$. It follows from [27] that $\mathcal{H}$ is a closed subspace of $L^2(\mu)$ and that M_χ is a bounded subnormal operator on $\mathcal{H}$. Furthermore, $\{S\}'$, where $S=M_\chi|_{\mathcal{H}}$, consists of those multiplication operators M_ψ, where $\psi\in\mathcal{H}\cap L^\infty(\mu) = H^\infty(G)$. Clearly then, the map defined by

$$\pi(f) = M_f \text{ on } \mathcal{H}$$

is a unital representation. In [3,4] the spectral mapping theorems associated with this particular unital representation are the focal points.

Recall that a point $\lambda\in\partial G$ is inessential if there is a $\delta>0$ such that every $f\in H^\infty(G)$ extends analytically to the open disc $\Delta(\lambda,\delta)$. The remaining points on ∂G are called essential boundary points. If π is a unital representation defined on $H^\infty(G)$ and λ is an inessential boundary point, then π can be extended to $H^\infty(G\cup\Delta(\lambda,\delta))$ via the formula

$$\tilde{\pi}(h) \equiv \pi(h|_G)$$

for all $h \in H^\infty(G \cup \Delta)$. Thus, there is no loss in generality in assuming, as we will do in the rest of the paper without comment, that <u>every point on ∂G is essential</u>.

The paper is organized in the following fashion. In Chapter 2, we show that if π is a unital representation with domain $H^\infty(G)$, then π is unique provided that either S is a pure subnormal operator, or $\mu(\partial G)=0$. (A pure subnormal operator is one that has no nontrivial invariant subspace on which it is normal.)

In Chapter 3 we establish as a corollary to a more general result that any unital representation π is weak-star, weak-star continuous, provided that $S=\pi(\chi)$ is pure. In Chapter 4 we describe some spectral mapping theorems for a given unital representation. Our results describe (under various hypotheses) the spectrum of $\pi(f)$, denoted $\sigma(\pi(f))$, and the essential spectrum of $\pi(f)$, denoted $\sigma_e(\pi(f))$.

We have already discussed the outline of Chapter 5. This section illustrates how important the hypothesis of purity in the theorems of Chapters 2 and 3 are. As a corollary to our principal theorem, we show there are many unital representations π from $H^\infty(D)$ into $L^\infty(m)$. Throughout the paper D will denote the open unit disc and m will denote normalized Lebesgue measure on ∂D. Given a Blaschke product b whose zeros accumulate everywhere on ∂D and a function $f \in L^\infty(m)$ with $\|f\| \le 1$, we shall show there exists a unital representation π such that $\pi(b)=f$ (consult Example 40). This last fact then answers a uniqueness question found in [8]. The authors of this latter work ask: "If B is a polynomially bounded operator such that $B^k \to 0$ in the weak operator topology, can there exist two different norm-continuous representations each of which sends 1 to 1 and χ to B?" The answer is yes. If B is the bilateral shift (a normal operator), then there are many such representations. (On the other hand, if B is the unilateral shift (a pure subnormal operator), then there is only one unital representation.)

In Chapter 6 we investigate the question of whether, for a given region G, there is a probability measure μ on ∂G and a unital representation of $H^\infty(G)$ into $L^\infty(\mu)$ that is an isometry? The problem is completely solved for simply connected domains. A region of this last type supports an isometric unital representation if and only if it is nicely connected. The representations that are isometries are classified. What happens for an arbitrary domain? We do not know.

In Chapter 7 we define the notion of when two unital representations are partially subordinate and relate this concept to some issues of the earlier sections. In particular, we investigate the lattice structure of a representation and answer the question of when a representation is one-to-one. The material in the last chapter has already been discussed.

We have tried to keep the material in the last four chapters as self-contained as possible. (There are, however, times when we need to draw on some of the results in Chapters 2 and 3). Our primary reason for doing this, as indicated earlier, is that the problems addressed in these sections are (can be viewed as) purely function-theoretic ones; we hope that readers, who may not have an interest in the theory of subnormal operators, will still find some value in this material.

