MODERN GEOMETRICAL OPTICS

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Preface

This book is the result of more than fourteen years' continuous labor. I am submitting the material now for publication, not because I feel that it represents the ideas in perfected form, but because there has to be an end of revising and supplementing if the book is to reach its readers.

One problem, which led to frequent rewriting, was the choice of the method of presentation. The best way to write the book would have been to start with the law of Fermat, or the equivalent Formula (14.4), and derive from it all the laws systematically. This would lead to an esthetically more pleasing treatment, but one that would require a considerable amount of mathematical knowledge on the part of the reader.

Those who most need a thorough knowledge of the theory of optics are the designers of optical lenses. I have a great admiration for the achievements of commercial designers, but lens design is still more an art than a science and, unfortunately, the acquaintance of the average designer with higher mathematics is limited. Since I hope that, in addition to setting forth a consistent theoretical system, this book will be of use in the practical design of optical instruments, it was necessary to accomplish the desired results with a minimum of mathematical technique. Moreover, the mathematical methods that are used are explained in detail in an appendix prepared by my colleague, Dr. Erich Marchand. I have limited the problems to those which can be treated by these methods.

The aim of the book is nothing less than to develop a mathematical model of an optical system that is complex enough so that all the characteristics of the geometrical optical image can be obtained from it and at the same time simple enough to be intelligible. It may be mentioned that the same model might also be used as a basis for calculating the diffraction image, since it gives the phase relationships in the exit pupil, but this is outside the scope of this book.

Part I concerns the problem of tracing rays through an optical system. Much of the extensive literature on this subject is now obsolete; the particular type of formulae used always depends on the computational tools at the disposal of the designer, and while the manuscript was in preparation, methods of computation that reduce many fold the time required for tracing a ray came into widespread use. The formulae in this book were developed to be suitable for high-speed electronic computers. To these formulae are added independent controls that are carried along in order to check the correctness of the machine operations.

Part II gives the first-order approximation theory (Gaussian optics), for which a new tool, the so-called Gaussian brackets, is introduced. This new tool enables us to investigate the effect of a variation of system data on the constructional elements of the system (focal length, back focus, magnification, etc.). We can thereby compute an approximate system (a system of thin lenses separated by finite distances) which has, in the realm of Gaussian optics, the desired specifications and is also corrected for Petzval sum and longitudinal and lateral color aberrations.

The ideas of Hamilton are derived in Part III as well as other laws of optical image formation. Special emphasis is laid on the study of the different *types* of imagery.

Concentric systems are taken up in Part IV, and it is shown that, for this limited field, a complete mathematical treatment can be given so that all optical systems with specified characteristics can be specified by the system data. If it were possible to extend the methods of this chapter to systems with rotational symmetry, the problem of lens design would change from an art to a science.

Part V emphasizes the specific form which the methods and general laws of Part III assume in systems with symmetry of rotation. A general theorem (22.5) of many applications enables us to investigate the limitations of optical image formation.

In Part VI most of the results of Allvar Gullstrand's work are derived by the method developed in this book. This gives rise to an approximation theory for normal systems along a principal ray.

The results are used in Part VII to develop a new image-error theory that combines third- and fifth-order aberrations for the neighborhood of the axis.

Finally, in Part VIII we develop a mathematical model of an optical system. The results of tracing a few rays, both meridional and skew, from several object points are fitted to a fifth-order formula equivalent to the theory discussed in Part VI. This formula is then used as an interpolation formula to calculate the intersection points of a large number of rays with one or more image planes. The rays are evenly spaced over the exit pupil, and therefore the plots of the intersection points, called spot diagrams, give a record of the distribution of light over the selected image planes. These spot diagrams are treated as vector sums of simple diagrams, and they serve as a new tool in lens design, giving a record of the behavior of all rays from each of the selected object points.

The last chapter indicates how the methods can be applied to the case of inhomogeneous media. The mathematical techniques are developed in an appendix, while other appendices give numerical examples and some remarks on the history of geometrical optics. The book ends with an extensive bibliography of source material.

Some of the mathematical tools used in this book are worthy of special mention. The treatment is simplified by considerations of symmetry. The introduction of the concept of the diapoint (the intersection of the image ray with the plane that passes through the object point and the axis of symmetry) also results in a simplification. The treatment of the chromatic aberrations is facilitated by using dispersion formulae that are linear functions of the indices.

A future edition of this book should contain some of the integral laws of optical systems, so important for energy considerations, a more detailed study of the dependence of image errors on the system data, and difference formulae for ray tracing. A chapter should also be added on the evaluation of the image of lines and objects of nonuniform density. In recent years much interest has been aroused in this field, and the many papers now being published will undoubtedly further the development of our science.

