Lecture Notes in Mathematics

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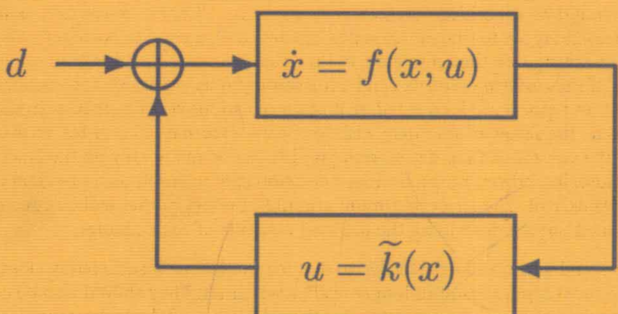

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Roberto Conti

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Nonlinear and Optimal Control Theory

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Preface

Mathematical Control Theory is a branch of Mathematics having as one of its main aims the establishment of a sound mathematical foundation for the control techniques employed in several different fields of applications, including engineering, economy, biology and so forth. The systems arising from these applied Sciences are modeled using different types of mathematical formalism, primarily involving Ordinary Differential Equations, or Partial Differential Equations or Functional Differential Equations. These equations depend on one or more parameters that can be varied, and thus constitute the control aspect of the problem. The parameters are to be chosen so as to obtain a desired behavior for the system. From the many different problems arising in Control Theory, the C.I.M.E. school focused on some aspects of the control and optimization of nonlinear, not necessarily smooth, dynamical systems. Two points of view were presented: Geometric Control Theory and Nonlinear Control Theory. The C.I.M.E. session was arranged in five six-hours courses delivered by Professors A.A. Agrachev (SISSA-ISAS, Trieste and Steklov Mathematical Institute, Moscow), A.S. Morse (Yale University, USA), E.D. Sontag (Rutgers University, NJ, USA), H.J. Sussmann (Rutgers University, NJ, USA) and V.I. Utkin (Ohio State University Columbus, OH, USA).

We now briefly describe the presentations.

Agrachev's contribution began with the investigation of second order information in smooth optimal control problems as a means of explaining the variational and dynamical nature of powerful concepts and results such as Jacobi fields, Morse's index formula, Levi-Civita connection, Riemannian curvature. These are primarily known only within the framework of Riemannian Geometry. The theory presented is part of a beautiful project aimed at investigating the connections between Differential Geometry, Dynamical Systems and Optimal Control Theory.

The main objective of Morse's lectures was to give an overview of a variety of methods for synthesizing and analyzing logic-based switching control systems. The term "logic-based switching controller" is used to denote a controller whose subsystems include not only familiar dynamical components (integrators, summers, gains, etc.) but logic-driven elements as well. An important category of such control systems are those consisting of a process to be controlled, a family of fixed-gain or variable-gain candidate controllers, and an "event-drive switching logic" called a supervisor whose job is to determine in real time which controller should be applied to the process. Examples of supervisory control systems include re-configurable systems, and certain types of parameter-adaptive systems.

Sontag's contribution was devoted to the input to state stability (ISS) paradigm which provides a way of formulating questions of stability with respect to disturbances, as well as a method to conceptually unify detectability, input/output stability, minimum-phase behavior, and other systems properties. The lectures discussed the main theoretical results concerning ISS and related notions. The proofs of the results showed in particular connections to relaxations for differential inclusions, converse Lyapunov theorems, and nonsmooth analysis.

Sussmann's presentation involved the technical background material for a version of the Pontryagin Maximum Principle with state space constraints and very weak technical hypotheses. It was based primarily on an approach that used generalized differentials and packets of needle variations. In particular, a detailed account of two theories of generalized differentials, the "generalized differential quotients" (GDQs) and the "approximate generalized differential quotients" (AGDQs), was presented. Then the resulting version of the Maximum Principle was stated.

Finally, Utkin's contribution concerned the Sliding Mode Control concept that for many years has been recognized as one of the key approaches for the systematic design of robust controllers for complex nonlinear dynamic systems operating under uncertainty conditions. The design of feedback control in systems with sliding modes implies design of manifolds in the state space where control components undergo discontinuities, and control functions enforcing motions along the manifolds. The design methodology was illustrated by sliding mode control to achieve different objectives: eigenvalue placement, optimization, disturbance rejection, identification.

