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Blocks and Families for Cyclotomic Hecke Algebras

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Preface

This book contains a thorough study of symmetric algebras, covering topics such as block theory, representation theory and Clifford theory. It can also serve as an introduction to the Hecke algebras of complex reflection groups. Its aim is the study of the blocks and the determination of the families of characters of the cyclotomic Hecke algebras associated to complex reflection groups.

I would like to thank my thesis advisor, Michel Broué, for his advice. These Springer Lecture Notes were, after all, his idea. I am grateful to Jean Michel for his help with the implementation and presentation of the programming part. I would like to thank Gunter Malle for his suggestion that I generalize my results on Hecke algebras, which led to the notion of "essential algebras". I also express my thanks to Cédric Bonnafé, Meinolf Geck, Nicolas Jacon, Raphaël Rouquier and Jacques Thévenaz for their useful comments. Finally, I thank Thanos Tsouanas for copy-editing this manuscript.

Introduction

The finite groups of matrices with coefficients in $\mathbb{Q}$ generated by reflections, known as *Weyl groups*, are a fundamental building block in the classification of semisimple complex Lie algebras and Lie groups, as well as semisimple algebraic groups over arbitrary algebraically closed fields. They are also a foundation for many other significant mathematical theories, including braid groups and Hecke algebras.

The Weyl groups are particular cases of *complex reflection groups*, finite groups of matrices with coefficients in a finite abelian extension of $\mathbb{Q}$ generated by "pseudo-reflections" (elements whose vector space of fixed points is a hyperplane) — if the coefficients belong to $\mathbb{R}$, then these are the finite Coxeter groups.

The work of Lusztig on the irreducible characters of reductive groups over finite fields (cf. [45]) has demonstrated the important role of the "families of characters" of the Weyl groups concerned. However, only recently was it realized that it would be of great interest to generalize the notion of families of characters to the complex reflection groups, or more precisely to the cyclotomic Hecke algebras associated to complex reflection groups.

On the one hand, the complex reflection groups and their associated cyclotomic Hecke algebras appear naturally in the classification of the "cyclotomic Harish-Chandra series" of the characters of the finite reductive groups, generalizing the role of the Weyl group and its traditional Hecke algebra in the principal series (cf. [19, 20]). Since the families of characters of the Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group, we can hope that the families of characters of the cyclotomic Hecke algebras play a key role in the organization of families of unipotent characters more generally.

On the other hand, for some complex reflection groups (non-Coxeter) W, some data have been gathered which seem to indicate that behind the group W, there exists another mysterious object — the *Spets* (cf. [21, 52]) — that could play the role of the "series of finite reductive groups with Weyl group W". In some cases, one can define the unipotent characters of the Spets, which are controlled by the "spetsial" Hecke algebra of W, a generalization of the classical Hecke algebra of the Weyl groups.

The main obstacle for this generalization is the lack of Kazhdan-Lusztig bases for the non-Coxeter complex reflection groups. However, more recent results of Gyoja [41] and Rouquier [58] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. Gyoja has shown (case by case) that the partition into "p-blocks" of the Iwahori-Hecke algebra of a Weyl group W coincides with the partition into families, when p is the unique bad prime number for W. Later, Rouquier proved that the families of characters of a Weyl group W are exactly the blocks of characters of the Iwahori-Hecke algebra of W over a suitable coefficient ring, the "Rouquier ring".

Broué, Malle and Rouquier (cf. [22]) have shown that we can define the *generic Hecke algebra* $\mathcal{H}(W)$ associated to a complex reflection group W as a quotient of the group algebra of the braid group of W. The algebra $\mathcal{H}(W)$ is an algebra over a Laurent polynomial ring in a set of indeterminates $\mathbf{v} = (v_i)_{0 \leq i \leq m}$ whose cardinality m depends on the group W. A *cyclotomic Hecke algebra* is an algebra obtained from $\mathcal{H}(W)$ via a specialization of the form $v_i \mapsto y^{n_i}$, where y is an indeterminate and $n_i \in \mathbb{Z}$ for all $i = 0, 1, \ldots, m$. The blocks of a cyclotomic Hecke algebra over the Rouquier ring are the *Rouquier blocks* of the cyclotomic Hecke algebra. Thus, the Rouquier blocks generalize the notion of the families of characters to all complex reflection groups.

