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Gershon Kresin
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Sharp Real-Part Theorems

A Unified Approach

$$|f(z)| \leq \frac{2|z|}{R-|z|} \sup_{|\zeta|<R} \Re f(\zeta)$$

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Preface

We present a unified approach to various sharp pointwise inequalities for analytic functions in a disk with the real part of the function on the circumference as the right-hand side. We refer to these inequalities as "real-part theorems" in reference to the first assertion of such a kind, the celebrated Hadamard's real-part theorem (1892). The inequalities in question are frequently used in the theory of entire functions and in the analytic number theory.

We hope that collecting these inequalities in one place, as well as generalizing and refining them, may prove useful for various applications. In particular, one can anticipate rich opportunities to extend these inequalities to analytic functions of several complex variables and solutions of partial differential equations.

The text contains revisions and extensions of recent publications of the authors [56]-[58] and some new material as well. The research of G. Kresin was supported by the KAMEA program of the Ministry of Absorption, State of Israel, and by the College of Judea and Samaria, Ariel. The work of V. Maz'ya was supported by the Liverpool University and the Ohio State University. The authors record their thanks to these institutions.

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Ariel, Israel — *Gershon Kresin*
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May, 2006

Introduction

Estimates for analytic functions and their derivatives play an important role in complex analysis and its applications. Among these estimates which enjoy a great variety, there are the following two closely related classes having a wide range of applications.

The estimates of the first class contain only modulus of the analytic function in the majorant part of an inequality. In particular, they embrace Cauchy's inequalities, maximum modulus principle, Schwarz lemma, Hadamard three circles theorem (see, for example, Titchmarsh [87], Ch. 2, 5), Bohr's theorem [18], estimates for derivatives due to Landau, Lindelöf, F. Wiener (see Jensen [51]), Makintyre and Rogosinski [69], Rajagopal [79, 80], Szász [86]. In addition to that, the first class embraces estimates of Schwarz-Pick type for derivatives of arbitrary order obtained by Anderson and Rovnyak [11], Avkhadiev and Wirths [12], Bénéteau, Dahlner and Khavinson [13], MacCluer, Stroethoff and Zhao [66, 67], Ruscheweyh [83]. Among other known estimates of the same nature are generalizations to analytic operator-valued functions of a Schwarz-Pick type inequality for derivatives of arbitrary order by Anderson and Rovnyak [11] and Carathéodory's inequality for the first derivative by Yang [90, 91].

During last years the so called Bohr's inequality attracted a lot of attention. A refined form of Bohr's result [18], as stated by M. Riesz, I. Schur, F. Wiener (see Landau [62], K. I, § 4), claims that any function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{1}$$

analytic and bounded in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$, obeys the inequality

$$\sum_{n=0}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} |f(\zeta)|, \tag{2}$$

where $|z| \leq R/3$. Moreover, the value $R/3$ of the radius cannot be improved.

Multi-dimensional analogues and other generalizations of Bohr's theorem are treated in the papers by Aizenberg [1, 2, 9], Aizenberg, Aytuna and Djakov [3, 4], Aizenberg and Tarkhanov [5], Aizenberg, Liflyand and Vidras [7], Aizenberg and Vidras [8], Boas and Khavinson [15], Boas [16], Defant, Garcia and Maestre [30], Dineen and Timoney [32, 33], Djakov and Ramanujan [35], Kaptanoğlu [52].

Various interesting recent results related to Bohr's inequalities were obtained by Aizenberg, Grossman and Korobeinik [6], Bénéteau and Korenblum [14], Bénéteau, Dahlner and Khavinson [13], Bombieri and Bourgain [20], Guadarrama [43], Defant and Frerick [31].

Certain problems of functional analysis connected with Bohr's theorem are examined by Defant, Garcia and Maestre [29], Dixon [34], Glazman and Ljubič [40], Nikolski [72], Paulsen, Popescu and Singh [73], Paulsen and Singh [74].