The C.I.M.E. course was attended by fifty five participants from several countries. Both graduate students and senior mathematicians intensively followed the lectures, seminars and discussions in a friendly and co-operative atmosphere.

As Editors of these Lectures Notes we would like to thank the persons and institutions that contributed to the success of the course. It is our pleasure to thank the Scientific Committee of C.I.M.E. for supporting our project: the Director, Prof. Pietro Zecca and the Secretary, Prof. Elvira Mascolo for their support during the organization. We would like also to thank Carla Dionisi for her valuable and efficient work in preparing the final manuscript for this volume.

Our special thanks go to the lecturers for their early preparation of the material to be distributed to the participants, for their excellent performance in teaching the courses and their stimulating scientific contributions.

We dedicate this volume to our teacher Prof. Roberto Conti, one of the pioneers of Mathematical Control Theory, who contributed in a decisive way to the development and to the international success of Fondazione C.I.M.E.

Siena and Firenze, May 2006

Paolo Nistri
Gianna Stefani

Contents

Geometry of Optimal Control Problems and Hamiltonian Systems

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Preface

These notes are based on the mini-course given in June 2004 in Cetraro, Italy, in the frame of a C.I.M.E. school. Of course, they contain much more material that I could present in the 6 h course. The idea was to explain a general variational and dynamical nature of nice and powerful concepts and results mainly known in the narrow framework of Riemannian Geometry. This concerns Jacobi fields, Morse's index formula, Levi-Civita connection, Riemannian curvature and related topics.

I tried to make the presentation as light as possible: gave more details in smooth regular situations and referred to the literature in more complicated cases. There is an evidence that the results described in the notes and treated in technical papers we refer to are just parts of a united beautiful subject to be discovered on the crossroads of Differential Geometry, Dynamical Systems, and Optimal Control Theory. I will be happy if the course and the notes encourage some young ambitious researchers to take part in the discovery and exploration of this subject.

Acknowledgments. I would like to express my gratitude to Professor Gamkrelidze for his permanent interest to this topic and many inspiring discussions and to thank participants of the school for their surprising and encouraging will to work in the relaxing atmosphere of the Mediterranean resort.

1 Lagrange Multipliers' Geometry

1.1 Smooth Optimal Control Problems

In these lectures we discuss some geometric constructions and results emerged from the investigation of smooth optimal control problems. We will consider problems with integral costs and fixed endpoints. A standard formulation of such a problem is as follows: Minimize a functional

$$J_{t_0}^{t_1}(u(\cdot)) = \int_{t_0}^{t_1} \varphi(q(t), u(t))\, dt, \tag{1}$$

where

$$\dot{q}(t) = f(q(t), u(t)), \quad u(t) \in U, \quad \forall t \in [t_0, t_1], \tag{2}$$

$q(t_0) = q_0$, $q(t_1) = q_1$. Here $q(t) \in \mathbb{R}^n$, $U \subset \mathbb{R}^k$, a *control function* $u(\cdot)$ is supposed to be measurable bounded while $q(\cdot)$ is Lipschitzian; scalar function φ and vector function f are smooth. A pair $(u(\cdot), q(\cdot))$ is called an *admissible pair* if it satisfies differential (2) but may violate the boundary conditions.

We usually assume that Optimal Control Theory generalizes classical Calculus of Variations. Unfortunately, even the most classical geometric variational problem, the length minimization on a Riemannian manifold, cannot be presented in the just described way. First of all, even simplest manifolds, like spheres, are not domains in $\mathbb{R}^n$. This does not look as a serious difficulty: we slightly generalize original formulation of the optimal control problem assuming that $q(t)$ belongs to a smooth manifold M instead of $\mathbb{R}^n$. Then $\dot{q}(t)$ is a tangent vector to M, i.e., $\dot{q}(t) \in T_{q(t)}M$ and we assume that $f(q,u) \in T_qM$, $\forall q, u$. Manifold M is called the *state space* of the optimal control problem.