In [18], Broué and Kim presented an algorithm for the determination of the Rouquier blocks of the cyclotomic Hecke algebras of the groups $G(d, 1, r)$ and $G(e, e, r)$. Later, Kim (cf. [42]) generalized this algorithm to include all the groups of the infinite series $G(de, e, r)$. However, it was realized recently that their algorithm does not work in general, unless d is a power of a prime number. Moreover, the Rouquier blocks of the spetsial cyclotomic Hecke algebra of many exceptional irreducible complex reflection groups have been calculated by Malle and Rouquier in [53]. In this book, we correct and complete the determination of the Rouquier blocks for all cyclotomic Hecke algebras and all complex reflection groups.

The key in our study of the Rouquier blocks has been the proof of the fact that they have the property of "semi-continuity" (the name is due to C. Bonnafé). Every complex reflection group W determines some numerical data, which in turn determine the "essential" hyperplanes for W. To each essential hyperplane H, we can associate a partition $\mathcal{B}(H)$ of the set of irreducible characters of W into blocks. Given a cyclotomic specialization $v_i \mapsto y^{n_i}$, the Rouquier blocks of the corresponding cyclotomic Hecke algebra depend only on which essential hyperplanes the integers n_i belong to. In particular, they are unions of the blocks associated with the essential hyperplanes to which the integers n_i belong, and they are minimal with respect to that property.

The property of semi-continuity also appears in works on Kazhdan-Lusztig cells (cf. [9,10,40]) and on Cherednik algebras (cf. [38]). The common appearance of this as yet unexplained phenomenon implies a connection between these structures and the Rouquier blocks, for which the reason is not yet apparent, but promises to be fruitful when explored thoroughly. In particular, due to the known relation between Kazhdan-Lusztig cells and families of characters for Coxeter groups, this could be an indication of the existence of Kazhdan-Lusztig bases for the (non-Coxeter) complex reflection groups.

Another indication of this fact comes from the determination of the Rouquier blocks of the cyclotomic Hecke algebras of all complex reflection groups, obtained in the last chapter of this book with the use of the theory of "essential hyperplanes". In the case of the Weyl groups and their usual Hecke algebra, Lusztig attaches to every irreducible character two integers, denoted by a and A, and shows (cf. [46], 3.3 and 3.4) that they are constant on the families. In an analogous way, we can define integers a and A attached to every irreducible character of a cyclotomic Hecke algebra of a complex reflection group. Using the classification of the Rouquier blocks, it has been proved that the integers a and A are constant on the "families of characters" of the cyclotomic Hecke algebras of all complex reflection groups (see end of Chapter 4).

The first chapter of this book is dedicated to commutative algebra. The need for the results presented in this chapter (some of them are well-known, but others are completely new) arises form the fact that when we are working on Hecke algebras of complex reflection groups, we work over integrally closed rings, which are not necessarily unique factorization domains.

In the second chapter, we present some classical results of block theory and representation theory of symmetric algebras. We see that the *Schur elements* associated to the irreducible characters of a symmetric algebra play a crucial role in the determination of its blocks. Moreover, we generalize the results known as "Clifford theory" (cf., for example, [29]), which determine the blocks of certain subalgebras of symmetric algebras, to the case of "twisted symmetric algebras of finite groups". Finally, we give a new criterion for a symmetric algebra to be split semisimple.

In the third chapter, we introduce the notion of "essential algebras". These are symmetric algebras whose Schur elements have a specific form: they are products of irreducible polynomials evaluated on monomials. We obtain many results on the block theory of these algebras, which we later apply to the Hecke algebras, after we prove that they are essential in Chapter 4. In particular, we have our first encounter with the phenomenon of semi-continuity (see Theorem 3.3.2).

