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Relativistic Electrodynamics and Differential Geometry

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Preface

The aim of this book is to provide a short but complete exposition of the logical structure of classical relativistic electrodynamics written in the language and spirit of coordinate-free differential geometry. The intended audience is primarily mathematicians who want a bare-bones account of the foundations of electrodynamics written in language with which they are familiar and secondarily physicists who may be curious how their old friend looks in the new clothes of the differential-geometric viewpoint which in recent years has become an important language and tool for theoretical physics. This work is not intended to be a textbook in electrodynamics in the usual sense; in particular no applications are treated, and the focus is exclusively the equations of motion of charged particles. Rather, it is hoped that it may serve as a bridge between mathematics and physics.

Many non-physicists are surprised to learn that the correct equation to describe the motion of a classical charged particle is still a matter of some controversy. The most mentioned candidate is the Lorentz-Dirac equation † . However, it is experimentally unverified, is known to have no physically reasonable solutions in certain circumstances, and its usual derivations raise serious foundational issues. Such difficulties are not extensively discussed in most electrodynamics texts, which quite naturally are oriented toward applying the well-verified part of the subject to concrete problems. Some authors claim that the supposed difficulties are irrelevant or easily resolved, others mention them briefly in passing, and others simply ignore them. This book focuses on them but takes no position. Rather, it attempts to present the basic issues as clearly and precisely as possible so that the reader can draw his own conclusions.

As to background, it is assumed that the reader is familiar with the language of modern mathematics and has an elementary acquaintance with electromagnetic theory, at least at the level of a good freshman physics course. In addition, a working knowledge of special relativity and elementary abstract differential geometry are highly desirable prerequisites which will in any event have to be acquired along the way. The necessary concepts from each are presented in the first two chapters, the first oriented toward mathematicians and the second toward physicists and mathematicians who are not experts in differential geometry. However, these are intended as refresher courses to establish a framework within which to develop the theory rather than as texts for beginners, and the reader who has never studied special relativity or is totally unfamiliar with differential geometry may find it hard going. If so, the obvious remedy is to spend a few weeks or months with one of the many good texts on these subjects and begin again.

Chapter 3 formulates the part of electrodynamics which deals with continuous distributions of charge, while Chapter 4 treats radiation and presents the usual motivation leading to the Lorentz-Dirac equation. These two chapters are an expository synthesis of standard material. Only Section 4.4, which applies differential-geometric ideas to clarify radiation calculations usually done in less transparent ways, has any claim to novelty of content as opposed to exposition.

† Not to be confused with the quantum-mechanical Dirac wave equation for the electron.

The last chapter, which is of a more specialized and speculative nature, explores further difficulties with the usual formulation of electrodynamics and discusses alternate approaches. Much of this chapter is drawn from the research literature, and some of it appears to be new. Particularly noteworthy is Section 5.5, which presents a proof (based on ideas of Hsing and Driver) of Eliezer's Theorem on nonexistence of physical solutions of the Lorentz-Dirac equation for one-dimensional symmetric motion of opposite charges. This theorem implies that if the Lorentz-Dirac equation holds, then two oppositely charged point particles released at rest can never collide, and in fact will eventually flee from each other at velocities asymptotic to that of light. Since no one seems to believe that real particles will actually behave this way, many (but by no means all) interpret this result as casting serious doubt on the Lorentz-Dirac equation. Surprisingly, few physics texts even mention this result, though it has been extensively discussed in the research literature.

Mathematicians often complain that physics texts are hard to read because of frequent looseness of language and lack of careful definitions. Physicists grumble about the insufficient attention to motivation, excessive concern with generality, and plodding definition-theorem-proof-corollary-definition style too common in mathematics texts. I have tried to avoid all of these, but style is largely a matter of taste and compromise, and it would be miraculous if my taste were to everyone's liking. No doubt some physicists will consider the style overly cautious and pedantic while some mathematicians will find it too loose.