In estimates of the second class the majorant involves the real part of the analytic function. Among these inequalities are the Hadamard-Borel-Carathéodory inequality for analytic functions in D_R with $\Re f$ bounded from above

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\}, \tag{3}$$

frequently called the Borel-Carathéodory inequality, and the Carathéodory-Plemelj inequality for analytic functions in D_R with bounded $\Re f$

$$|\Im f(z) - \Im f(0)| \leq \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re\{f - f(0)\}||_\infty \tag{4}$$

(see, for example, Burckel [23], Ch. 5, 6 and references there), where $|z| = r < R$. The same class includes Carathéodory's inequality for derivatives at the center of a disk [24], M. Riesz' theorem on conjugate harmonic functions [81] and many other estimates (see, for example, Jensen [51], Koebe [53], Lindelöf [64], Rajagopal [78], Ruscheweyh [82], Yamashita [89]). The sharp constant in M. Riesz inequality for analytic functions in the half-plane was obtained by Gohberg and Krupnik [41], Pichorides [75] and Cole (see Gamelin [37]). Note that sharp constants in parametric M. Riesz inequalities for analytic functions in the half-plane and in the disk were found in the paper of Hollenbeck, Kalton and Verbitsky [48], where a wide range of questions relating Fourier and Hilbert transforms was treated.

We note that different sources give different formulations of inequalities containing the real part as a majorant. In fact, Cartwright ([27], Ch. 1), Holland ([47], Ch. 3), Levin ([63], Lect. 2), Titchmarsh ([87], Ch. 5) formulate the Hadamard-Borel-Carathéodory inequality for functions which are analytic in $\overline{D}_R$. Unlike them, in the books by Burckel ([23], Ch. 6), Ingham ([50], Ch. 3), Littlewood ([65], Ch. 1) and Polya and Szegö ([76], III, Ch. 6) the same estimate is derived for functions which are analytic in D_R and have the real part bounded from above.

The Hadamard-Borel-Carathéodory inequality is used in an essential fashion in the theory of entire functions (see, e.g. the books Boas [17], Ch. 1 and Holland [47], Ch. 4). In particular, this inequality and its variants are applied for factorization of entire functions (see Hadamard [44]), in the proof of the Little Picard theorem (see Borel [21], Zalcman [92]) and in approximation of entire functions (see Elkins [36]).

The Hadamard-Borel-Carathéodory inequality is of use also in the analytic number theory (see Ingham [50], Ch. 3) and in mathematical physics (see Maharana [68]).

During the last years, generalizations of the Hadamard real-part theorem (the first form of the Hadamard-Borel-Carathéodory inequality) for holomorphic functions in domains on a complex manifold (see Aizenberg, Aytuna and Djakov [3]), the Carathéodory inequality for derivatives (see Aizenberg [9]) in several complex variables, and an extension of the Hadamard-Borel-Carathéodory inequality for analytic multifunctions (see Chen [28]) appeared.

The estimates in one of the classes mentioned above have their analogues in the other class. For example, this relates Bohr's theorem as well as its analogues containing the real part (see Aizenberg, Aytuna and Djakov [3], Paulsen, Popescu and Singh [73], Sidon [85], Tomić [88]).

Sharp pointwise estimates, being a classical object of analysis, occupy a special place in analytic function theory. In a way, they provide the best description of the pointwise behaviour of analytic functions from a given space.

The subject matter of this book is sharp pointwise estimates for analytic functions and their derivatives in a disk in terms of the real part of the function on the boundary circle. We consider various inequalities of this type from one point of view which reveals their intimate relations.

All inequalities with sharp constants to be obtained result from the analysis of Schwarz integral representation

$$f(z) = i\,\Im f(0) + \frac{1}{2\pi R}\int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z}\,\Re f(\zeta)|d\zeta|,$$

where $|z| < R$. The sharp estimates for the increment of an analytic function are written in a parametric form, where the role of the parameter is played by an arbitrary real valued function $\alpha(z)$ in D_R.

The book contains seven chapters.

In Chapter 1 we obtain sharp estimate for analytic functions in D_R with $\Re f$ bounded from above

$$\Re\{e^{i\alpha(z)}(f(z) - f(0))\} \leq \frac{2r(R - r\cos\alpha(z))}{R^2 - r^2} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\}, \tag{5}$$

where $r = |z| < R$, and α is a real valued function on D_R. This estimate implies various forms of the Hadamard-Borel-Carathéodory inequality and some other similar inequalities. The sharpness of inequality (5) is proved with the help of a parameter dependent family of test functions, each of them being analytic in $\overline{D}_R$.