It is in the fourth chapter that we define for the first time the braid group, the generic Hecke algebra and the cyclotomic Hecke algebras associated to a complex reflection group. We show that the generic Hecke algebra of a complex reflection group is essential, by proving that its Schur elements are of the required form. Applying the results of Chapter 3, we obtain that the Rouquier blocks (*i.e.*, the families of characters) of the cyclotomic Hecke algebras have the property of semi-continuity and only depend on some "essential" hyperplanes for the group, which are determined by the generic Hecke algebra.

In the fifth and final chapter of this book, we present the algorithms and the results of the determination of the families of characters for all irreducible complex reflection groups. The use of Clifford theory is essential, since it allows us to restrict ourselves to the study of only certain cases of complex reflection groups. The computations were made with the use of the GAP package CHEVIE (cf. [37]) for the exceptional irreducible complex reflection groups, whereas combinatorial methods were applied to the groups of the infinite series. In particular, we show that the families of characters for the latter can be obtained from the families of characters of the Weyl groups of type B, already determined by Lusztig.

Contents

Chapter 1
On Commutative Algebra

The first chapter contains some known facts and some novel results on Commutative Algebra which are crucial for the proofs of the results of Chapters 3 and 4. The former are presented here without their proofs (with the exception of Theorem 1.4.1) for the convenience of the reader.

In the first section of this chapter, we define the localization of a ring and give some main properties. The second section is dedicated to integrally closed rings. We study particular cases of integrally closed rings, such as valuation rings, discrete valuation rings and Krull rings. We use their properties in order to obtain results on Laurent polynomial rings over integrally closed rings. We state briefly some results on completions of rings in Section 1.3. In the fourth section, we introduce the notion of *morphisms associated with monomials*. They are morphisms which allow us to pass from a Laurent polynomial ring A in $m+1$ indeterminates to a Laurent polynomial ring B in m indeterminates, while mapping a specific monomial to 1. Moreover, we prove (Proposition 1.4.9) that every surjective morphism from A to B which maps each indeterminate to a monomial is associated with a monomial. We call *adapted morphisms* the compositions of morphisms associated with monomials. They play a key role in the proof of the main results of Chapters 3 and 4. Finally, in the last section of the first chapter, we give a criterion (Theorem 1.5.6) for a polynomial to be irreducible in a Laurent polynomial ring with coefficients in a field.

Throughout this chapter, all rings are assumed to be commutative with 1. Moreover, if R is a ring and $x_0, x_1, \ldots, x_m$ is a set of indeterminates on R, then we denote by $R[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ the Laurent polynomial ring in $m+1$ indeterminates $R[x_0, x_0{}^{-1}, x_1, x_1{}^{-1}, \ldots, x_m, x_m{}^{-1}]$.

1.1 Localizations

Definition 1.1.1. Let R be a commutative ring with 1. We say that a subset S of R is a *multiplicatively closed set* if $0 \notin S$, $1 \in S$ and every finite product of elements of S belongs to S.

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In the set $R \times S$, we introduce an equivalence relation such that (r, s) is equivalent to (r', s') if and only if there exists $t \in S$ such that $t(s'r - sr') = 0$. We denote the equivalence class of (r, s) by r/s. The set of equivalence classes becomes a ring under the operations such that the sum and the product of r/s and r'/s' are given by $(s'r + sr')/ss'$ and rr'/ss' respectively. We denote this ring by $S^{-1}R$ and we call it the *localization* of R at S. If S contains no zero divisors of R, then any element r of R can be identified with the element $r/1$ of $S^{-1}R$ and we can regard the latter as an R-algebra.

Remarks.

- *If S is the set of all non-zero divisors of R, then $S^{-1}R$ is called* the total quotient ring *of R. If, moreover, R is an integral domain, the total quotient ring of R is* the field of fractions *of R.*
- *If R is Noetherian, then $S^{-1}R$ is Noetherian.*
- *If $\mathfrak{p}$ is a prime ideal of R, then the set $S := R - \mathfrak{p}$ is a multiplicatively closed subset of R and the ring $S^{-1}R$ is simply denoted by $R_{\mathfrak{p}}$.*

The proofs for the following well known results concerning localizations can be found in [11].