The decision to do electrodynamics in the general context of a Lorentzian manifold without explicitly introducing general relativity (i.e. the Einstein equation) was also largely a matter of taste. It certainly is not necessary to leave Minkowski space to present all the main ideas of electrodynamics, and, unfortunately, some of the important ideas and methods extend to general spacetimes only at the expense of considerable mathematical complication or physical obscurity. On balance, however, it seemed that enough does extend easily to make the relatively small extra effort worthwhile. More importantly, I feel that some of the ideas are actually clearer if one temporarily forgets the vector space structure of Minkowski space. In the end, I did it the way I should have liked to have seen it when I was learning it for the first time.

I have tried to make the notation as coordinate-free as practical while keeping it close to traditional physics notation. The only real departure from the latter is the use of the superscript "$*$" to indicate the operation which identifies a vector with a linear form: relative to traditional physics notation, if $u = u^i$, then $u* = u_i$. The physicist who feels comfortable with abstract differential geometry and wants to dive right in to Chapter 4 or 5 should be able to do so after scanning the table of notations.

I sincerely thank all who helped, directly or indirectly, with this book. In particular, I am grateful for the hospitality of the Mathematics Department of the University of California, Berkeley, where much of the writing was done. I appreciated helpful conversations with J. D. Jackson and R. Sachs and am indebted to H. Cendra and M. Mayer for many useful suggestions. Naturally, I alone am responsible for any errors, and notification of any such will be gratefully received.

Stephen Parrott

July 11, 1986

Contents

Chapter 1

Special Relativity

1.1 Coordinatizations of spacetime.

Classical Newtonian physics is formulated in terms of the notions of "distance" and "time". These are taken as primitive concepts whose meaning is supposed to be self-evident and universally agreed upon. The theory of relativity does not recognize "distance" and "time" as concepts whose meaning is self-evident. Instead, they are considered as physical quantities which must be carefully defined in terms of other, still more elementary concepts.

Of course, relativity must itself be based on some primitive concepts, and the fundamental conceptual unit recognized by the theory of relativity is that of *event*. The term *event* refers to a definite happening or occurrence, such as the explosion of a bomb or the emission of a photon by an atom. Intuitively, an "event" is "something which happens at a definite place at a definite time". The set **E** of all events is called *spacetime*.

This interpretation of the term "event" makes it reasonable to try to assign to each event e a quadruple of numbers (t_e, x_e, y_e, z_e) in which the last three components x_e, y_e, z_e specify, in some sense which we choose not to make precise yet, the point in "space" at which the event occurred, and the first component t_e is associated with the "time" at which it occurred. † If such a map $e \longmapsto (t_e, x_e, y_e, z_e)$ is a bijection (i.e. a 1-1 correspondence) from **E** onto the four-dimensional real vector space R^4, it will be called a *coordinatization* of **E**. We shall also occasionally refer to *local coordinatizations* which only map a subset of **E** onto R^4.

As a concrete illustration of how such a coordinatization might be carried out, consider four observers O, A, B, and C, each equipped with a clock and a device to measure angles between light rays. Let the observers adjust their positions so that light rays sent from A, B, and C to O are measured by O as mutually orthogonal and so that the round trip time of a light ray sent from O to any of the observers A, B, C and reflected back to O depends neither on the observer to whom the beam is sent nor on the time (as measured by O's clock) that the beam is sent. Intuitively, this means that the observers A, B, and C are at rest with respect to O, are equidistant from O, and are located on the axes of an orthogonal coordinate system with origin at O. (See Figure 1.)

Suppose we are given an event e to be coordinatized, such an explosion which emits a flash of light of infinitesimal duration. We assume that the four observers are constantly exchanging light signals, so that the angles α, β, γ, and δ in the diagram can be measured. Now imagine that O, A, B, and C are points in three-dimensional Euclidean space R^3, and choose the unit of distance in R^3 so that the distance from O to A, B, or C is equal to one. Then it is a simple matter to use standard trigonometry to compute three numbers x_e, y_e, and z_e which would be the coordinates of a point in R^3 such that the lines drawn in R^3 from this point to O, A, B, and C form the same angles α, β, γ, δ shown in the diagram. Let us call these three numbers x_e, y_e, z_e the "space"

† Since the theory of relativity does not recognize "space" and "time" as primitive concepts, statements like this must be taken as purely poetic descriptions. A problem in introducing relativity is that the Newtonian view of the world is so intertwined with the language we speak that it is often difficult to formulate statements in everyday language which do not use Newtonian concepts in a relativistically inadmissible way. As soon as we have replaced the Newtonian concepts of "space" and "time" with the relativistic ones of "event" and "coordinatization", this linguistic difficulty will disappear.

