Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1258

Joachim Weidmann

Spectral Theory of Ordinary Differential Operators

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1258

Joachim Weidmann

Spectral Theory of Ordinary Differential Operators

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Author

Joachim Weidmann
Fachbereich Mathematik, Universität Frankfurt
Robert-Mayer-Straße 6–10, Postfach 111932
6000 Frankfurt/Main 11, Federal Republic of Germany

Mathematics Subject Classification (1980): 35B25, 35PXX, 47A40, 47E05, 81C10

ISBN 3-540-17902-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17902-X Springer-Verlag New York Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in other ways, and storage in data banks. Duplication of this publication or parts thereof is only permitted under the provisions of the German Copyright Law of September 9, 1965, in its version of June 24, 1985, and a copyright fee must always be paid. Violations fall under the prosecution act of the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1987
Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–1034 please contact your bookseller or Springer-Verlag.

Vol. 1035: The Mathematics and Physics of Disordered Media. Proceedings, 1983. Edited by B.D. Hughes and B.W. Ninham. VII, 432 pages. 1983.

Vol. 1036: Combinatorial Mathematics X. Proceedings, 1982. Edited by L.R.A. Casse. XI, 419 pages. 1983.

Vol. 1037: Non-linear Partial Differential Operators and Quantization Procedures. Proceedings, 1981. Edited by S.I. Andersson and H.-D. Doebner. VII, 334 pages. 1983.

Vol. 1038: F. Borceux, G. Van den Bossche, Algebra in a Localic Topos with Applications to Ring Theory. IX, 240 pages. 1983.

Vol. 1039: Analytic Functions, Błażejewko 1982. Proceedings. Edited by J. Ławrynowicz. X, 494 pages. 1983

Vol. 1040: A. Good, Local Analysis of Selberg's Trace Formula. III, 128 pages. 1983.

Vol. 1041: Lie Group Representations II. Proceedings 1982–1983. Edited by R. Herb, S. Kudla, R. Lipsman and J. Rosenberg. IX, 340 pages. 1984.

Vol. 1042: A. Gut, K. D. Schmidt, Amarts and Set Function Processes. III, 258 pages. 1983.

Vol. 1043: Linear and Complex Analysis Problem Book. Edited by V.P. Havin, S.V. Hruščëv and N.K. Nikol'skii. XVIII, 721 pages. 1984.

Vol. 1044: E. Gekeler, Discretization Methods for Stable Initial Value Problems. VIII, 201 pages. 1984.

Vol. 1045: Differential Geometry. Proceedings, 1982. Edited by A.M. Naveira. VIII, 194 pages. 1984.

Vol. 1046: Algebraic K-Theory, Number Theory, Geometry and Analysis. Proceedings, 1982. Edited by A. Bak. IX, 464 pages. 1984.

Vol. 1047: Fluid Dynamics. Seminar, 1982. Edited by H. Beirão da Veiga. VII, 193 pages. 1984.

Vol. 1048: Kinetic Theories and the Boltzmann Equation. Seminar, 1981. Edited by C. Cercignani. VII, 248 pages. 1984.

Vol. 1049: B. Iochum, Cônes autopolaires et algèbres de Jordan. VI, 247 pages. 1984.

Vol. 1050: A. Prestel, P. Roquette, Formally p-adic Fields. V, 167 pages. 1984.

Vol. 1051: Algebraic Topology. Aarhus 1982. Proceedings. Edited by I. Madsen and B. Oliver. X, 665 pages. 1984.

Vol. 1052: Number Theory, New York 1982. Seminar. Edited by D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn and M.B. Nathanson. V, 309 pages. 1984.

Vol. 1053: P. Hilton, Nilpotente Gruppen und nilpotente Räume. V, 221 pages. 1984.

Vol. 1054: V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. VII, 237 pages. 1984.

Vol. 1055: Quantum Probability and Applications to the Quantum Theory of Irreversible Processes. Proceedings, 1982. Edited by L. Accardi, A. Frigerio and V. Gorini. VI, 411 pages. 1984.

Vol. 1056: Algebraic Geometry. Bucharest 1982. Proceedings, 1982. Edited by L. Bădescu and D. Popescu. VII, 380 pages. 1984.