Proposition 1.1.2. *Let A and B be two rings with multiplicative sets S and T respectively and f an homomorphism from A to B such that $f(S)$ is contained in T. Then there exists a unique homomorphism f' from $S^{-1}A$ to $T^{-1}B$ such that $f'(a/1) = f(a)/1$ for every $a \in A$. Let us suppose now that T is contained in the multiplicatively closed set of B generated by $f(S)$. If f is surjective (respectively injective), then f' is also surjective (respectively injective).*

Corollary 1.1.3. *Let A and B be two rings with multiplicative sets S and T respectively such that $A \subseteq B$ and $S \subseteq T$. Then $S^{-1}A \subseteq T^{-1}B$.*

Proposition 1.1.4. *Let A be a ring and S, T two multiplicative sets of A such that $S \subseteq T$. We have $S^{-1}A = T^{-1}A$ if and only if every prime ideal of R that meets T also meets S.*

The following proposition and its corollary give us information about the ideals of the localization of a ring R at a multiplicatively closed subset S of R.

Proposition 1.1.5. *Let R be a ring and let S be a multiplicatively closed subset of R. Then*

(1) Every ideal $\mathfrak{b}'$ of $S^{-1}R$ is of the form $S^{-1}\mathfrak{b}$ for some ideal $\mathfrak{b}$ of R.
(2) Let $\mathfrak{b}$ be an ideal of R and let f be the canonical surjection $R \twoheadrightarrow R/\mathfrak{b}$. Then $f(S)$ is a multiplicatively closed subset of $R/\mathfrak{b}$ and the homomorphism from $S^{-1}R$ to $(f(S))^{-1}(R/\mathfrak{b})$ canonically associated with f is surjective with kernel $\mathfrak{b}' = S^{-1}\mathfrak{b}$. By passing to quotients, an isomorphism between $(S^{-1}R)/\mathfrak{b}'$ and $(f(S))^{-1}(R/\mathfrak{b})$ is defined.

(3) *The application* $\mathfrak{b}' \mapsto \mathfrak{b}$, *restricted to the set of maximal (respectively prime) ideals of* $S^{-1}R$, *is an isomorphism (for the relation of inclusion) between this set and the set of maximal (respectively prime) ideals of* R *that do not meet* S.
(4) *If* $\mathfrak{q}'$ *is a prime ideal of* $S^{-1}R$ *and* $\mathfrak{q}$ *is the prime ideal of* R *such that* $\mathfrak{q}' = S^{-1}\mathfrak{q}$ *(we have* $\mathfrak{q} \cap S = \emptyset$*), then there exists an isomorphism from* $R_\mathfrak{q}$ *to* $(S^{-1}R)_{\mathfrak{q}'}$ *which maps* r/s *to* $(r/1)/(s/1)$ *for* $r \in R$, $s \in R - \mathfrak{q}$.

Corollary 1.1.6. *Let* R *be a ring,* $\mathfrak{p}$ *a prime ideal of* R *and* $S := R - \mathfrak{p}$. *For every ideal* $\mathfrak{b}$ *of* R *which does not meet* S, *let* $\mathfrak{b}' := \mathfrak{b}R_\mathfrak{p}$. *Assume that* $\mathfrak{b}' \neq R_\mathfrak{p}$. *Then*