I want to thank Dr. Fred Perrin and Dr. Erich Marchand for

their help in the difficult task of editing and proofreading this book; Miss Nancy McClure for her help in preparing the index; Miss Helene Donnelly and Miss McClure for preparing and checking the numerical examples; Mr. James Watts for the preparation of the difficult optical drawings; as well as Mr. Stephen Insalaco, who prepared the drawings for Chapter 12.

I am greatly indebted to my publisher for his patience and to him and the printer for their painstaking care in setting in type such difficult material.

Finally, I want to thank the Eastman Kodak Company, who not only gave me the time to write this book, but put all their many facilities at my disposal.

Rochester, New York M. H.
March 1, 1958

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PART I. RAY-TRACING

CHAPTER ONE

The Laws of Refraction and Reflection

Geometrical optics is based on two simple physical assumptions. The first has to do with the behavior of light within a given medium and the other with its behavior in passing from one medium to another.

The velocity of light depends on the medium through which the light passes. Since optical systems are commonly used in air, we may take the velocity in air as the standard and call it c; if the velocity in another medium is designated v, the ratio c/v is defined as the ***refractive index*** of the second medium with respect to air. The index with respect to vacuum would be only slightly different, since the index of air with respect to free space is only 1.0003.

Except in free space, the velocity of light and therefore the refractive index of the medium varies with wavelength λ, by which is meant the wavelength in space or, practically, in air. This variation of index with wavelength, known as ***dispersion***, is negligible for air but is great enough to be very serious for the materials used in optical systems. Unless the contrary is stated or implied, the light will be assumed to be *monochromatic*, which means that the refractive index of a given medium is constant.

If the refractive index is independent of the location of the point under consideration in the medium, the latter is said to be *homogeneous*; if the index is independent of the direction of travel of the light through the point, the medium is said to be ***isotropic***. Except when the contrary is stated, all media will be assumed to be both homogeneous and isotropic.

The first basic assumption of geometrical optics is that, in a homogeneous, isotropic medium, light of a given wavelength travels in straight lines called *rays*. The second basic assumption is the law of refraction and reflection. Light rays change their direction in passing from one homogeneous, isotropic medium to another of different refractive index. The surface separating two such media will be assumed to be smooth, that is, it will be considered to be continuous with a continuously varying tangential plane. At such a surface, a light ray is generally split into two parts, one being *refracted* into the second medium and the other being *reflected* backwards into the first medium.

The laws relating to reflection and refraction have a long and interesting history.

The law of reflection—that the angle of reflection equals the angle of incidence—appears as early as 300 B.C. in the "Catoptrics" of *Euclid*. This is very probably the first book on optics, although there is some doubt of its authenticity.

Around 60 B.C., Hero of Alexandria "derived" the law of reflection from a minimum principle: "The light path is the shortest way between two of its points." It is interesting that he draws only planes and convex mirrors to prove his point, and not concave mirrors, for which the principle does not always apply.

The phenomenon of refraction was also known to the Greeks. There is a famous passage in *Plato's* "Timaios" in which Plato tries to prove the unreliability of our perceptions by demonstrating that a stick submerged in water seems to be shorter than in air, whereas, by withdrawing it, one can immediately see that its length has not changed.

Ptolemy of Alexandria (A.D. 150) tried to find the law of refraction by measuring the angle between the incident and the refracted rays for combinations of air and glass, air and water, and water and glass at 10 intervals for the incident angle. *Boegehold* has shown that all his values may not have been observed because they obey too accurately a second-order interpolation formula.

Witelo (Vitellius), about 1270, edited a ten-volume "Handbook on Optics" containing tables for the combinations studied by Ptolemy. The values in these tables certainly could not have been obtained experimentally because some of them go beyond the angle of total reflection. These erroneous tables were unfortunate for *Kepler* when he studied the phenomenon of refraction, about 1610. His account of his numerous unsuccessful attempts to find the law of refraction is worth reading. Eventually he reduced the problem to one of finding a surface which refracts a parallel bundle of rays so that they come to a focus (cf. Chapter 5). He investigated a hyperbolic surface of rotation and obtained the right answer, but dismissed it on the grounds that it did not lead to Witelo's values.