Vol. 1057: Bifurcation Theory and Applications. Seminar, 1983. Edited by L. Salvadori. VII, 233 pages. 1984.

Vol. 1058: B. Aulbach, Continuous and Discrete Dynamics near Manifolds of Equilibria. IX, 142 pages. 1984.

Vol. 1059: Séminaire de Probabilités XVIII, 1982/83. Proceedings. Edité par J. Azéma et M. Yor. IV, 518 pages. 1984.

Vol. 1060: Topology. Proceedings, 1982. Edited by L. D. Faddeev and A.A. Mal'cev. VI, 389 pages. 1984.

Vol. 1061: Séminaire de Théorie du Potentiel. Paris, No. 7. Proceedings. Directeurs: M. Brelot, G. Choquet et J. Deny. Rédacteurs: F. Hirsch et G. Mokobodzki. IV, 281 pages. 1984.

Vol. 1062: J. Jost, Harmonic Maps Between Surfaces. X, 133 pages. 1984.

Vol. 1063: Orienting Polymers. Proceedings, 1983. Edited by J.L. Ericksen. VII, 166 pages. 1984.

Vol. 1064: Probability Measures on Groups VII. Proceedings, 1983. Edited by H. Heyer. X, 588 pages. 1984.

Vol. 1065: A. Cuyt, Padé Approximants for Operators: Theory and Applications. IX, 138 pages. 1984.

Vol. 1066: Numerical Analysis. Proceedings, 1983. Edited by D.F. Griffiths. XI, 275 pages. 1984.

Vol. 1067: Yasuo Okuyama, Absolute Summability of Fourier Series and Orthogonal Series. VI, 118 pages. 1984.

Vol. 1068: Number Theory, Noordwijkerhout 1983. Proceedings. Edited by H. Jager. V, 296 pages. 1984.

Vol. 1069: M. Kreck, Bordism of Diffeomorphisms and Related Topics. III, 144 pages. 1984.

Vol. 1070: Interpolation Spaces and Allied Topics in Analysis. Proceedings, 1983. Edited by M. Cwikel and J. Peetre. III, 239 pages. 1984.

Vol. 1071: Padé Approximation and its Applications, Bad Honnef 1983. Prodeedings. Edited by H. Werner and H.J. Bünger. VI, 264 pages. 1984.

Vol. 1072: F. Rothe, Global Solutions of Reaction-Diffusion Systems. V, 216 pages. 1984.

Vol. 1073: Graph Theory, Singapore 1983. Proceedings. Edited by K.M. Koh and H.P. Yap. XIII, 335 pages. 1984.

Vol. 1074: E.W. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations. III, 143 pages. 1984.

Vol. 1075: H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points. IX, 159 pages. 1984.

Vol. 1076: Infinite-Dimensional Systems. Proceedings, 1983. Edited by F. Kappel and W. Schappacher. VII, 278 pages. 1984.

Vol. 1077: Lie Group Representations III. Proceedings, 1982–1983. Edited by R. Herb, R. Johnson, R. Lipsman, J. Rosenberg. XI, 454 pages. 1984.

Vol. 1078: A.J.E.M. Janssen, P. van der Steen, Integration Theory. V, 224 pages. 1984.

Vol. 1079: W. Ruppert. Compact Semitopological Semigroups: An Intrinsic Theory. V, 260 pages. 1984

Vol. 1080: Probability Theory on Vector Spaces III. Proceedings, 1983. Edited by D. Szynal and A. Weron. V, 373 pages. 1984.

Vol. 1081: D. Benson, Modular Representation Theory: New Trends and Methods. XI, 231 pages. 1984.

Vol. 1082: C.-G. Schmidt, Arithmetik Abelscher Varietäten mit komplexer Multiplikation. X, 96 Seiten. 1984.

Vol. 1083: D. Bump, Automorphic Forms on GL (3,IR). XI, 184 pages. 1984.

Vol. 1084: D. Kletzing, Structure and Representations of Q-Groups. VI, 290 pages. 1984.

Vol. 1085: G.K. Immink, Asymptotics of Analytic Difference Equations. V, 134 pages. 1984.

Vol. 1086: Sensitivity of Functionals with Applications to Engineering Sciences. Proceedings, 1983. Edited by V. Komkov. V, 130 pages. 1984

Vol. 1087: W. Narkiewicz, Uniform Distribution of Sequences of Integers in Residue Classes. VIII, 125 pages. 1984.

