Search Problems

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SEARCH PROBLEMS

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Preface

Over the past thirty years there has been an increasing number of contributions to the topic of searching, both in theoretically as well as in applications-oriented journals. It is remarkable that very different kinds of problems are classified as search problems and that researchers from various fields often know little about results achieved in areas with which they are not familiar.

In this book we attempt to present the extensive literature in such a way that the reader can quickly understand the range of questions and obtain a survey of them which is as comprehensive as possible.

Our goal was to treat the most important works in the field from the viewpoint of the current state of the science. However, we make no claim at all at completeness since the restraints of this book make such an attempt futile. For some works, which actually deserved to be presented in detail, we therefore settled for presenting the results. The interested researcher will then be in a position to find his own path through the literature. The book should be useful to the expert as a reference.

But our primary concern is to provide to every reader who is willing and able to engage in abstract, formal thought access to the basic ideas, methods and results in the field which have not yet appeared in book form but which deserve broader distribution due to their importance. Part 2 of the book is intended primarily for this broader group of readers. The only requirement here is a solid background in basic mathematics. All proofs are presented in great detail.

Knowledge of elementary probability theory is necessary to understand Part 4. Part 3 requires knowledge equivalent to a basic course in stochastics. In both these Parts the proofs are more concise.

All the individual Parts can be read and used independently for teaching purposes.

We wish to thank Mrs Waltraud Blenski for typing the manuscript.

We thank Beatrix and Christa for having found us.

Bielefeld, May 1979

R. Ahlswede
I. Wegener

Preface to the English Edition

The German edition of the book appeared in 1979. In 1982 the Russian edition was published by MIR. It includes also a supplement, *Information-theory Methods in Search Problems*, which was written by Maljutov and contains primarily those results obtained in the Soviet Union, to which we had no access. There, in particular, fascinating connections between multi-user information theory and the theory of screening design of experiments are emphasized. We think that some readers will appreciate that we included here the references from that supplement.

In recent years some striking improvements of earlier results have been obtained and, even more importantly, new connections of search to other areas and also new types of search problems have come into focus.

Under the heading 'Further reading' we mention articles and books which inform the researcher about new developments and results which seem to carry the seed for further discoveries. This selection certainly reflects the authors' research interests and judgements. However, we feel that an attempt to give some orientation to the reader, even at the risk of being proved wrong at a later time, is more valuable than an encyclopedic collection. It is conceivable that there is important work in progress which escaped our attention. We are grateful for any advice in this and other respects, which could be considered in another edition.

A systematic treatment of all the material referred to would require a major revision and extension of the book. This would not only cause further delay of this English edition, but also would not conform with the mostly elementary character of the book. Among conflicting demands we have kept as our highest priority the readability of the text for a large community. The time for another book will have come when most search problems can be put into the frame of a unified theory of search.

It is a pleasure to thank on this occasion Prof. M. V. Maljutov for his supplement of the Russian Edition. We are indebted to John Wiley & Sons Ltd for including the book in their series and to Dr Spuhler from B.G. Teubner of Stuttgart as well as to Mrs C. Farmer from John Wiley & Sons Ltd for all their help and patience during the course of publication. We also thank Mrs M. Matz for her secretarial assistance.

Finally, we express our gratitude to Mrs J. E. Wotschke for undertaking the tasks of translation and typing with great diligence and devotion.

November 1986

R. Ahlswede
I. Wegener

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PART 1

Introductory Remarks and Definitions

1

Introduction

The following is attributed to Lessing: 'What we are is nothing; what we seek is everything', but we couldn't find it in his collected works. Our intention here is not to philosophize on this remark, anyway. But if we did, we would probably have to modify it to: 'Searching is everything.' We only want to emphasize how important searching is to thinking people.

Without having to think long and hard, anyone can see how much time is spent in a day looking for something. It could be an article in a department store, television programs, what to wear, misplaced items at home or at work, books in the library, a good alibi or even the introduction to a book. Or deeper things, such as the right decision, knowledge, or even one's identity.

Independently of the importance each of us attaches to the various goals of searching and the method of searching, this list has certainly made clear that no one can seriously deny the importance of searching to human existence.

A basic question for the scientist is how good working hypotheses can be found and how to set up experiments so that they can help in making decisions. The theoretician searches for theories and methods of developing them. Anyone who has spent any amount of time on such activities develops a certain feeling that this field of creative endeavor and fantasy is subject to certain rules. Finding these rules is very difficult. It is notable that Descartes never finished his *Regulae ad directionem ingenii* (it appeared fifty years after his death) and Leibniz gave up his plans for a *Kunst des Erfindens* (*Art of Inventing*).

In mathematics, especially, an important task is to make conjectures (working hypotheses) and to prove or disprove them. Every mathematician has available to him a certain reservoir of methods of procedure which he uses consciously and sometimes unconsciously. To what extent can this process be made more transparent? Can rules be established to enable us to find the goal faster? Polya (*How to Solve It*, 1945) made some very interesting contributions to this subject.