(1) *Let* f *be the canonical surjection* $R \twoheadrightarrow R/\mathfrak{b}$. *The ring homomorphism from* $R_\mathfrak{p}$ *to* $(R/\mathfrak{b})_{\mathfrak{p}/\mathfrak{b}}$ *canonically associated with* f *is surjective and its kernel is* $\mathfrak{b}'$. *Thus it defines, by passing to quotients, a canonical isomorphism between* $R_\mathfrak{p}/\mathfrak{b}'$ *and* $(R/\mathfrak{b})_{\mathfrak{p}/\mathfrak{b}}$.
(2) *The application* $\mathfrak{b}' \mapsto \mathfrak{b}$, *restricted to the set of prime ideals of* $R_\mathfrak{p}$, *is an isomorphism (for the relation of inclusion) between this set and the set of prime ideals of* R *contained in* $\mathfrak{p}$. *Therefore,* $\mathfrak{p}R_\mathfrak{p}$ *is the only maximal ideal of* $R_\mathfrak{p}$.
(3) *If now* $\mathfrak{b}'$ *is a prime ideal of* $R_\mathfrak{p}$, *then there exists an isomorphism from* $R_\mathfrak{b}$ *to* $(R_\mathfrak{p})_{\mathfrak{b}'}$ *which maps* r/s *to* $(r/1)/(s/1)$ *for* $r \in R$, $s \in R - \mathfrak{b}$.

The notion of localization can also be extended to the modules over the ring R.

Definition 1.1.7. Let R be a ring and S a multiplicatively closed set of R. If M is an R-module, then we call *localization of* M *at* S and denote by $S^{-1}M$ the $S^{-1}R$-module $M \otimes_R S^{-1}R$.

1.2 Integrally Closed Rings

Theorem-Definition 1.2.1 *Let* R *be a ring,* A *an* R*-algebra and* a *an element of* A. *The following properties are equivalent:*

(i) *The element* a *is a root of a monic polynomial with coefficients in* R.
(ii) *The subalgebra* $R[a]$ *of* A *is an* R*-module of finite type.*
(iii) *There exists a faithful* $R[a]$*-module which is an* R*-module of finite type.*

If $a \in A$ *verifies the conditions above, we say that it is* integral *over* R.

Definition 1.2.2. Let R be a ring and A an R-algebra. The set of all elements of A that are integral over R is an R-subalgebra of A containing R; it is called the *integral closure of* R *in* A. We say that R is *integrally closed in* A, if R is an integral domain and coincides with its integral closure in A. If now R is an integral domain and F is its field of fractions, then the integral closure of R in F is named simply the *integral closure* of R, and if R is integrally closed in F, then R is said to be *integrally closed.*

The following proposition ([12], §1, Proposition 13) implies that the transfer theorem holds for integrally closed rings (Corollary 1.2.4).

Proposition 1.2.3. *If R is an integral domain, let us denote by $\bar{R}$ the integral closure of R. Let $x_0, \ldots, x_m$ be a set of indeterminates over R. Then the integral closure of $R[x_0, \ldots, x_m]$ is $\bar{R}[x_0, \ldots, x_m]$.*

Corollary 1.2.4. *Let R be an integral domain. Then $R[x_0, \ldots, x_m]$ is integrally closed if and only if R is integrally closed.*

Corollary 1.2.5. *If K is a field, then every polynomial ring $K[x_0, \ldots, x_m]$ is integrally closed.*

The next proposition ([12], §1, Proposition 16) along with its corollaries treats the integral closures of localizations of rings.

Proposition 1.2.6. *Let R be a ring, A an R-algebra, $\bar{R}$ the integral closure of R in A and S a multiplicatively closed subset of R which contains no zero divisors. Then the integral closure of $S^{-1}R$ in $S^{-1}A$ is $S^{-1}\bar{R}$.*

Corollary 1.2.7. *Let R be an integral domain, $\bar{R}$ the integral closure of R and S a multiplicatively closed subset of R. Then the integral closure of $S^{-1}R$ is $S^{-1}\bar{R}$.*

Corollary 1.2.8. *If R is an integrally closed domain and S is a multiplicatively closed subset of R, then $S^{-1}R$ is also integrally closed.*

Example 1.2.9. Let K be a finite field extension of $\mathbb{Q}$ and $\mathbb{Z}_K$ the integral closure of $\mathbb{Z}$ in K. Obviously, the ring $\mathbb{Z}_K$ is integrally closed. Let $x_0, x_1, \ldots, x_m$ be indeterminates. Then the ring $\mathbb{Z}_K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ is also integrally closed.