Vol. 1088: A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, Characterization of Distributions by the Method of Intensively Monotone Operators. X, 175 pages. 1984.

Vol. 1089: Measure Theory, Oberwolfach 1983. Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XIII, 327 pages. 1984.

continued on page 305

Preface

Hermann Weyl's celebrated work from 1910 "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen" together with the developement of quantum mechanics around 1925 initiated a continuous and extremely fruitful research activity in the spectral theory of *Sturm-Liouville operators*. Although the general theory in some sense had reached its final shape with the proof of the spectral representation and the Weyl-Titchmarsh formula for the spectral matrix, many fascinating results about special operators have been contributed by a large number of mathematicians up to the present days.

Wide parts of the theory were generalized long time ago to certain even order operators mainly by I.M. Glazman, K. Kodaira and M.A. Neumark. On the other hand by S.D. Conte, B.W. Roos, W.C. Sangren and E.C. Titchmarsh results which are almost identical to those in the Sturm-Liouville case have been found for certain first order differential expressions operating on $\mathbb{C}^2$-valued functions (*Dirac systems*). But there was no general frame including all these different types, although it seemed obvious that there were many common features.

The starting point for writing these notes was the intention to present a general theory of ordinary differential operators, covering operators of arbitrary order n operating on $\mathbb{C}^m$-valued functions for arbitrary m. This is the content of about two thirds of the present text. In the remaining part we apply this theory to Sturm-Liouville operators and Dirac systems, studying mainly oscillation theory and absolute continuity of the spectrum. Most of the results can be found in the literature in some form; but there are also some new results, mainly connected with the problem of existence of self-adjoint realizations with separated boundary conditions (section 4), multiplicity of the spectrum (section 10), and the absolute continuity of the spectrum (sections 10, 15 and 16). The proofs are functional analytic in spirit wherever possible.

此为试读,需要完整PDF请访问: www.ertongbook.com

The text is almost completely self-contained. Besides some fundamental facts from various fields of analysis which are used without reference we only need a number of results from the abstract theory of self-adjoint operators in Hilbert spaces; for all these results we refer to the author's book [70]. This should make the subject easily accessible to mathematicians interested in applications to physical problems as well as to physicists with some mathematical background.

Many people helped me during the preparations of the manuscript. Discussions with auditors of several lecture series on this subject at the Universities of Munich, Frankfurt and Pretoria (R.S.A.) lead to many improvements. My collaborators Andreas Orth, Günter Stolz, Thomas Poerschke, and Werner Stork read the manuscript at different stages and with their criticism contributed a lot to its final form. Christel Quaß typed parts of an earlier version of the text, and finally Martina Eismann typed the complete manuscript into the computer and never lost patience with my permanent wishes for corrections and changes. It is a pleasure to thank all of them for their invaluable assistance.

Frankfurt am Main, March 1987 J. Weidmann

Contents

Introduction

Many physical systems are described by differential equations (or systems) of the form

$$i \frac{\partial}{\partial t} \psi(x,t) = \mathcal{A}\psi(x,t) \tag{1}$$

(e.g. *Schrödinger equation*) or

$$\frac{\partial^2}{\partial t^2} \psi(x,t) = -\mathcal{A}\psi(x,t) \tag{1'}$$

(e.g. *Wave equation*), where $\mathcal{A}$ is a (in general partial) differential expression with respect to the x-variable. In many cases it is quite natural to consider (1) and (1') as differential equations in an L_2-Hilbert space H of *states* of the system:

$$i \frac{d}{dt} f(t) = Af(t), \tag{2}$$

$$\frac{d^2}{dt^2} f(t) = - Af(t), \tag{2'}$$

where A represents an operator in the Hilbert space H. If A is self-adjoint then by means of the spectral calculus the solutions of the initial value problems

$$i \frac{d}{dt} f(t) = Af(t), \quad f(0) = f_0 \in D(A), \tag{3}$$

$$\frac{d^2}{dt^2} f(t) = - Af(t), \quad f(0) = f_0 \in D(A), \quad f'(0) = f_1 \in D(A^{\frac{1}{2}}) \tag{3'}$$

can be given by

$$f(t) = \exp(-it\, A)\, f_0, \tag{4}$$

respectively

$$f(t) = \cos\,(tA^{\frac{1}{2}})\, f_0 + A^{-\frac{1}{2}} \sin\,(tA^{\frac{1}{2}})\, f_1. \tag{4'}$$

Therefore we have a complete description of the solutions of the above initial value problems if the spectral resolution of A is known.