Now we will try to give a natural formulation of the length minimization problem as an optimal control problem on a Riemannian manifold M. Riemannian structure on M is (by definition) a family of Euclidean scalar products $\langle \cdot, \cdot \rangle_q$ on T_qM, $q \in M$, smoothly depending on q. Let $f_1(q), \ldots, f_n(q)$ be an orthonormal basis of T_qM for the Euclidean structure $\langle \cdot, \cdot \rangle_q$ selected in such a way that $f_i(q)$ are smooth with respect to q. Then any Lipschitzian curve on M satisfies a differential equation of the form:

$$\dot{q} = \sum_{i=1}^{n} u_i(t) f_i(q), \tag{3}$$

where $u_i(\cdot)$ are measurable bounded scalar functions. In other words, any Lipschitzian curve on M is an admissible trajectory of the control system (3). The Riemannian length of the tangent vector $\sum\limits_{i=1}^{n} u_i f_i(q)$ is $\left(\sum\limits_{i=1}^{n} u_i^2\right)^{1/2}$. Hence the length of a trajectory of system (3) defined on the segment $[t_0, t_1]$ is $\ell(u(\cdot)) = \int_{t_0}^{t_1} \left(\sum\limits_{i=1}^{n} u_i^2(t)\right)^{1/2} dt$. Moreover, it is easy to derive from the Cauchy–Schwarz inequality that the length minimization is equivalent to the minimization of the functional $J_{t_0}^{t_1}(u(\cdot)) = \int_{t_0}^{t_1} \sum\limits_{i=1}^{n} u_i^2(t)\, dt$. The length minimization problem is thus reduced to a specific optimal control problem on the manifold of the form (1), (2).

Unfortunately, what I have just written was wrong. It would be correct if we could select a smooth orthonormal frame $f_i(q),\ q \in M,\ i = 1, \ldots, n$. Of course, we can always do it locally, in a coordinate neighborhood of M but, in general, we cannot do it globally. We cannot do it even on the two-dimensional sphere: you know very well that any continuous vector field on the two-dimensional sphere vanishes somewhere. We thus need another more flexible formulation of a smooth optimal control problem.

Recall that a *smooth locally trivial bundle* over M is a submersion $\pi : V \to M$, where all *fibers* $V_q = \pi^{-1}(q)$ are diffeomorphic to each other and, moreover, any $q \in M$ possesses a neighborhood O_q and a diffeomorphism $\Phi_q : O_q \times V_q \to \pi^{-1}(O_q)$ such that $\Phi_q(q', V_q) = V_{q'},\ \forall q' \in O_q$. In a less formal language one can say that a smooth locally trivial bundle is a smooth family of diffeomorphic manifolds V_q (the fibers) parameterized by the points of the manifold M (the base). Typical example is the tangent bundle $TM = \bigcup\limits_{q \in M} T_qM$ with the canonical projection π sending T_qM into q.

Definition. A smooth control system with the state space M is a smooth mapping $f : V \to TM$, where V is a locally trivial bundle over M and $f(V_q) \subset T_qM$ for any fiber $V_q,\ q \in M$. An admissible pair is a bounded[1] measurable mapping $v(\cdot) : [t_0, t_1] \to V$ such that $t \mapsto \pi(v(t)) = q(t)$ is a Lipschitzian curve in M and $\dot{q}(t) = f(v(t))$ for almost all $t \in [t_0, t_1]$. Integral cost is a functional $J_{t_0}^{t_1}(v(\cdot)) = \int\limits_{t_0}^{t_1} \varphi(v(t))\, dt$, where φ is a smooth scalar function on V.

Remark. The above more narrow definition of an optimal control problem on M was related to the case of a *trivial bundle* $V = M \times U,\ V_q = \{q\} \times U$. For the length minimization problem we have $V = TM,\ f = \mathrm{Id},\ \varphi(v) = \langle v, v \rangle_q,\ \forall v \in T_qM,\ q \in M$.

Of course, any general smooth control system on the manifold M is locally equivalent to a standard control system on $\mathbb{R}^n$. Indeed, any point $q \in M$ possesses a coordinate neighborhood O_q diffeomorphic to $\mathbb{R}^n$ and a mapping $\Phi_q : O_q \times V_q \to \pi^{-1}(O_q)$ trivializing the restriction of the bundle V to O_q; moreover, the fiber V_q can be embedded in $\mathbb{R}^k$ and thus serve as a set of control parameters U.