Chapter 2 deals with a sharp estimate of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by the L_p-norm of $\Re f - \Re f(0)$ on the circle $|\zeta| = R$, where $|z| < R, 1 \le p \le \infty$, and α is a real valued function on D_R. In particular, we give explicit formulas for sharp constants in inequalities for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ with $p = 1, 2, \infty$. We find also the sharp constant in the upper estimate of $|\Im f(z) - \Im f(0)|$ by $||\Re f - \Re f(0)||_p$ for $1 \le p \le \infty$ which generalizes the classical Carathéodory-Plemelj estimate (4) with $p = \infty$. The evaluation of sharp constants is reduced to finding the minimum value of integrals depending on a real parameter entering the integrand.

In Chapter 3 we give sharp estimates of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by the L_p-norm of $\Re f - c$ on the circle $|\zeta| = R$, where $|z| < R, 1 \le p \le \infty$, and α is a real valued function on D_R. Here c is a real constant. More specifically, we obtain similar sharp estimates formulated in terms of the best approximation of $\Re f$ by a real constant on the circle $|\zeta| = R$. As corollaries, we give explicit formulas for sharp constants in inequalities for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ with $p = 1, 2, \infty$. In particular, an estimate containing $||\Re f - c||_1$ in the right-hand side implies

$$|\Re\{e^{i\alpha}(z)(f(z)-f(0))\}| \le \frac{2r(R + r|\cos\alpha(z)|)}{R^2 - r^2} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

which contains Hadamard-Borel-Carathéodory inequality (3) and similar estimates for the real and imaginary parts.

Other corollaries of the main results in Chapters 2 and 3 are estimates for $\big|\log|f(z)|\big|, |z| < R$, by the L_p-norm of $\log|f|$ on the circle $|\zeta| = R$, where f is an analytic zero-free function in D_R. The results of Chapters 1-3 also imply sharp inequalities for $|f'(z)|$ with various characteristics of the real part of f on the disk in the right-hand side.

Using previous results, in Chapter 4 we obtain sharp estimates for directional derivatives (in particular, for the modulus of the gradient) of a harmonic function in and outside the disk D_R, and in the half-plane. Here the majorants contain either characteristics of a harmonic function (interior estimates for derivatives), or characteristics of its directional derivative. In the last case we differ between estimates with a fixed and with a varying direction. In particular, using an estimate for $|f'(z)|$ inside of the disk D_R, obtained in Chapter 3, we derive a refined inequality (see, for comparison, Protter and Weinberger, [77], Ch. 2, Sect. 13) for the gradient of a harmonic function inside of the bounded domain.

In Chapter 5 we find estimates with the best constants of $|f^{(n)}(z)|$ for $n \ge 1$ by the L_p-norm of $\Re\{f - \mathcal{P}_m\}$ on the circle $|\zeta| = R$, where $\mathcal{P}_m$ is a polynomial of degree $m \le n-1$, $|z| < R, 1 \le p \le \infty$. For $z = 0$ explicit sharp constants are found for all $p \in [1, \infty]$. In particular, from the above mentioned sharp estimates for $|f^{(n)}(z)|$ with $p = 1$, we derive inequalities analogous to the Hadamard real-part theorem, as well as to the Carathéodory and Landau inequalities. Sharp inequality for $|f^{(n)}(z)|$ similar to Hadamard's real-part theorem is known (see, for example, Ingham [50], Ch. 3 and Rajagopal [78]). Unlike the approach used in these works, the method developed in Chapter 5 yields sharp estimates for the modulus of derivative formulated in terms of L_p-characteristics of the real part. The last section contains sharp parametric inequalities for $|f^{(n)}(z)|$.