In general it cannot be expected that the spectral resolution of a self-adjoint partial differential operator is known explicitly. But, if the differential expression $\mathcal{A}$ has certain symmetry properties (e.g. rotational symmetry of the potential in a one particle Schrödinger equation), then it may be possible to apply a *separation of variables* which leads to a decomposition of the Hilbert space H into an orthogonal sum of Hilbert spaces H_j $(j \in J)$ which reduce A and have the property that the restriction A_j of A to H_j is unitarily equivalent to an ordinary differential operator T_j in a space $L_2(I)$, $I \subset \mathbb{R}$. In section 1 we give a number of examples from mathematical physics to which this procedure is applicable.

The ordinary differential expression occurring most frequently in applications is the *Sturm-Liouville expression*

$$\tau u(x) = \frac{1}{r(x)} \{ -(p(x)u'(x))' + q(x)u(x) \}, \qquad x \in (a,b).$$

The study of the self-adjoint operators associated with Sturm-Liouville expressions goes back at least to H. Weyl [72]. Many authors have contributed to this theory; we only mention F. Rellich, Ph. Hartman, A. Wintner, K. Kodaira and E.C. Titchmarsh. *Weyl's alternative* (limit point case and limit circle case) allows a complete description of all self-adjoint realizations of τ in the weighted L_2-space $L_2(a,b;r)$. The general structure of the resolvents, of the spectral resolutions, and of the spectral representations are known, and the formulae of Weyl-Titchmarsh-Kodaira allow to calculate the spectral resolution and the spectral representation explicitely. For a huge number of special cases many specific results about spectral properties have been proved (semi-boundedness, pure point spectrum, absolute continuity, absence of singular continuous spectrum etc.).

For the Dirac operator with spherically symmetric potential a separation of variables (cf. Appendix to section 1) leads to a family τ_j $(j \in \mathbb{Z}\setminus\{0\})$ of first order differential expressions for $\mathbb{C}^2$-valued functions on $(0,\infty)$. The corresponding self-adjoint operators in $L_2(0,\infty)^2$ have been studied by S.D. Conte, B.W. Roos and W.C. Sangren [9, 54, 55, 56], E.C. Titchmarsh [61 - 65]. The theory of these *Dirac systems* was developed in complete analogy to the theory of Sturm-Liouville operators, including Weyl's alternative and the Weyl-Titchmarsh-Kodaira formula. Detailed studies of the spectral properties for potentials which are sufficiently general for applications to physically interesting problems were given by J. Weidmann [68, 71] and H. Behncke [2, 3, 4]. A theory which allows to treat Sturm-Liouville operators and Dirac systems simultaneously did not exist so far.

In these notes we give a general and rather complete theory of self-adjoint ordinary differential operators of arbitrary order n operating on $\mathbb{C}^m$-valued functions for arbitrary $m \in \mathbb{N}$. The general form of these formally self-adjoint differential expressions is given in (1.1). It specifies for

$$n = 1: \quad \tau u(x) = r(x)^{-1}\{ (q_0(x)u(x))' - q_0^*(x)u'(x) + p_0(x)u(x) \},$$

$$n = 2: \quad \tau u(x) = r(x)^{-1}\{-(p_1(x)u'(x))'+(q_0(x)u(x))'-q_0^*(x)u'(x)+p_0(x)u(x) \},$$

etc., where $r(.)$, $p_j(.)$ and $q_j(.)$ are $m{\times}m$-matrix valued, $p_j(x)^* = p_j(x)$, and $r(x)$ is positive definite. It is known (cf. I.M. Glazman [15]) that every formally self-adjoint differential expression with sufficiently smooth coefficients can be written in the form (1.1). But we allow quite singular coefficients, such that the differential expressions cannot be evaluated term by term; we have to use the *quasi derivatives* which are introduced in section 2 (sometimes such differential expressions are called *quasi differential expressions*). For the above special cases τu must be evaluated as follows:

$$n = 1: \quad \tau u(x) = r(x)^{-1}\{ (q_0(x) - q_0^*(x))u'(x) + (q_0'(x) + p_0(x))u(x) \},$$

$$n = 2: \quad \tau u(x) = r(x)^{-1}\{-(p_1(x)u'(x)-q_0(x)u(x))'- q_0^*(x)u'(x) + p_0(x)u(x) \}.$$