The credit for discovering the law of refraction must be divided between the Dutch physicist *Snell* (Snellius, 1591–1626) and the French mathematician, physicist, and philosopher *Descartes*. Unfortunately, Snell's book that was said to have contained the law was destroyed by fire, so that Descartes' "Dioptrique" (1637) is the first extant publication containing the refraction law. Descartes has been posthumously attacked by I. Voss, who claimed that Descartes had seen Snell's book before its publication. Possibly; but recent studies of some of Descartes' letters by H. Boegehold indicate that Descartes probably knew the law of refraction before his visit to Leyden, where he first became acquainted with Snell.

The *law of refraction*, as discovered by Snell and Descartes, may be stated as follows:

The incident ray and the refracted ray lie in a plane that contains the normal to the refracting surface at the point of incidence (the incidence normal), and the directions of the two rays are related by the equation

$$n \sin i = n' \sin i', \tag{1.1}$$

where n and n' are the refractive indices of the first and second media respectively, i is the angle between the incident ray and

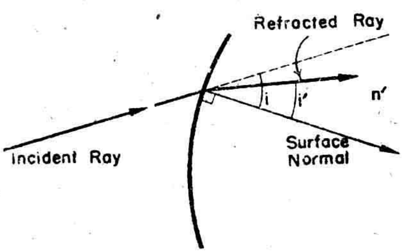

Fig. 1.1. Refraction.

the surface normal (Fig. 1.1), and i', the angle between the refracted ray and the normal. The light will always be assumed to travel from left to right in the drawings (except in certain cases of reflection, to be considered later) and the surface normals will always be drawn from left to right. The positive direction of a ray will always be taken as the direction in which the light travels. Then i will be defined as the angle between the direction of the incidence normal and the positive direction of the incident ray, and i' will be defined correspondingly with respect to the refracted ray. Since two intersecting lines form four angles, we shall assume that $0 \leq i \leq \pi$. There is no lack of generality if we assume that $\sin i$ and $\sin i'$ are positive since the calculation formulae involve only the squares of these quantities. (One exception will be made to obtain the traditional formula for meridional rays from our development.)

It should be noted that Equation (1.1) has two solutions, the sum of the two being 180° or π. Only one makes sense physically, however, because the refracted ray must enter the *second* medium. The ambiguity is avoided if one adds to Equation (1.1) the condition that both $\cos i$ and $\cos i'$ shall have the same sign in the case of refraction.

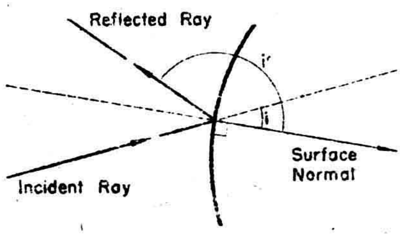

Fig. 1.2. Reflection.

The complete *law of reflection* can be stated as follows: The incident ray and the reflected ray lie in a plane containing the incidence normal, and this normal bisects the angle between the two rays.

This law can be treated as a special case of the refraction law. The direction of the incident ray will be assumed to be from left to right in the drawings, whereas the reflected ray will go from right to left after one or an odd number of reflections. If we set $n = n'$ in Equation (1.1), we obtain

$$\sin i = \sin i'. \tag{1.2}$$

This equation again has two possible solutions, namely, $i' = i$ and $i' = \pi - i$, but only the second solution makes sense physically since the reflected ray must return into the *same* medium as the incident ray. This requirement can be expressed mathematically by demanding that the cosine change its sign, or that simultaneously (Fig. 1.2)

$$\sin i' = \sin i \tag{1.3}$$

and

$$\cos i' = -\cos i,$$

whence

$$i' = \pi - i. \tag{1.4}$$

The author considers that this treatment of the law of reflection is more logical than the widely used convention of postulating a negative refractive index in the case of reflection, since a negative index could have no physical significance.

The case of ***normal incidence***, when $i = 0$, gives $i' = 0$ for the refracted ray from Equation (1.1) and $i' = \pi$ for the reflected ray from Equations (1.2) and (1.3). The reflected and the refracted rays therefore lie along the direction of the surface normal.

When a ray goes from an optically dense medium into a less dense medium, $n > n'$ and Equation (1.1) shows that sin i' may then become greater than unity. This is mathematically impossible, so, for incidence angles greater than i_c as defined by

$$\sin i_c = n'/n, \tag{1.5}$$

no light is refracted and all the incident light is reflected. The angle i_c is called the *critical angle*, and any ray with a greater angle of incidence is *totally reflected*. This phenomenon was discovered by Kepler.

The laws of refraction and reflection are the only physical laws required for geometrical optics. All that follows can be considered to be simply an evaluation of their consequences.