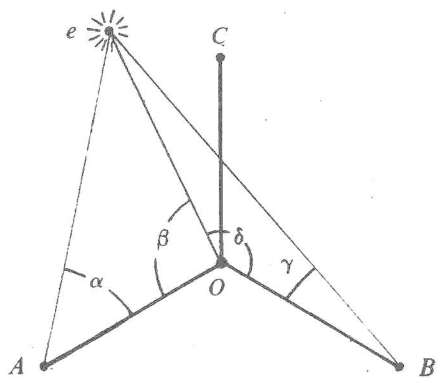

Figure 1-1. The straight lines are the paths of light rays.

coordinates of the event e. To assign a "time" coordinate to e, define a constant c as twice the reciprocal of the round-trip time of a light signal from O to A and back (so that c is the average velocity of light on this particular round-trip), let t_0 denote the time on O's clock at which the flash of light emitted by the event being coordinatized reached O, and define

$$t_e := t_0 - \frac{1}{c}(x_e{}^2 + y_e{}^2 + z_e{}^2)^{1/2} ,$$

That is, t_e would be the time at which the flash was emitted if the velocity of light in our imaginary Euclidean space were always c.

We have thus defined a map $e \longmapsto (t_e, x_e, y_e, z_e)$ from E to R^4. To postulate that this map is a a bijection is a physical assumption with far-reaching implications which we shall not examine further here, partly because we do not want to base our treatment of special relativity on this particular coordinatization, and partly because it is assumed that the reader already has at least a passing acquaintance with the physical bases of this theory. Note that this coordinatization used only *local* measurements of angles and time; the only time interval measured was measured by the clock at O, and this clock was never compared with a clock elsewhere. Thus there is no need within the above framework for the Newtonian concept of an absolute time with respect to which all clocks are synchronous; all that is needed is the operationally defined "time as measured by O's clock".

Naturally, there are a great many other reasonable ways to try to coordinatize E, and we do not claim that particular method just described has any special merit other than its conceptual simplicity. The details of the coordinatization will not be important for our purposes; all that we shall need to begin our treatment of special relativity is the assumption that there is *some* coordinatization with special properties which we shall specify in the next section, most notably that light always travels in straight lines with constant velocity. This is a convenient and efficient way to rapidly develop special relativity, but it does slight many interesting and important physical issues, and we urge the reader who is not well acquainted with the physical ideas to read at least the first few chapters in any of the many good physics texts on the subject, some of which are recommended at the end of this chapter.

To avoid misunderstanding, we also remark that although computations using ordinary Euclidean trigonometry were used in the above coordinatization, they served only as a tool to assign coordinates, and none of the geometrical properties of Euclidean space are assumed, *a* *priori,* to carry over to a "space slice" of spacetime. For instance, in Euclidean space, the sum of the angles in a triangle is always 180°, but there is no reason that the angles of a triangle $OA-AB-BO$ of light rays in Figure 1 should sum to 180°.

1.2. Lorentz coordinatizations.

The theory of special relativity is based on the assumption that there exist coordinatizations of the set **E** of all events, called *Lorentz coordinatizations,* with special properties which we shall now describe. Let $e \longmapsto (t_e, x_e, y_e, z_e)$ be a coordinatization of **E**. Consider an observer who carries with him a clock, and suppose that at every instant τ of time as measured by that clock he is able to determine his coordinates $(t(\tau), x(\tau), y(\tau), z(\tau))$. (It is *not* assumed that $t(\tau) = \tau$.) We shall call the observer *stationary* at x_0, y_0, z_0 with respect to the given coordinatization if $x(\tau) = x_0$, $y(\tau) = y_0$, $z(\tau) = z_0$ for all times τ. It will be helpful to think of space as densely populated with stationary observers, each carrying a "standard clock". The "standard clocks" are conceived as identically constructed (for example, one might use excited hydrogen atoms which emit spectral lines of characteristic frequencies and can in principle serve as clocks by counting successive wavecrests). The first property which we demand of a Lorentz coordinatization is the following.