For $m = 1$ a similar class of ordinary differential expressions has been introduced by N.W. Everitt and A. Zettl [14, 74]. Differential expressions of the above form containing only the even order terms $(p_j(x)u^{(j)}(x))^{(j)}$ have already been studied much earlier by I.M. Glazman [14], K. Kodaira [40] and M.A. Neumark [46] (see also E. Müller-Pfeiffer [45] and the references given there); this theory was also developed along

此为试读,需要完整PDF请访问: www.ertongbook.com

the lines of the Sturm-Liouville theory, but a result comparable to Weyl's alternative does not exist (cf. I.M. Glazman [14]).

The organization of the book is as follows:

After describing several typical examples covered by our theory we start in section 2 with some basic facts about our class of differential expressions. The *quasi derivatives* are introduced in order to transform the differential equations $(\tau-\lambda)u = f$ into first order systems. Together with a quite general existence and uniqueness theorem for linear first order systems with locally integrable coefficients this enables us to state our general assumptions on the coefficients which will be used throughout the following sections.

In section 3 the *maximal operator* T associated with τ is defined in a natural way to be the "differential operator" defined by τ with the largest possible domain. The *minimal operator* T_0' (or its closure T_0, the *closed minimal operator*) is such that $T_0'^* = T$, i.e. the adjoint of every operator with "essentially smaller" domain would not be a "differential operator" any more. Therefore every *self-adjoint realization* A of τ must be a restriction of the maximal operator T and an extension of the minimal operator T_0, i.e. $T_0 \subset A \subset T$.

In section 4 the deficiency indices of T_0 are determined by means of the L_2-properties of the solutions of $(\tau-\lambda)u = 0$ near the boundary points a and b. By means of von Neumann's theory this solves the problem of existence of self-adjoint extensions of T_0 and allows to construct all of them as restrictions of T by means of boundary conditions. A large part of this section is devoted to the question if there exist self-adjoint extensions of T_0 with separated boundary conditions. For *regular* problems (cf. section 3) all self-adjoint extensions of T_0 can be given explicitely.

The form of the solutions of the inhomogeneous equation $(\tau-\lambda)u = f$ is studied in section 5. Among others the results allow to prove that, if for some $\lambda_0 \in \mathbb{C}$ all solutions of $(\tau-\lambda_0)u = 0$ and of $(\tau-\overline{\lambda}_0)u = 0$ are square integrable near a, resp. b, then this holds for every $\lambda \in \mathbb{C}$ *(quasi regular* at a, resp. b); we also give a functional analytic proof of this fact going back to I.M. Glasman [14]. Specializing to the case $p = n \times m = 2$ Weyl's *limit point / limit circle alternative* follows. In this case a complete description of all self-adjoint extensions of T_0 can be given. Some of the most important limit point / limit circle criteria are proved in section 6, separately for Sturm-Liouville expressions and Dirac systems; these cover most of the physically relevant problems. In an appendix operators of *Sturm-Liouville type* (i.e. operators of the form of a Sturm-Liouville operator, but with $m \geq 1$) are studied with respect to semi-boundedness.

The general form of the resolvent of self-adjoint extensions of T_0 is studied in section 7; the calculation of the resolvent turns out to be especially simple in the case of separated boundary conditions. In section 8 the representation of the resolvent is used to find the general form of the spectral representation and of the spectral resolution. This involes a spectral matrix $\varrho(.)$ containing complete information about the spectrum. It can be calculated by means of the Weyl-Titchmarsh-Kodaira formula; several versions of which are proved in section 9.

It is obvious from the general form of the spectral representation that the spectral multiplicity is at most $p = n \ast m$. In section 10 we prove that the multiplicity is smaller under several very general assumptions. We also prove a simple result about the absence of singular continuous spectrum (Theorem 10.14) which has an immediate application to periodic operators in section 12.