Yes, working locally we do not obtain new systems with respect to those in $\mathbb{R}^n$. Nevertheless, general intrinsic definition is very useful and instructive even for a purely local geometric analysis. Indeed, we do not need to fix specific coordinates on M and a trivialization of V when we study a control system defined in the intrinsic way. A change of coordinates in M is actually a smooth transformation of the state space while a change of the trivialization results in the feedback transformation of the control system. This means that an intrinsically defined control system represents actually the whole class of

[1] The term "bounded" means that the closure of the image of the mapping is compact.

systems that are equivalent with respect to smooth state and feedback transformations. All information on the system obtained in the intrinsic language is automatically invariant with respect to smooth state and feedback transformations. And this is what any geometric analysis intends to do: to study properties of the object under consideration preserved by the natural transformation group.

We denote by $L_\infty([t_0, t_1]; V)$ the space of measurable bounded mappings from $[t_0, t_1]$ to V equipped with the L_∞-topology of the uniform convergence on a full measure subset of $[t_0, t_1]$. If V were an Euclidean space, then $L_\infty([t_0, t_1]; V)$ would have a structure of a Banach space. Since V is only a smooth manifold, then $L_\infty([t_0, t_1]; V)$ possesses a natural structure of a smooth Banach manifold modeled on the Banach space $L_\infty([t_0, t_1]; \mathbb{R}^{\dim V})$.

Assume that $V \to M$ is a locally trivial bundle with the n-dimensional base and m-dimensional fibers; then V is an $(n+m)$-dimensional manifold.

Proposition 1.1. *Let $f : V \to TM$ be a smooth control system; then the space $\mathcal{V}$ of admissible pairs of this system is a smooth Banach submanifold of $L_\infty([t_0, t_1]; V)$ modeled on $\mathbb{R}^n \times L_\infty([t_0, t_1]; \mathbb{R}^m)$.*

Proof. Let $v(\cdot)$ be an admissible pair and $q(t) = \pi(v(t))$, $t \in [t_0, t_1]$. There exists a Lipschitzian with respect to t family of local trivializations $R_t : O_{q(t)} \times U \to \pi^{-1}(O_{q(t)})$, where U is diffeomorphic to the fibers V_q. The construction of such a family is a boring exercise which we omit.

Consider the system

$$\dot{q} = f \circ R_t(q, u), \quad u \in U. \tag{4}$$

Let $v(t) = R_t(q(t), u(t))$; then R_t, $t_0 \le t \le t_1$, induces a diffeomorphism of an L_∞-neighborhood of $(q(\cdot), u(\cdot))$ in the space of admissible pairs for (4) on a neighborhood of $v(\cdot)$ in $\mathcal{V}$. Now fix $\bar{t} \in [t_0, t_1]$. For any $\hat{q}$ close enough to $q(\bar{t})$ and any $u'(\cdot)$ sufficiently close to $u(\cdot)$ in the L_∞-topology there exists a unique Lipschitzian path $q'(\cdot)$ such that $\dot{q}'(t) = f \circ R_t(q'(t), u'(t)))$, $t_0 \le t \le t_1$, $q'(\bar{t}) = \hat{q}$; moreover the mapping $(\hat{q}, u'(\cdot)) \mapsto q'(\cdot)$ is smooth. In other words, the Cartesian product of a neighborhood of $q(\bar{t})$ in M and a neighborhood of $u(\cdot)$ in $L_\infty([t_0, t_1], U)$ serves as a coordinate chart for a neighborhood of $v(\cdot)$ in $\mathcal{V}$. This finishes the proof since M is an n-dimensional manifold and $L_\infty([t_0, t_1], U)$ is a Banach manifold modeled on $L_\infty([t_0, t_1], \mathbb{R}^m)$. □

An important role in our study will be played by the "evaluation mappings" $F_t : v(\cdot) \mapsto q(t) = \pi(v(t))$. It is easy to show that F_t is a smooth mapping from $\mathcal{V}$ to M. Moreover, it follows from the proof of Proposition 1.1 that F_t is a submersion. Indeed, $q(t) = F_t(v(\cdot))$ is, in fact a part of the coordinates of $v(\cdot)$ built in the proof (the remaining part of the coordinates is the control $u(\cdot)$).