The two parts of the law of refraction—relating to (*a*) the magnitude and (*b*) the coplanarity of angles i and i'—can be combined into a single formula in vector notation.* This is the simplest form for analyzing optical problems.

* The fundamental operations of vector analysis are outlined in the Appendix.

Let $\vec{s}$ be a vector in the direction of the entering ray, its length being equal to the refractive index n of the first medium. Let $\vec{s}'$ be a vector in the direction of the refracted ray, its length being equal to the refractive index n' of the second medium. Let the direction of $\vec{s}$ and $\vec{s}'$ be positive when the vectors point to the right and let $\vec{\sigma}$ be a vector of unit length normal to the refracting surface at the point of incidence and drawn to the right (Fig. 1.3). Then the complete law of refraction can be written as

$$\vec{s} \times \vec{\sigma} = \vec{s}' \times \vec{\sigma} = \vec{\jmath}, \tag{1.6}$$

where the symbol "$\times$" designates the cross or vector product.

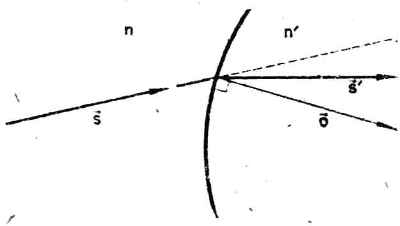

Fig. 1.3. Refraction (vector description).

That this equation represents both parts of the refraction law can be easily seen. The product $\vec{s} \times \vec{\sigma}$ represents a vector normal to the plane of incidence while $\vec{s}' \times \vec{\sigma}$ represents a vector normal to the plane of refraction. The equation states that the two planes coincide, thus fulfilling the first part of the law. Furthermore, the magnitude of $\vec{s}' \times \vec{\sigma}$ is equal to the value of $n \sin i$ and the magnitude of $\vec{s}' \times \vec{\sigma}$ is equal to $n' \sin i'$, and thus the equality of the vector products in Equation (1.6) gives the equation

$$n \sin i = n' \sin i'. \tag{1.7}$$

But this is merely a restatement of Equation (1.1) and thus Equation (1.6) fulfills the second part of the refraction law.

The problem involved in applying the refraction law is to find $\vec{s}$ when $\vec{s}$ and $\vec{\sigma}$ are given. Now (1.6) can be written

$$(\vec{s}' - \vec{s}) \times \vec{\sigma} = 0, \tag{1.8}$$

which means that the vector $(\vec{s}' - \vec{s})$ lies along the direction of $\vec{\sigma}$ since the sine of the angle between $(\vec{s}' - \vec{s})$ and $\vec{\sigma}$ must equal zero. That being the case, it must be possible to multiply the unit vector $\vec{o}$ by some scalar Γ to obtain $(\vec{s}' - \vec{s})$, or

$$\vec{s}' - \vec{s} = \Gamma \vec{o}. \tag{1.9}$$

The quantity Γ is frequently called the *astigmatic constant* in the literature because it was first found in connection with the formulae for tracing astigmatism along a ray (*cf.* Chapter 25). We prefer to call it the *deviation constant.*

By the rules of scalar multiplication, it follows from Equation (1.9) that

$$\Gamma = \vec{o}\vec{s}' - \vec{o}\vec{s} = n' \cos i' - n \cos i. \tag{1.10}$$

Introducing the refraction law,

$$\begin{aligned} \vec{o}\vec{s}' &= n' \cos i' = \sqrt{n'^2 - n'^2 \sin^2 i'} \\ &= \sqrt{n'^2 - n^2 \sin^2 i} = \sqrt{n'^2 - n^2 + n^2 \cos^2 i} \\ &= \sqrt{n'^2 - n^2 + (\vec{o}\vec{s})^2}, \end{aligned} \tag{1.11}$$

where the sign of the root is positive in the case of refraction. Therefore

$$\Gamma = \sqrt{n'^2 - n^2 + (\vec{o}\vec{s})^2} - \vec{o}\vec{s}. \tag{1.12}$$

With Γ found, the direction of the refracted ray can be determined from Equation (1.9) to be

$$\vec{s}' = \vec{s} + \Gamma \vec{o}. \tag{1.13}$$

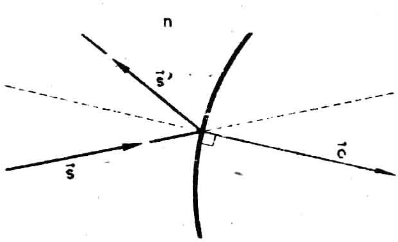

Fig. 1.4. Reflection (vector description).