(1) We assume that the coordinatization is such that *stationary* standard clocks measure "coordinate time" $t(\tau)$. That is, for any stationary standard clock with coordinates $(t(\tau), x_0, y_0, z_0)$ as above, $t(\tau) = \tau$ for all τ.

This implies, in particular, that the rate of a stationary standard clock does not depend on its spatial location. This assumption might seem self-evident or at least innocuous, but the delicacy of the situation is shown by the fact that it is only approximately true in the real world. The theory of general relativity (which does not make this assumption) predicts, and experiment confirms, that identical clocks located at different places in a gravitational field will run at different rates. Special relativity only approximates physical reality in a sense quite closely analogous to the way that a tangent plane approximates a surface. The surface of the ocean appears flat so long as one does not have to navigate long distances, and special relativity provides a description of reality which is accurate in a laboratory small enough that differences in the gravitational field can be neglected.

Suppose we have a coordinatization satisfying (1) and for each triple of real numbers (x, y, z), a stationary observer with this triple for spatial coordinates. Imagine a pulse of light travelling through space. Think of the pulse as of infinitesimal duration and spatial extent, so that it is like a moving particle. For instance, turning on a flashlight with a very narrow beam for a very short time would approximate such a pulse. Given spatial coordinates (x, y, z), we may ask the stationary observer at (x, y, z) at what time t on his standard clock the pulse passed through his point (x, y, z), assuming that it passed through that point at all. It is reasonable to suppose that for any given number t, exactly one point, which we denote $(x(t), y(t), z(t))$ will report a passage at time t. Since all the stationary standard clocks measure coordinate time, this assumption just means that at any given coordinate time, the pulse is somewhere, and that it can't be in two different places at the same coordinate time. We now introduce the second property of a Lorentz coordinatization:

(2) Light always moves in straight lines with unit velocity. This means that if we set $\vec{r}(t) := (x(t), y(t), z(t))$, with $x(t), y(t), z(t)$ defined as above, then the function $t \longmapsto \vec{r}(t)$ is of the form $\vec{r}(t) = \vec{v}t + \vec{r}_0$, where $\vec{v}, \vec{r}_0 \in R^3$ are constant vectors, and $\vec{v}$ is a unit vector.

A coordinatization $e \longmapsto (t_e, x_e, y_e, z_e)$ of spacetime **E** satisfying (1) and (2) will be called a *Lorentz coordinatization*, or *Lorentz coordinate system*, or *Lorentz frame.* The first of two fundamental physical assumptions of special relativity is that *there exists a Lorentz coordinatization for spacetime.*

The assumption that $\vec{v}$ is a unit vector is considerably more than a normalization. The physical content of this assumption is not only that the speed $|\vec{v}|$ at which a pulse of light travels is finite, but also that it is the *same* under all conditions. For instance, $|\vec{v}|$ must be the same for a flash emitted by the headlight of a speeding motorcycle as for a signal from the flashlight of a person standing on the road. Given this, choosing the velocity of light to be unity is of course just a normalization. Physically it means that we are choosing as a unit of length the distance that light travels in a unit of time (e.g. if time is measured in years, then distance is measured in light-years); mathematically it means that we replace a coordinatization $e \longmapsto (t_e, x_e, y_e, z_e)$ in which the vectors $\vec{v}$ in (2) have length c by the new coordinatization $e \longmapsto (t_e, x_e/c, y_e/c, z_e/c)$. We shall use this normalization throughout, so the velocity of light will never enter explicitly into our formulae.

Given a Lorentz coordinatization $e \longmapsto (t_e, x_e, y_e, z_e)$ of **E**, it is often convenient to identify **E** with R^4 via the coordinatization map, and we shall do this without comment when it is not likely to cause confusion. Having fixed such a coordinatization, it is permissible to speak of the "position" (x_e, y_e, z_e) of an event or of the "time" t_e at which it occurred. The time t_e will be called *coordinate time.* to distinguish it from other time measurements, such as time intervals measured by observers undergoing accelerated motion. Of course, "coordinate time" is not an absolute notion but is only defined relative to a given coordinatization.