Ph. Hartman and A. Wintner [20] have proved that a $\lambda \in \mathbb{R}$ belongs to the essential spectrum of every self-adjoint realization of a Sturm-Liouville expression if the equation $(\tau-\lambda)u = 0$ has **no** solution which is square integrable near a (or b). This result easily extends to the general case with $p = n \ast m = 2$. Several extensions to arbitrary τ are given in section 11. On the other hand, if τ is regular at a and for **every** λ from an interval I there exists a square integrable solution of $(\tau-\lambda)u = 0$, then the spectrum is a pure point spectrum and nowhere dense in I; this also generalizes a result of Ph. Hartman - A. Wintner [22] for Sturm-Liouville operators.

The spectral properties of differential operators with periodic coefficients are studied in section 12. We get absolutely continuous *band spectrum*. For the case $p = n \ast m = 2$ the connection between the bands and the eigenvalues of the regular problems on a periodicity interval with periodic and semi-periodic boundary conditions is given. In an appendix we study operators with periodic coefficients on the half-line.

In the remaining sections we turn to the special case $p = n \times m = 2$. In section 13, 14 and 15 we study oscillation theory for regular respectively singular Sturm-Liouville operators. We prove the connection between the "number" of the eigenvalue and the zeros of the corresponding eigenfunction for regular problems as well as the connection between oscillation properties and the essential spectrum in the singular case. And finally we use oscillation methods in order to prove absolute continuity of the positive spectrum of certain Sturm-Liouville operators. These results are essentially contained in J. Weidmann [67, 71]. New is only the fact that the operator of multiplication with the variable in $L_2(0,\infty)$ is unitarily equivalent to some part of these Sturm-Liouville operators. These results are applicable to every reasonable one particle Schrödinger operator with spherically symmetric potential without any restriction of the behaviour of the potential near the origin (cf. section 17.F).

Similar results for Dirac systems are proved in section 16. In order to do this we develop an oscillation theory for Dirac systems. The technical details are much simpler than for Sturm-Liouville operators. These results are also essentially taken from J. Weidmann [68, 71]. The absolute continuity result for Dirac systems as well as for Sturm-Liouville operators has also been proved by E. Heinz [24] using completely different methods (limiting absorption).

In the final section 17 we apply many of our results in order to study a number of more or less explicitely solvable problems. The first example shows that the one dimensional Fourier transform can easily be recovered as the spectral representation of the simplest first order operator $T = -i\, d/dx$ on $\mathbb{R}^1$. All other examples are concerned with special cases of Sturm-Liouville operators and Dirac systems: Coefficients which are constant near infinity, periodic coefficients, Sturm-Liouville expressions with non-definite main part, and the Sturm-Liouville expressions and Dirac systems occurring after separation of the one particle Schrödinger operator and Dirac operator with spherically symmetric potentials.

1. Formally self-adjoint differential expressions

We shall study operators generated by means of *formal differential expressions* τ of the form

$$\tau u(x) = r(x)^{-1} \left\{ \sum_{j=0}^{[\frac{n}{2}]} (-1)^j (p_j(x)u^{(j)}(x))^{(j)} \right. \tag{1.1}$$
$$\left. + \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \left[(q_j(x)u^{(j)}(x))^{(j+1)} - (q_j^*(x)u^{(j+1)}(x))^{(j)}\right] \right\},$$

where

- u are $\mathbb{C}^m$ -valued functions defined on (a,b), $-\infty \le a < b \le \infty$,
- the symbol $[\alpha]$ stands for the largest integer less than or equal to α; therefore the natural number n is the *order* of the differential expression τ,
- the coefficients r, p_j and q_j are $m\times m$-matrix valued functions on (a,b), $r(x)$ is positive definite, and the $p_j(x)$ are hermitian.

Further assumptions on the coefficients will be stated in the following section.

The factors $(-1)^j$ are chosen such that:

(α) for $m = 1$, $n = 2$ and $q_0 = 0$ we have the well known *Sturm-Liouville* differential expression

$$\tau u(x) = \frac{1}{r(x)} \{ -(p_1(x)u'(x))' + p_0(x)u(x) \},$$

(β) for $m = 1$, $n = 1$, $p_0 = 0$, $q_0 = \frac{1}{2i}$ and $r = 1$ we have the 1-dimensional *momentum operator*

$$\tau u(x) = \frac{1}{i} u'(x),$$

(γ) for $n = 2k$, $p_k(x) > 0$ (positive definite) and some additional conditions on the lower order terms the operators defined by τ will be bounded from below, (δ) the definition of the quasi derivatives in section 2 is comparatively simple.