These general remarks were meant to increase our awareness of the complexity and range of the topic 'searching'. A cynic might reduce this to the statement, 'A philosopher is someone who knows neither what he is searching for nor how he is searching.' As disciples of Hilbert we cannot be satisfied with that since we are haunted by his voice: 'We must know and we shall know.' And this establishes the scope of this book. For we shall treat only search problems for which we know what our goal is and which search methods are available. Within this framework we look for search strategies which are as good as possible (successful, fast, economical, simple). In Chapter 2 we shall discuss an example to explain this.

We want to use our concepts and classifications to make a contribution to working out the essential, common points of the various search problems. This ought to help the reader to understand more quickly the problems of this nature and get a quick grasp of the most recent results. At the same time, we would like to inspire as many readers as possible to engage in research. By contrasting the various search problems and the methods for their solution, we hope finally to improve the exchange of information between scientists in the various fields. The necessity for this is underscored simply by the fact that certain results are 'discovered' again and again.

We intentionally dispense with a uniform and universally polished theory of searching because this leads to premature conclusions in such a dynamic field and tempts one to leave further developments to the *perpetuum mobile* of the mathematical apparatus. We are interested in maintaining a certain consciousness of the problems and not in preventing controversies on the various assumptions and working methods. That is why we chose 'Search Problems' and not 'The Theory of Searching' for the title of this book.

Once the reader has understood the book, he can throw it away with a clear conscience since he will always be able to find it again. But if he doesn't find it, then maybe someone else who really should read it will. Anyone who has serious trouble understanding the book or parts thereof can seek consolation in Matthew 7, 7.

2

A sample search model

In this chapter we would like to present the search model that we will investigate in Part 2 of this book (with the exception of Chapter 7, Section 3). The more general models which are treated later will be explained in their respective chapters. The search problem which we use to explain the model is investigated in more detail in Chapter 7, Section 2.

In World War II, all Americans drafted into the Army were examined for syphilis. Blood from each man was subjected to the Wasserman test, in which it was tested for antibodies which are found in the blood of people who have syphilis. During this mass examination, people realized that it could be more effective to examine the blood of several men at once. If the joint blood test contained no antibodies, then none of the men examined had syphilis. Otherwise, at least one of the men must have the disease. Now, we are interested in a method with which the set of all sick men can be determined as 'quickly' as possible.

The men to be examined are numbered consecutively $1, \ldots, N$. Each subset of $\{1, \ldots, N\}$ can be the set of sick men. In our search model, we generally assume that there is a finite set of possible results. This set is called the search domain $\mathscr{X}$. In our example, $\mathscr{X}$ is the power set $\mathscr{P}(\{1, \ldots, N\})$ of $\{1, \ldots, N\}$.

What possibilities do we have in this search? In our example, we can subject the blood of the men in A jointly to the Wasserman test for all $A \subseteq \{1, \ldots, N\}$. There are two possible results, depending on whether a person in A is ill or not. In general, we model the actions in a search by error-free tests, t, where t is a mapping of $\mathscr{X}$ into a set R. If $i \in \mathscr{X}$ is being sought, then t gives the result $t(i)$. If, on the other hand, t has the result $r \in R$, then the object being sought must be in $t^{-1}(r)$. The test $t_A : \mathscr{P}(\{1, \ldots, N\}) \to \{0, 1\}$ with $t_A(B) = 1 \Leftrightarrow A \cap B \neq \emptyset$ corresponds to the joint examination of persons in $A \subseteq \{1, \ldots, N\}$. The blood of the men in A contains antibodies (test result 1) if and only if the set B of ill men has a nonempty intersection with the set A of examined men.

How can we plan our search? We can define a sequence of tests $t_1, \ldots, t_m$ where we intend to conduct these tests consecutively. If $i \in \mathscr{X}$ is being sought, then we obtain the result sequence $e(i) = (t_1(i), \ldots, t_m(i))$. If the result sequences $e(i)$ $(i \in \mathscr{X})$ are distinct, then the strategy $s = (t_1, \ldots, t_m)$ is successful since we can, from the result sequences, uniquely arrive at the element being sought. For our example, in the case $N=8$ the strategies

$$s' = (t_{\{1\}}, \ldots, t_{\{8\}}),\ s'' = (t_{\{1,\ldots,8\}}, t_{\{1\}}, \ldots, t_{\{8\}})$$

are obviously successful.

For $i \in \mathscr{X}$, the search time for $s = (t_1, \ldots, t_m)$ is $l(i) = k$ when we can infer from $t_1(i), \ldots, t_k(i)$ but not from $t_1(i), \ldots, t_{k-1}(i)$ that i is being sought. For s', the search time is always 8, while for s'' the search time is 1 when all men are healthy. The search time is 8 when only the 8th man is ill, otherwise the search time is 9.