In Newtonian physics, one describes the history of a particle by a function $t \longmapsto \vec{r}(t)$ from R^1 to R^3, where $\vec{r}(t)$ represents the position of the particle at time t. The point $(t, \vec{r}(t)) \in R^4$ corresponds to an event, so in the context of relativity theory it is natural to think of the history of the particle as the set of events

$$\{\ (t, \vec{r}(t))\ \mid\ t \in R^1\ \}\ .$$

This set of events is called the *worldline* of the particle. It is customary to think of the worldline as a parametrized curve $s \longmapsto e(s) \in \mathbf{E}$ (so that if $e(s)$ has coordinates $(t(s), \vec{x}(s))$ relative to the coordinatization in which we are working, then $\vec{x}(s) = \vec{r}(t(s))$). Of course, we expect on physical grounds that coordinate time itself can always be used as a parameter, but there are many situations in which it is advantageous to use a different parameter, such as time as measured by a standard clock moving with the particle.

1.3. Minkowski space.

Define a symmetric bilinear form $<\cdot,\cdot>$ on R^4 as follows. If $u, v \in R^4$ have components $u = (u^0, u^1, u^2, u^3)$ and $v = (v^0, v^1, v^2, v^3)$, then

$$(1) \qquad <u, v> \ := \ u^0v^0 - u^1v^1 - u^2v^2 - u^3v^3\ .$$

(The use of upper indices for the components of vectors is traditional in relativity theory.) This bilinear form is variously called the *Lorentz metric,* or *Minkowski metric,* or *metric tensor,* and it plays a fundamental role in relativity theory. Note that it is non-degenerate, which means that the only vector v in R^4 which satisfies $<u, v> = 0$ for all vectors u in R^4 is $v = 0$. The vector space R^4 equipped with this metric is called *Minkowski space* and will be denoted as a bold-face **M**.

We often view R^4 as $R^1 \times R^3$ in the obvious way, and we denote vectors $\vec{r} \in R^3$ by the traditional notation of vector calculus. In particular, the usual inner product of two vectors $\vec{r}$, $\vec{s}$ in R^3 is denoted as $\vec{r}\cdot\vec{s}$, so that if $\vec{r} = (r^1, r^2, r^3)$ and $\vec{s} = (s^1, s^2, s^3)$, then

$$\vec{r}\cdot\vec{s} := r^1 s^1 + r^2 s^2 + r^3 s^3 .$$

A vector $v \in \mathbf{M}$ is called

timelike if $<v, v> > 0$,

null if $<v, v> = 0$, and

spacelike if $<v, v> < 0$.

The physical significance of these terms will be explained in Section 6.

If we identify spacetime **E** with **M** via a given Lorentz coordinatization, assumption (2) of the previous section states that the worldline of an idealized infinitesimal pulse of light is of the form

$$t \longmapsto (t, \vec{v}t + \vec{r}_0) = (1, \vec{v})t + (0, \vec{r}_0) , \qquad \text{with} \quad \vec{v}\cdot\vec{v} = 1 .$$

Since $<(1, \vec{v}), (1, \vec{v})> = 1 - \vec{v}\cdot\vec{v} = 0$, the worldlines of such pulses may be neatly characterized in terms of the Lorentz metric as the lines

$$t \longmapsto at + b \in \mathbf{M} \qquad \text{with } <a, a> = 0 .$$

Such a line is called a *null line.* †

The union of all null lines passing through a given point $b \in \mathbf{M}$ is called the *light cone* at b and may be characterized as

$$\{ p \in \mathbf{M} \mid <p-b, p-b> = 0 \} .$$

The light cone at the origin of **M** is sometimes simply called "the light cone". The *forward* light cone at a point $b \in \mathbf{M}$, (respectively, *backward* light cone at b) is the set of $p = (p^0, p^1, p^2, p^3)$ in the light cone at b such that $p^0 - b^0 > 0$ (resp. $p^0 - b^0 < 0$).