We close this chapter by asking a question that deals with an issue not addressed in this paper.

Question 2. If S is a pure subnormal operator acting on the Hilbert space $\mathcal{H}$ and π is an algebra homomorphism from $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ with $\pi(1)=1$ and $\pi(\chi)=S$, then is π norm continuous?

If S has a cyclic vector, then S is unitarily equivalent to M_χ on $H^2(\mu)$ for some compactly supported measure in the plane. (The Hilbert space $H^2(\mu)$ consists of the closure of the polynomials in $L^2(\mu)$.) A result of Yoshino [41] states that $\{S\}' = \{M_\psi : \psi \in H^2(\mu) \cap L^\infty(\mu)\}$ if S has a cyclic vector. Hence, every operator belonging to $\{S\}'$ is subnormal.

If we assume S has a cyclic vector in Question 2, then the answer is yes. Recall the theorem that says any algebra homomorphism from one Banach algebra into a semi-simple commutative Banach algebra is continuous [12, Prop.4.2]. So to prove the fact, it suffices to show the only quasinilpotent operator in $\{S\}'$ is zero. If $\psi \in H^2(\mu) \cap L^\infty(\mu)$ and M_ψ is quasinilpotent, then $\sigma(M_\psi) = \{0\}$. Hence, we see that $M_\psi = 0$ if we recall the fact that subnormal operators have the property that their spectral radius is equal to their norm. (Yes, the last argument did not use the assumption of purity.)

CHAPTER II

UNIQUENESS OF REPRESENTATIONS

Let S be a subnormal operator on $\mathcal{H}$ and let N be its minimal normal extension on $\mathcal{K}$. The spectral measure for N will be denoted by E. Suppose π_1 and π_2 are two unital representations from $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ such that $\pi_i(\chi)=S$ for $i=1,2$. Note that in the next three lemmas and the corollary, we do not assume S is a pure operator (in particular, the case S=N is allowed).

Lemma 3. (Assume the notation in the preceding paragraph.) Let W be a bounded region containing G and let $g \in H^\infty(W)$. Let $\lambda \in W$ and let $\Delta(r) = \Delta(\lambda, r)$ be the open disc centered at λ with radius r. Then

$$\overline{\lim_{r\to 0}} \frac{\|E(\Delta(r))(\pi_1(g)-\pi_2(g))\|}{r^2} = 0.$$

Proof. Fix $g \in H^\infty(W)$ and $\lambda \in W$ and choose $r>0$ such that $\Delta(r) \subset W$. Let $h_\lambda(z)$ be the first three terms of the Taylor series expansion of g about λ; i.e.,

$$h_\lambda(z) = g(\lambda) + g'(\lambda)(z-\lambda) + \frac{g''(\lambda)(z-\lambda)^2}{2!}.$$

Then $g-h_\lambda = (\chi-\lambda)^3 q_\lambda$ where the supremum norm of q_λ on W, denoted $\|q_\lambda\|$, is bounded by M, where M depends only on some universal constants, $\|g\|$ and the distance of λ to ∂W. (To see this estimate on M use the Cauchy estimates on the derivatives of g at λ and the inequality

$$|q_\lambda| \le (|g| + |h_\lambda|)(|\chi-\lambda|)^{-3}.)$$

We then have

$$\|E(\Delta)\,[\pi_1(g)-\pi_2(g)]\|$$

$$\leq \|E(\Delta)\,[\pi_1(g)-h_\lambda(S)]\| + \|E(\Delta)[h_\lambda(S)-\pi_2(g)]\|$$

$$= \|E(\Delta)(S-\lambda)^3\pi_1(q_\lambda)\| + \|E(\Delta)(S-\lambda)^3\pi_2(q_\lambda)\|$$