Our concept of strategy discussed so far turns out to be too specialized in many cases. Before we can conduct the kth test, we already know the results of the first $k-1$ tests. Strategies that use this information are called sequential, while the simple strategies shown above are called nonsequential. Let T be the set of permitted error-free tests and STOP the action to terminate the search. A sequential strategy $s = (s_1, s_2, \ldots)$ looks like this: $s_1 \in T^* := T \cup \{\text{STOP}\}$. If $s_1 = t$, then we begin with test t, while if $s_1 = \text{STOP}$ we immediately terminate the search. If $s_1 = t$ and r is a possible result of t, then $s_2(r) \in T^*$ indicates what we should do next if the first test has the result r. If $s_1, \ldots, s_{k-1}$ are already defined, then s_k ought to be defined on the set of sequences $(r_1, \ldots, r_{k-1})$ of results which are possible in the first $k-1$ tests. This set can, of course, be empty (e.g., when $s_{k-1} \equiv \text{STOP}$). The next action when we have obtained the results $r_1, \ldots, r_{k-1}$ is $s_k(r_1, \ldots, r_{k-1}) \in T^*$.

Obviously, every nonsequential strategy $s = (t_1, \ldots, t_m)$ can be considered to be a sequential strategy: $s_1 \equiv t_1, \ldots, s_m \equiv t_m$, $s_{m+1} \equiv \text{STOP}$. It makes sense to define sequential strategies in such a way that we can stop the search when we have found the object being sought.

For the following successful, sequential strategy s^*, there is no corresponding nonsequential strategy. Let $N=8$. We partition the 8 persons into four pairs $\{1,2\}, \ldots, \{7,8\}$. Next the blood of the persons in a pair is tested jointly. Pairs in whose blood no antibodies were found are healthy. For the other pairs we test the blood of the person with the odd number. If this person is healthy, then the other person of the pair must be ill. Thus, these pairs, too, are classified. For all the other pairs, the blood of the person with the even number is also examined. As an exercise, the reader should compute the search time of s^* if exactly persons 1, 4, 7 and 8 are ill.

Since $\mathscr{X}$ is finite, we can limit ourselves to finite, sequential strategies, i.e., strategies s for which there is a k with $s_k \equiv \text{STOP}$. For $i \in \mathscr{X}$, there is then a unique sequence $e(i) = (e_1(i), \ldots, e_{l(i)}(i))$ of results with $l(i) < k$, where the strategy, s, if i is being sought, is terminated after $l(i)$ tests with the result sequence $e(i)$. s is said to be successful if for each $i \in \mathscr{X}$ no other object yields the result sequence $e(i)$ after $l(i)$ tests. $l(i)$ is then the search time of s for i. The maximum search time of s is $\max\{l(i) \mid i \in \mathscr{X}\}$. Our goal could be to exhibit a sequential or nonsequential successful strategy with the smallest possible maximum search time.

This problem is trivial for our example. If all persons are ill, then we can determine this only after the blood of every single person has been tested. The maximum search time of any strategy is thus at least N. On the other hand, the successful strategy $t_{\{1\}}, \ldots, t_{\{N\}}$ has a maximum search time of N and is therefore optimal.

In the mass screening of the American military, the proportion q of persons with syphilis in the group of people examined could be estimated rather accurately after some time period. In addition, since there was hardly any information on the individual citizens, the following assumptions could be justified: The '*a priori* probability' of having syphilis is the same for everybody and therefore equal to q. Whether a person has syphilis or not is independent of the state of health of the other persons. Therefore, the *a priori* probability that exactly the persons in $A \subset \{1, \ldots, N\}$ have syphilis is $p(A) = q^{|A|}(1-q)^{n-|A|}$. In general, an *a priori* distribution on $\mathscr{X}$ is a mapping $p: \mathscr{X} \to [0,1]$ with $\Sigma_{i \in \mathscr{X}} p(i) = 1$.

If it is possible for a search problem to determine (to estimate) an *a priori* distribution, then $E(s)$, the expected search time of a successful strategy s, can be expressed as $E(s) = \Sigma_{i \in \mathscr{X}} p(i) l(i)$. Our goal now is to find a sequential (nonsequential) successful strategy with a minimal expected search time. For our medical search problem, this task remains unsolved (Chapter 7, Section 2). Thus, we have already presented a problem here which an interested researcher can continue to work on. We would like to add the remark that the strategy s^* given above is better than s' if and only if $q < \frac{3}{2} - \frac{1}{2}\sqrt{5} \approx 0.38$ (Chapter 7, Section 2).

In more general search models, the cardinality of the search domain does not have to be finite. Then the tests can also have more than finitely many results. Often, the test result is not uniquely determined by the $i \in \mathscr{X}$ being sought. The test result can be a random quantity due to the structure of the test, external influences or even human error. In some cases, there are also successful strategies which will identify the object for us with absolute certainty. Otherwise, we have to be satisfied with correctly (or almost correctly) identifying with a high degree of probability the object being sought. The strategies are then compared using other criteria of performance. Finally, different tests can result in different costs. Instead of the search time, we then investigate the search costs. We treat these more general models in the later chapters.