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† Sometimes, this remark is encapsulated by statements like "the worldlines of photons are null lines". A quantum mechanician (or should it be quantum mechanic?) might well object to the use of the term "photon" in this context. "Photons" are quantum-mechanical constructs which have no proper classical analog, and their quantum-mechanical description is very different from the classically conceivable idealization of an infinitesimal pulse of light. In particular, one can't "see", or otherwise detect, an individual photon at different times in its history in the same way that one can follow the motion of a material particle or a pulse of light. For this reason, we speak of light pulses rather than photons.

1.4. Lorentz transformations.

Having identified **E** with Minkowki space **M** via a given Lorentz coordinatization, we may seek new Lorentz coordinatizations by transforming the given coordinatization by a bijection $F: \mathbf{M} \longrightarrow \mathbf{M}$. That is, if we write $F(t, x, y, z) = (t', x', y', z')$, where t', x', y', and z' are functions of t, x, y, z, and if we assign the new coordinates (t', x', y', z') to the point whose old coordinates were (t, x, y, z), then we have produced a new coordinatization of **E**, and we may ask for which F the new coordinatization will be a Lorentz coordinatization. This question is partly mathematical and partly physical.

As noted in the last section, the worldline of a light pulse with respect to the original coordinatization is some null line $t \longmapsto at+b$, $<a, a> = 0$, in **M**, and all null lines occur in this way. With respect to the new coordinatization, the worldline is the image curve

$$t \longmapsto F(at+b) = (t'(at+b), x'(at+b), y'(at+b), z'(at+b)).$$

If the new coordinatization is to be a Lorentz coordinatization, then the new worldline must also be a null line. That is, if F defines a new Lorentz coordinatization, then it must map null lines to null lines. Moreover, it is clear that any two Lorentz coordinatizations are related in this way by some such F. Thus the mathematical part of the study of Lorentz coordinatizations may be regarded as the study of those bijections $F: \mathbf{M} \longrightarrow \mathbf{M}$ which preserve the set of all null lines. However, the definition of Lorentz coordinatization was not a purely mathematical one because it included the physical assumption that stationary "standard clocks" measure coordinate time. Even if we have an F which maps null lines to null lines, there is no guarantee that real physical clocks will measure coordinate time in the new coordinatization. We shall first examine the mathematical question and then the physical one.

Some obvious mappings which send null lines to null lines are:

(i) translations $v \longmapsto v+c$ $(v \in \mathbf{M})$ by a fixed $c \in \mathbf{M}$,

(ii) multiplications $v \longmapsto sv$ by a nonzero scalar s, and

(iii) linear transformations L which preserve the metric in the sense that

$$<Lu, Lv> = <u, v> \qquad \text{for all } u, v \in \mathbf{M}.$$

The translations and multiplications by a scalar are rather trivial, both mathematically and physically, and do not play an important role in the theory. The metric-preserving linear transformations are called *Lorentz transformations*, and we now turn to their study.

It is easy to write down examples of Lorentz transformations. For instance, if we view **M** as $R^4 = R^1 \oplus R^3$ in the obvious way, then the direct sum $R = I \oplus U$ of the identity I on R^1 with any orthogonal transformation U on R^3 will be a Lorentz transformation. We shall call such transformations *spatial rotations* with respect to the original Lorentz coordinatization which identified **E** with **M**. (Note that orthogonal transformations in R^3 include not only rotations about an axis, all of which have unit determinant 1, but also orientation-reversing orthogonal transformations such as reflections through a plane, which have determinant -1. Thus "spatial rotations" on **M** include spatial reflections.) More generally, if L is any Lorentz transformation, then $L^{-1}RL$ is of the above form with respect to the new coordinatization defined by L and is also called a spatial rotation with respect to that coordinatization.