$$\leq \|E(\Delta)(S-\lambda)^3\|[\|\pi_1(q_\lambda)\| + \|\pi_2(q_\lambda)\|]$$

$$\leq \|(N-\lambda)^3E(\Delta)\|\ M(\|\pi_1\| + \|\pi_2\|)$$

$$\leq \|(N-\lambda)^3E(\Delta)\|\ M(\|\pi_1\| + \|\pi_2\|)$$

$$\leq r^3\ M\,(\|\pi_1\| + \|\pi_2\|). \blacksquare$$

Lemma 4. With the same notation and assumptions as in Lemma 3, we have

$$\|E(W)\ (\pi_1(g)-\pi_2(g))\| = 0$$

for every $g \in H^\infty(W)$.

Proof. If not, then by the regularity of E there exists a compact set $K \subset W$ such that

$$\eta \equiv \|E(K)(\pi_1(g)-\pi_2(g)\| > 0.$$

Let d be the diameter of K. Construct a square that has sides of length d and that contains K. Partition the square into four congruent squares each of which has sides of length d/2. One of these four squares, say T_1, has the property that

$$\|E(T_1 \cap K)(\pi_1(g)-\pi_2(g))\| \geq \eta/4.$$

Continuing this process by induction, we construct a sequence $\{T_n\}$ of squares with the following properties:

$T_{n+1} \subset T_n$;

the sides of T_n have length $d/2^n$;

and

$$\|E(T_n \cap K)(\pi_1(g)-\pi_2(g))\| \geq \eta/4^n.$$

Let λ be such that $\{\lambda\} = \cap T_n$. Since K is compact and $T_n \cap K$ is nonempty for every n, it follows that $\lambda \in K$.

Let $r_n = d/2^{n-1}$. Then $T_n \subset \Delta(\lambda, r_n)$ and

$$\|E(\Delta(\lambda,r_n))(\pi_1(g)-\pi_2(g))\|$$
$$\geq \|E(T_n \cap K)(\pi_1(g)-\pi_2(g))\|$$
$$\geq \eta/4^n.$$

Thus,

$$\|E(\Delta(\lambda,r_n))(\pi_1(g)-\pi_2(g))\| \geq \eta r_n^2/4d^2.$$

Letting $n \to \infty$, we see that the conclusion of Lemma 3 is contradicted. ■

In a way, the result of Lemma 4 is unsatisfactory. The lemma implies that $E(W)\pi_1(g)$ and $E(W)\pi_2(g)$ are equal; but, intuition tells one (based on many examples) that $\pi(g)$ should be multiplication by g on the space $E(W)\mathcal{H}$ for $g \in H^\infty(W)$. (We continue to use the setting of Lemmas 3 and 4.) Note, however, $E(W)\mathcal{H}$ may not be contained in $\mathcal{H}$ and, therefore, vectors in the subspace $E(W)\mathcal{H}$ are not necessarily in the domain of $\pi(g)$. However, by an appropriate orthogonal decomposition of the space $\mathcal{K}$, we can place our beliefs on solid ground.

Let μ be a scalar-valued spectral measure for N. Then [10, Chapter 2, Section 9] there exists a sequence (possibly finite) of measures $\{\mu_i\}$ such that $\mu=\mu_1$, $\mu_{i+1} << \mu_i$ for all i and N is (unitarily equivalent to) the operator $\oplus_i M_\chi$ on the space $\oplus_i L^2(\mu_i)$. Assume that π is a unital representation defined on $H^\infty(G)$ and N is the minimal normal extension of $\pi(\chi)$.

It follows easily from the fact that π is a homomorphism that $\bar{G} \supset \sigma(S)$. Recalling the facts that $\sigma(S) \supset \sigma(N)$, that $\partial\sigma(S) \subset \partial\sigma(N)$ and that $\sigma(N) = \text{spt } \mu$, we see spt $\mu_i \subset \bar{G}$ for all i.