1.2.1 Lifting Prime Ideals

Definition 1.2.10. Let R, R' be two rings and let $h : R \to R'$ be a ring homomorphism. We say that a prime ideal $\mathfrak{a}'$ of R' *lies over* a prime ideal $\mathfrak{a}$ of R, if $\mathfrak{a} = h^{-1}(\mathfrak{a}')$.

The next result is [12], §2, Proposition 2.

Proposition 1.2.11. *Let $h : R \to R'$ be a ring homomorphism such that R' is integral over R. Let $\mathfrak{p}$ be a prime ideal of R, $S := R - \mathfrak{p}$ and $(\mathfrak{p}'_i)_{i \in I}$ the family of all the prime ideals of R' lying over $\mathfrak{p}$. If $S' = \bigcap_{i \in I}(R' - \mathfrak{p}'_i)$, then $S^{-1}R' = S'^{-1}R'$.*

The following corollary of Proposition 1.2.11 deals with a case we will encounter in a following chapter, where there exists a unique prime ideal lying over the prime ideal $\mathfrak{p}$ of R. In combination with Proposition 1.2.6, Proposition 1.2.11 implies the following:

Corollary 1.2.12. *Let R be an integral domain, A an R-algebra, $\bar{R}$ the integral closure of R in A. Let $\mathfrak{p}$ be a prime ideal of R and $S := R - \mathfrak{p}$. If there exists a unique prime ideal $\bar{\mathfrak{p}}$ of $\bar{R}$ lying over $\mathfrak{p}$, then the integral closure of $R_{\mathfrak{p}}$ in $S^{-1}A$ is $\bar{R}_{\bar{\mathfrak{p}}}$.*

1.2.2 Valuations

Definition 1.2.13. Let R be a ring and Γ a totally ordered abelian group. We call a *valuation* of R with values in Γ any application $v : R \to \Gamma \cup \{\infty\}$ which satisfies the following properties:

(V1) $v(xy) = v(x) + v(y)$ for $x \in R, y \in R$.
(V2) $v(x + y) \geq \inf\{v(x), v(y)\}$ for $x \in R, y \in R$.
(V3) $v(1) = 0$ and $v(0) = \infty$.

In particular, if $v(x) \neq v(y)$, property (V2) gives $v(x + y) = \inf\{v(x), v(y)\}$ for $x \in R, y \in R$. Moreover, from property (V1), we have that if $z \in R$ with $z^n = 1$ for some integer $n \geq 1$, then $nv(z) = v(z^n) = v(1) = 0$ and thus $v(z) = 0$. Consequently, $v(-x) = v(-1) + v(x) = v(x)$ for all $x \in R$.

Now let F be a field and let $v : F \to \Gamma$ be a valuation of F. The set A of $a \in F$ such that $v(a) \geq 0$ is a local subring of F. Its maximal ideal $\mathfrak{m}(A)$ is the set of $a \in A$ such that $v(a) > 0$. For all $a \in F - A$, we have $a^{-1} \in \mathfrak{m}(A)$. The ring A is called *the ring of the valuation* v on F.

We will now introduce the notion of a valuation ring. For more information about valuation rings and their properties, see [13]. Some of them will also be discussed in Chapter 2, Section 2.4.

Definition 1.2.14. Let R be an integral domain contained in a field F. Then R is a *valuation ring* if for all non-zero element $x \in F$, we have $x \in R$ or $x^{-1} \in R$. Consequently, F is the field of fractions of R.

If R is a valuation ring, then it has the following properties:

- It is an integrally closed local ring.
- The set of the principal ideals of R is totally ordered by inclusion.
- The set of the ideals of R is totally ordered by inclusion.

Let R be a valuation ring and F its field of fractions. Let us denote by $R^{\times}$ the set of units of R. Then the set $\Gamma_R := F^{\times}/R^{\times}$ is an abelian group,