For a more interesting example, let v be a real number with $-1 < v < 1$, define

(1) $$\gamma(v) := (1-v^2)^{-1/2},$$

and consider the linear transformation L defined by $L(t, x, y, z) = (t', x', y', z')$, where

(2)
$$t' := \gamma(v)(t - vx)$$
$$x' := \gamma(v)(x - vt)$$
$$y' := y$$
$$z' := z \quad .$$

It is routine to check that L is a Lorentz transformation, which is called a *boost* in the x-direction with velocity v .

The factor $\gamma(v)$ occurs so often that we shall permanently define $\gamma(v)$ as above, and sometimes we write simply γ when it is obvious what v is. Also, if $\vec{v}$ is a vector in R^3, we write $\gamma(\vec{v})$ for $(1-\vec{v}\cdot\vec{v})^{-1/2}$.

Inverting the above to solve for t, x, y, z yields

(3)
$$t = \gamma(v)(t'+vx')$$
$$x = \gamma(v)(x'+vt')$$
$$y = y'$$
$$z = z' ,$$

so L^{-1} is a boost in the same direction with opposite velocity.

To see the physical meaning of the boost L , consider a particle at rest with respect to the new coordinatization defined by L , say at rest at the point whose new spatial coordinates are $x' = k$, $y' = 0 = z'$. Then at time t in the old coordinatization, the old x-component of position is given by the second equation of (2) as

(4)
$$x = \gamma^{-1}k + vt \quad .$$

This shows that a particle at rest with respect to the new coordinatization is travelling at velocity v in the original coordinatization. Thus the new coordinatization defined by the boost L describes the coordinates of events as observed from a spatial coordinate system which is moving at velocity v in the x-direction with respect to the old one.

The well-known Lorentz spatial contraction follows easily from (4). To derive this result, consider two particles at rest in the new coordinate system situated at positions $x' = k_1$ and $x' = k_2$, respectively. The distance between these particles as measured in the new system is, of course, k_2-k_1 , but at any time t in the old system, the difference in the old positions is given by (4) as

$$(\gamma^{-1}k_2 + vt) - (\gamma^{-1}k_1 + vt) = (k_2-k_1)\gamma^{-1}$$

If we think of the two particles as marking the ends of a measuring rod of length k_2-k_1 units in the new system, then the length of the rod in the old system is

(5)
$$(k_2-k_1)\gamma^{-1} = (k_2-k_1)(1-v^2)^{1/2} \leq (k_2-k_1) \quad .$$

To derive the famous "time dilation" , consider a clock at rest at the origin of the new system. Supppose that at time t' in the new system the clock emits a signal. The event which is the emission of the signal has new time coordinate t' and new space coordinate

$x'=0$. The same event has time coordinate t in the old system given by

$$t = (t' + vx')\gamma = t'\gamma \ .$$

Hence if the same clock emits one signal at new time t' and another at new time $t'+1$, the time diferencc between these two events acording to the old coordinatization will be

$$(6) \qquad (t'+1)\gamma - t'\gamma = \gamma \geq 1 \ .$$

This implies that the difference in old time coordinates of two events with the same new space coordinates will differ by a factor of $\gamma(v) = (1-v^2)^{-1/2}$ from the difference of the new time coordinates of the same event. In other words, from the point of view of the network of stationary observers in the old system, the moving clock (which is stationary in the new system) runs slower by a factor of $\gamma(v)^{-1} = (1-v^2)^{1/2}$ relative to the stationary clocks which it is passing.

At this point this is no more than a mathematical tautology which says nothing of substance about the real world. We could obtain the same result by applying a Lorentz transformation to the coordinates of a Newtonian world. The physical content of (6) will be supplied by the postulate, or physical fact, that standard clocks at rest in the new frame do measure coordinate time in that frame. This is the second fundamental assumption of special relativity:

(7) We assume that *applying a Lorentz transformation to a Lorentz coordinatization yields a new Lorentz coordinatization.* This implies, in particular, that a standard clock moving with uniform velocity in a particular Lorentz coordinatization will also serve as a standard clock (i.e. will keep coordinate time) in a new coordinate system in which it is at rest obtained by applying a Lorentz transformation to the old system.