<u>Lemma</u> 5. Let π be a unital representation of $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ where $\pi(\chi)=S$. Let W be a region containing G. Using a unitary operator, if necessary, we assume the minimal normal extension N of $\pi(\chi)$ is

$$\oplus_i M_\chi \text{ on } \oplus_i L^2(\mu_i).$$

(This decomposition is that described in the above.) Let $t = \oplus_i t_i$ be a vector in $\mathcal{H}$ and let $f \in H^\infty(W)$. Choose $\phi_i \in L^2(\mu_i)$ for each i so that

$$\pi(f)t = \oplus_i \phi_i$$

Then, for each i, we have

$$\phi_i = f t_i$$

almost everywhere $\mu_{i|W}$.

The measure $\mu_{i|W}$ is the restriction of μ_i to W. Clearly Lemma 5 implies Lemma 4. The proof of Lemma 5 relies on the following fact whose proof is left to the reader. (One way of proving this fact is to use the technique of the proof of Lemma 4.)

Fact 6. If μ is a positive compactly supported measure in the plane, then the set

$$\{w \in \mathbb{C}: \int \frac{d\mu(z)}{|z-w|^2} < \infty\}$$

has μ-measure zero.

Proof of Lemma 5. Suppose to the contrary that there exist an i_0 and $\varepsilon > 0$ such that

$$|\phi_{i_0} - f t_{i_0}| \geq \varepsilon$$

on a compact subset $E \subset W$ with $\mu_{i_0}(E) > 0$. We may assume, without loss of generality, $|t_{i_0}| \leq M$ on E for some positive constant M. Fix a point $w_0 \in E$ such that $\mu_{i_0}(E \cap U) > 0$ for each neighborhood U of w_0.

By the continuity of f and the boundedness of t_{i_0}, there exists a neighborhood U of w_0, $U \subset W$ such that

$$|f(w) - f(w_0)|\ |t_{i_0}| < \varepsilon/2$$

for all $w \in U$. Observe that, for each $w \in U$,

$$(\pi(f - f(w)))t = \oplus_i (\phi_i - f(w)t_i).$$

For each $w \in U$, we define a function $h_w \in H^\infty(W)$ via

$$h_w(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \in G \setminus \{w\} \\ f'(w) & z=w. \end{cases}$$

We compute:

$$\pi(f-f(w)) = \pi((\chi-w)h_w)$$

$$= (S-w)\pi(h_w)$$

$$= (N-w)\pi(h_w)$$

$$= ((\sum_i M_\chi)-w)\pi(h_w).$$

Hence, for each $w \in U$, the function

$$\bigoplus_i \frac{(\phi_i - f(w)t_i)}{\chi - w} = \pi(h_w)t$$

belongs to $\oplus_i L^2(\mu_i)$. But, everywhere on $E \cap U$, we have

$$|\phi_{i_0} - f(w)t_{i_0}| \geq |\phi_{i_0} - f t_{i_0}| - |f - f(w)|\,|t_{i_0}|$$

$$\geq \epsilon - \epsilon/2$$

$$= \epsilon/2.$$

Hence, it follows that $\frac{1}{\chi - w} \in L^2(\mu_{i_0}|_{U\cap E})$ for every $w \in U$. Applying Fact 6, we get that $\mu_{i_0}(U\cap E)=0$; a clear contradiction to the fact $\mu_{i_0}(U\cap E)>0$. ■

A uniqueness theorem for unital representations now follows for a wide variety of examples.

Corollary 7. Let S be a subnormal operator and let N be its minimal normal extension. If π_i for $i=1,2$ are two unital representations defined on $H^\infty(G)$ with $\pi_i(\chi)=S$, then $\pi_1=\pi_2$ provided that $E(\partial G) = 0$ where E is the spectral measure for N.

We see, from this corollary, that the representation defined in Example 1 is unique. As mentioned in the introduction (see Chapter 5) this uniqueness can fail if $E(\partial G)\neq 0$. But not if $\pi(\chi)$ is pure. Before we prove this last statement, we present two results from