1.2 Lagrange Multipliers

Smooth optimal control problem is a special case of the general smooth conditional minimum problem on a Banach manifold $\mathcal{W}$. The general problem consists of the minimization of a smooth functional $J : \mathcal{W} \to \mathbb{R}$ on the level sets $\Phi^{-1}(z)$ of a smooth mapping $\Phi : \mathcal{W} \to N$, where N is a finite-dimensional manifold. In the optimal control problem we have $\mathcal{W} = \mathcal{V}$, $N = M \times M$, $\Phi = (F_{t_0}, F_{t_1})$.

An efficient classical way to study the conditional minimum problem is the Lagrange multipliers rule. Let us give a coordinate free description of this rule. Consider the mapping

$$\bar{\Phi} = (J, \Phi) : \mathcal{W} \to \mathbb{R} \times N, \quad \bar{\Phi}(w) = (J(w), \Phi(w)), \ w \in \mathcal{W}.$$

It is easy to see that any point of the local conditional minimum or maximum (i.e., local minimum or maximum of J on a level set of Φ) is a critical point of $\bar{\Phi}$. I recall that w is a critical point of $\bar{\Phi}$ if the differential $D_w\bar{\Phi} : T_w\mathcal{W} \to T_{\bar{\Phi}(w)}(\mathbb{R} \times N)$ is *not* a surjective mapping. Indeed, if $D_w\bar{\Phi}$ were surjective then, according to the implicit function theorem, the image $\bar{\Phi}(O_w)$ of an arbitrary neighborhood O_w of w would contain a neighborhood of $\bar{\Phi}(w) = (J(w), \Phi(w))$; in particular, this image would contain an interval $((J(w) - \varepsilon, J(w) + \varepsilon), \Phi(w))$ that contradicts the local conditional minimality or maximality of $J(w)$.

The linear mapping $D_w\bar{\Phi}$ is not surjective if and only if there exists a nonzero linear form $\bar{\ell}$ on $T_{\bar{\Phi}(w)}(\mathbb{R} \times N)$ which annihilates the image of $D_w\bar{\Phi}$. In other words, $\bar{\ell}D_w\bar{\Phi} = 0$, where $\bar{\ell}D_w\bar{\Phi} : T_w\mathcal{W} \to \mathbb{R}$ is the composition of $D_w\bar{\Phi}$ and the linear form $\bar{\ell} : T_{\bar{\Phi}(w)}(\mathbb{R} \times N) \to \mathbb{R}$.

We have $T_{\bar{\Phi}(w)}(\mathbb{R} \times N) = \mathbb{R} \times T_{\Phi(w)}N$. Linear forms on $(\mathbb{R} \times N)$ constitute the adjoint space $(\mathbb{R} \times N)^* = \mathbb{R} \oplus T^*_{\Phi(w)}N$, where $T^*_{\Phi(w)}N$ is the adjoint space of $T_{\Phi(w)}M$ (the *cotangent space* to M at the point $\Phi(w)$). Hence $\ell = \nu \oplus \ell$, where $\nu \in \mathbb{R}$, $\ell \in T^*_{\Phi(w)}N$ and

$$\bar{\ell}D_w\bar{\Phi} = (\nu \oplus \ell)(d_wJ, D_w\Phi) = \nu d_wJ + \ell D_w\Phi.$$

We obtain the equation

$$\nu d_wJ + \ell D_w\Phi = 0. \tag{5}$$

This is the Lagrange multipliers rule: if w is a local conditional extremum, then there exists a nontrivial pair (ν, ℓ) such that (5) is satisfied. The pair (ν, ℓ) is never unique: indeed, if α is a nonzero real number, then the pair $(\alpha\nu, \alpha\ell)$ is also nontrivial and satisfies (5). So the pair is actually defined up to a scalar multiplier; it is natural to treat this pair as an element of the projective space $\mathbb{P}\left(\mathbb{R} \oplus T^*_{\Phi(w)}N\right)$ rather than an element of the linear space.