The first sentence of (7) also implies that the velocity of light in the new coordinate system, as measured by clocks stationary in that system, is the same as in the old system. In our treatment, this appears as a consequence of the fact that moving standard clocks keep coordinate time in the new system. Historically, the behavior of the clocks was deduced from the assumed constancy of the velocity of light rather than vice versa, since it is only fairly recently that the clock behavior could be directly verified, while the fact that the velocity of light is independent of uniform motion has been known for nearly a century.

The reader who is not familiar with special relativity may be tempted to consider as paradoxical the fact that by symmetry, a clock which is stationary with respect to the old coordinatization will be moving with respect to the new and will therefore run slow compared to the stationary clocks in the new system which it is passing. Thus it might seem that a stationary clock runs slower than a moving clock which in turn runs slower than a stationary clock, so that the stationary clock runs slower than itself! This is one of the more simple-minded versions of the so-called "clock paradox". The "solution" is the observation that statements like "clock A runs slower than clock B " have no absolute meaning in special relativity; such a statement acquires meaning only when it is specified how the clocks are to be compared, and no matter how one does this, the paradox disappears. Further discussions of such "paradoxes" can be found in nearly any text on special relativity, and the reader who is not familiar with the subject will probably have to puzzle through a few of them before he feels comfortable with it.

We add a few more words of caution for the beginner. It is tempting to try to interpret the time dilation (6) as implying that if two events occur with a time difference of t' time units in the new coordinatization, then the observed time difference in the old would be $t'\gamma$ units. This is not usually true, and our derivation of this conclusion relied on the very special hypothesis that the two events occurred *at the same space coordinate in the new system.* One must also be very careful about the use of everyday phrases such as "at the same time". It makes sense to say that two events occur at the same time in the new coordinate system (i.e. that the new time coordinates of the two events are equal) and it makes sense to say that the events occur at the same time in the old system, but the truth of one of these statements does not imply the truth of the other. In special relativity, the concept of simultaneity of events is only defined *relative to a given Lorentz coordinatization.*

The boost (2) relates two coordinate systems moving with uniform relative velocity v in the x-direction, and of course there is an analog of (2) for any spatial direction. Let $\vec{v}$ be a given vector in R^3 with $|\vec{v}| < 1$, and let

$$\vec{u} := \frac{\vec{v}}{(\vec{v}\cdot\vec{v})^{1/2}}$$

denote the unit vector in the direction of $\vec{v}$. The linear transformation B on $\mathbf{M} = R^1 \oplus R^3$ defined by $B(t, \vec{w}) = (t', \vec{w}')$, where

(8) $$t' := (t - \vec{w}\cdot\vec{v})\gamma(\vec{v}),$$

$$\vec{w}' := ((\vec{w}\cdot\vec{u})\vec{u} - \vec{v}t)\gamma(\vec{v}) + \vec{w} - (\vec{w}\cdot\vec{u})\vec{u}$$

is easily seen to be a Lorentz transformation such that a point which is stationary in the new coordinatization which it defines has velocity $\vec{v}$ in the original coordinatization. This transformation is called the *boost with velocity* $\vec{v}$ with respect to the original Lorentz coordinatization which identifies $\mathbf{E}$ with $\mathbf{M}$. Similarly, if L is any Lorentz transformation, then $L^{-1}BL$ is called a boost with respect to the new coordinatization defined by L.

The set of all Lorentz transformations is obviously a group under composition, so further Lorentz transformations can be obtained by composing boosts and spatial rotations. It is not hard to see that every Lorentz transformation L can be written as a composition $L = BRT$ of a boost B, a spatial rotation R, (both of which can be taken relative to the same original coordinatization) and a transformation T which is either the identity or multiplication by the scalar -1. (Exercise 21).

In linear algebra, linear transformations which preserve a non-degenerate real inner product are called *orthogonal,* though many texts restrict the inner product to be positive definite before making this definition. The determinant of any orthogonal transformation is always ± 1, and the same holds for indefinite inner products, though not all of the usual proofs for the positive definite case readily extend. (Exercise 2.13 outlines a proof of this fact.) In particular, Lorentz transformations have determinant ± 1. (A quick and dirty proof for this case can be obtained from the above decompositon $L = BRT$ and explicit computation that the determinant of a boost is always 1.)