In Chapter 6 we show that given a function (1) with $\Re f$ in the Hardy space $h_1(D_R)$ of harmonic functions on D_R, the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1$$

holds with the sharp constant, where $r = |z| < R$, $m \geq 1$, $q \in (0, \infty]$. This estimate implies sharp inequalities for the l_q-norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions, analytic functions with bounded $\Re f$, analytic functions with $\Re f$ bounded from above, as well as for analytic functions with $\Re f > 0$. Each of these estimates, specified for $q = 1$ and $m = 1$, improves a certain sharp Hadamard-Borel-Carathéodory type inequality. As corollaries, we obtain some sharp Bohr's type modulus and real part inequalities. Besides, we derive sharp Bohr's type estimates and theorems for non-concentric circles.

Chapter 7 is devoted to sharp estimates of $|f^{(n)}(z) - f^{(n)}(0)|$ for $n \geq 0$ by the L_p-norm of $\Re\{f - \mathcal{P}_m\}$ on the circle $|\zeta| = R$, where $\mathcal{P}_m$ is a polynomial of degree $m \leq n$, $|z| < R, 1 \leq p \leq \infty$. In particular, from the estimate for $|f^{(n)}(z) - f^{(n)}(0)|$ by the value $||\Re\{f - \mathcal{P}_m\}||_1$ in the right-hand side we obtain sharp estimates for the increment of derivatives of the type similar to Hadamard-Borel-Carathéodory, Carathéodory and Landau inequalities.

The sharpness of estimates for derivatives, similar to the Hadamard-Borel-Carathéodory, the Carathéodory and the Landau inequalities is proved in Chapters 5 and 7 using a family of test functions, analytic in $\overline{D}_R$. Besides, in these chapters, sharp pointwise estimates for the modulus of the derivatives and their increments are formulated in terms of the best approximation of the real part of f by the real part of polynomials $\mathcal{P}_m$ in the norm of $L_p(\partial D_R)$. In particular, for $p = 2$ the best constants are given in an explicit form.

The index and list of symbols are given at the end of the book.

The reader we have in mind should be familiar with the basics in complex function theory. The references are limited to works mentioned in the text.

Contents

1

Estimates for analytic functions bounded with respect to their real part

1.1 Introduction

Hadamard's real-part theorem is the following inequality

$$|f(z)| \leq \frac{Cr}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.1.1}$$

where $|z| = r < R$ and f is an analytic function on the closure $\overline{D}_R$ of the disk $D_R = \{z : |z| < R\}$ and vanishing at $z = 0$. This inequality was first obtained by Hadamard with $C = 4$ in 1892 [44].

A more general estimate for $|f(z)|$ with $f(0) \neq 0$ was obtained by Borel [21] and applied in his proof of Picard's theorem independent of modular functions. The inequality

$$|f(z)| \leq |\Im f(0)| + \frac{R+r}{R-r}|\Re f(0)| + \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta)$$

was found by Carathéodory (see Landau [60], pp. 275-277, [61], pp. 191-194). A detailed historic survey on these and other fundamental inequalities for analytic functions can be found in the paper by Jensen [51].

The following generalization of the real-part theorem with $C = 2$ resulting from (1.1.1) after replacing $f(z)$ by $f(z) - f(0)$,

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \max_{|\zeta|=R} \Re \{f(\zeta) - f(0)\}, \tag{1.1.2}$$

and its corollary

$$|f(z)| \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.1.3}$$

are often called the Borel-Carathéodory inequalities.

Sometimes, (1.1.2) and (1.1.3), as well as the related inequality for $\Re f$

$$\Re f(z) \le \frac{R-r}{R+r}\,\Re f(0) + \frac{2r}{R+r}\max_{|\zeta|=R}\Re f(\zeta), \tag{1.1.4}$$

are called Hadamard-Borel-Carathéodory inequalities (see, e.g., Burckel [23], Ch. 6 and references there).

In this chapter we obtain sharp estimates for

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\}$$

by the upper (or lower) bound of $\Re f$ on the disk D_R, where α is an arbitrary real valued function on D_R.

In Section 1.2 we give three known proofs of the real-part theorem: based on a conformal representation and the Schwarz lemma, on the Schwarz integral representation, and on a series expansion.