The pair (ν, ℓ) which satisfies (5) is called the *Lagrange multiplier* associated to the critical point w. The Lagrange multiplier is called *normal* if $\nu \neq 0$ and abnormal if $\nu = 0$. In these lectures we consider only normal Lagrange multipliers, they belong to a distinguished coordinate chart of the projective space $\mathbb{P}\left(\mathbb{R} \oplus T^*_{\Phi(w)}N\right)$.

Any normal Lagrange multiplier has a unique representative of the form $(-1, \ell)$; then (5) is reduced to the equation

$$\ell D_w \Phi = d_w J. \tag{6}$$

The vector $\ell \in T^*_{\Phi(w)}N$ from (6) is also called a normal Lagrange multiplier (along with $(-1, \ell)$).

1.3 Extremals

Now we apply the Lagrange multipliers rule to the optimal control problem. We have $\Phi = (F_{t_0}, F_{t_1}) : \mathcal{V} \to M \times M$. Let an admissible pair $v \in \mathcal{V}$ be a critical point of the mapping $\left(J_{t_0}^{t_1}, \Phi\right)$, the curve $q(t) = \pi(v(t))$, $t_0 \le t \le t_1$ be the corresponding trajectory, and $\ell \in T^*_{(q(t_0),q(t_1))}(M \times M)$ be a normal Lagrange multiplier associated to $v(\cdot)$. Then

$$\ell D_v \left(F_{t_0}, F_{t_1}\right) = d_v J_{t_0}^{t_1}. \tag{7}$$

We have $T^*_{(q(t_0),q(t_1))}(M \times M) = T^*_{q(t_0)}M \times T^*_{q(t_1)}M$, hence ℓ can be presented in the form $\ell = (-\lambda_{t_0}, \lambda_{t_1})$, where $\lambda_{t_i} \in T^*_{q(t_i)}M$, $i = 0, 1$. Equation (7) takes the form

$$\lambda_{t_1} D_v F_{t_1} - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t_1}. \tag{8}$$

Note that λ_{t_1} in (8) is uniquely defined by λ_{t_0} and v. Indeed, assume that $\lambda'_{t_1} D_v F_{t_1} - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t_1}$ for some $\lambda'_{t_1} \in T^*_{q(t_1)}M$. Then $(\lambda'_{t_1} - \lambda_{t_1}) D_v F_{t_1} = 0$. Recall that F_{t_1} is a submersion, hence $D_v F_{t_1}$ is a surjective linear map and $\lambda'_{t_1} - \lambda_{t_1} = 0$.

Proposition 1.2. *Equality (8) implies that for any $t \in [t_0, t_1]$ there exists a unique $\lambda_t \in T^*_{q(t)}M$ such that*

$$\lambda_t D_v F_t - \lambda_{t_0} D_v F_{t_0} = d_v J_{t_0}^{t} \tag{9}$$

and λ_t is Lipschitzian with respect to t.

Proof. The uniqueness of λ_t follows from the fact that F_t is a submersion as it was explained few lines above. Let us proof the existence. To do that we use the coordinatization of $\mathcal{V}$ introduced in the proof of Proposition 1.1, in particular, the family of local trivializations $R_t : O_{q(t)} \times U \to \pi^{-1}(O_{q(t)})$. Assume that $v(t) = R_t(q(t), u(t))$, $t_0 \le t \le t_1$, where $v(\cdot)$ is the referenced admissible pair from (8).

Given $\tau \in [t_0, t_1]$, $\hat{q} \in O_{q(\tau)}$ let $t \mapsto Q_\tau^t(\hat{q})$ be the solution of the differential equation $\dot{q} = R_t(q, u(t))$ which satisfies the condition $Q_\tau^\tau(\hat{q}) = \hat{q}$. In particular, $Q_\tau^t(q(\tau)) = q(t)$. Then Q_τ^t is a diffeomorphism of a neighborhood of $q(\tau)$ on a neighborhood of $q(t)$. We define a Banach submanifold $\mathcal{V}_\tau$ of the Banach manifold $\mathcal{V}$ in the following way:

$$\mathcal{V}_\tau = \{v' \in \mathcal{V} : \pi(v'(t)) = Q_\tau^t(\pi(v'(\tau))),\ \tau \le t \le t_1\}.$$