It is apparent that these differential expressions are *formally self-adjoint* in the following sense: If $(.,.)$ denotes the usual inner product in $\mathbb{C}^m$

$$(\xi,\eta) = \sum_{j=1}^{m} \overline{\xi}_j \eta_j \quad \text{for} \quad \xi = (\xi_1,\ldots,\xi_m),\ \eta = (\eta_1,\ldots,\eta_m),$$

then we have for "sufficiently smooth" functions $u,v : (a,b) \to \mathbb{C}^m$ with compact support

$$\int_a^b (r(x)\tau u(x),v(x))\,dx = \int_a^b (r(x)u(x),\tau v(x))\,dx.$$

So far it is not clear that there are "sufficiently many" functions u to which the differential expression τ can be applied. In the following sections we shall define a Hilbert space of $\mathbb{C}^m$-valued functions on (a,b) with the inner product

$$\langle u,v\rangle = \int_a^b (r(x)u(x),v(x))\,dx.$$

The formal self-adjointness means that τ generates a hermitian operator in this Hilbert space. We shall see that this operator is densely defined (i.e. symmetric). In many cases the existence of self-adjoint extensions can be shown.

We shall demonstrate now that many interesting applications lead to differential expressions of the above form.

Example 1.1 *The vibrating string.* We assume that an elastic string is spanned over the interval $[a,b]$, clamped at the end points a and b. Let $r(x) > 0$ be the mass density (mass per unit length) at the point x, $p(x) > 0$ the elasticity modulus at the point x, $u(x,t)$ the displacement of the string at the point x and time t. Then

$$r(x)u_{tt}(x,t) = (p(x)u_x(x,t))_x \quad \text{for} \quad a \le x \le b,\ t \ge 0\ ,$$

with the *boundary condition*

$$u(a,t) = u(b,t) = 0 \qquad \text{for} \quad t \ge 0.$$

It is expected that, in order to have uniqueness of the solution, we also need the *initial conditions*

$$\left.\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned}\right\} \text{ for } a \le x \le b.$$

For solutions of the form

$$u(x,t) = v(x)w(t)$$

(*separation of variables*) we have

$$u_{tt}(x,t) = v(x)w''(t),$$

$$(p(x)u_x(x,t))_x = (p(x)v'(x))'w(t),$$

and therefore

$$r(x)v(x)w''(t) = (p(x)v'(x))'w(t).$$

For $v(x) \neq 0$ and $w(t) \neq 0$ this implies

$$\frac{w''(t)}{w(t)} = \frac{(p(x)v'(x))'}{r(x)v(x)} \quad \text{for } x \in [a,b],\ t \ge 0.$$

Since both sides depend on different variables this equation can only hold if both sides are constant, say equal to $-\lambda$:

$$-\,w''(t) = \lambda w(t),$$

$$-\,(p(x)v'(x))' = \lambda r(x)v(x), \qquad v(a) = v(b) = 0.$$

If the second equation (including the boundary conditions) has a solution v for some λ, then setting

$$w(t) = A\cos(\lambda^{\frac{1}{2}}t) + B\,\lambda^{-\frac{1}{2}}\sin(\lambda^{\frac{1}{2}}t)$$

we get a solution $u(x,t) = v(x)w(t)$ of the original problem:

$$u(x,t) = \{\, A\cos(\lambda^{\frac{1}{2}}t) + B\,\lambda^{-\frac{1}{2}}\sin(\lambda^{\frac{1}{2}}t)\,\}\, v(x).$$

Clearly this solution satisfies the initial conditions

$$u(x,0) = Av(x),$$

$$u_t(x,0) = Bv(x).$$

The boundary value problem

$$\lambda v(x) = \tau v(x) := -r(x)^{-1}(p(x)v'(x))', \qquad v(a) = v(b) = 0$$

此为试读,需要完整PDF请访问: www.ertongbook.com