Section 1.3 is auxiliary. Using a lemma proved in Section 1.3, in Section 1.4 we derive the following sharp pointwise estimate

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \le \frac{2r(R-r\cos\alpha(z))}{R^2-r^2}\max_{|\zeta|=R}\Re\left\{f(\zeta)-f(0)\right\}, \tag{1.1.5}$$

where f is analytic in $\overline{D}_R$ and $|z|=r<R$.

The lower estimate for the constant in (1.1.5) is obtained with the help of a family of test functions which are analytic in $\overline{D}_R$. As a corollary of (1.1.5) we obtain the inequality with the same sharp constant for analytic functions f in D_R with $\Re f$ bounded from above

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \le \frac{2r(R-r\cos\alpha(z))}{R^2-r^2}\sup_{|\zeta|<R}\Re\{f(\zeta)-f(0)\}. \tag{1.1.6}$$

Sections 1.5-1.7 contain various corollaries of estimate (1.1.6). Among them, there are Hadamard-Borel-Carathéodory inequalities for the modulus as well as for the real and imaginary part of an analytic function, Harnack inequalities, and analogues of (1.1.6) for $\Re\{e^{i\alpha(z)}(f(z)-f(\xi))\}$ in the case of a disk and the half-plane.

1.2 Different proofs of the real-part theorem

Proofs of (1.1.1) with $C=2$ or (1.1.3) are given in Borel [22], Burckel ([23], Ch. 6), Cartwright ([27], Ch. 1), Holland [47], Ingham ([50], Ch. 3), Jensen [51], Levin ([63], L. 11), Lindelöf [64], Littlewood ([65], Ch. 1), Maz'ya and Shaposhnikova ([70], Ch. 9), Polya and Szegö ([76], III, Ch. 6), Rajagopal [78], Titchmarsh ([87], Ch. 5), Zalcman [92].

In this section we provide three different proofs of the real-part theorem with the constant $C = 2$. In all these proofs we assume that $f = u + iv$ is an analytic function in $\overline{D}_R$ with $f(0) = 0$. We introduce the notation

$$\mathcal{A}_f(R) = \sup_{|z|<R} \Re f(z) \tag{1.2.1}$$

to be used henceforth.

We recall that according to the Schwarz lemma, every analytic function f in D_R with $|f(z)| \le M$ and $f(0) = 0$ satisfies

$$|f(z)| \le MR^{-1}|z| \quad \text{for} \quad |z| < R$$

(see, for example, Littlewood [65], p. 112).

A combination of conformal mappings and the Schwarz lemma form the basis of the so called subordination principle, used, in particular, in the proof of the Hadamard-Borel-Carathéodory inequality and similar estimates (see Burckel [23], Ch. 6, § 5, Polya and Szegö [76], III, Ch. 6, § 2). The following proof is of the same nature.

Proof based on a conformal mapping and the Schwarz lemma (see Littlewood [65], pp. 113-114, Titchmarsh [87], p. 174-175). When proving the inequality

$$|f(z)| \le \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.2.2}$$

we may assume that $f \not\equiv 0$. Then, by the maximum principle for harmonic functions, $\mathcal{A}_f(R) > u(0) = 0$. The function

$$w = \psi(\zeta) = -2\mathcal{A}_f(R)\frac{\zeta}{1-\zeta}$$

performs the conformal mapping of the disk $|\zeta| < 1$ onto the half-plane $\Re w < \mathcal{A}_f(R)$ so that, $\psi(0) = 0$. Using the inverse mapping

$$\varphi(w) = \frac{w}{w - 2\mathcal{A}_f(R)},$$

consider the function

$$\omega(z) = \varphi(f(z)) = \frac{f(z)}{f(z) - 2\mathcal{A}_f(R)}, \qquad |z| < R. \tag{1.2.3}$$

According to the conformal representation theory, the function ω is analytic in D_R and $|\omega(z)| \le 1$. These properties of ω can be also justified by other arguments. The function ω is analytic in D_R, since the denominator in the right-hand side of (1.2.3) does not vanish. Furthermore, since

$$-2\mathcal{A}_f(R) + u(z) \le u(z) \le 2\mathcal{A}_f(R) - u(z),$$