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Nonlinear Evolution Equations – Global Behavior of Solutions

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PREFACE

These four chapters, divided into thirty five one hour lectures, contain the basic material used in my two semester "Seminar Course" at Brown University, during the academic year 1979-1980.

I am particularly endebted to H. Brezis for allowing me to rely in an essential manner on his lecture note book.

I wish to express my thanks to my colleagues of the Mathematics Department and the "Students" of my seminar for the favorable atmosphere in which these notes were written; more especially, I express my gratitude to C. M. Dafermos and W. A. Strauss for their unfailing encouragement during the preparation of this work.

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INTRODUCTION

The main purpose of this book is to survey some recent advances concerning the Cauchy problem, the periodic problem and asymptotic behavior of solutions for some nonlinear "time" -dependent partial differential equations or systems. Thus the equations will involve a privileged variable denoted by t, which generally describes $\mathbb{R}^+$ but can represent something else than physical time.

We take the point of view of topological dynamics which associates to such equations an ordinary differential equation in a Banach space of functions.

In order that this work be more or less self-contained, the nonlinear problems treated here are chosen in such a way that most of the relevant information can be derived through the only use of general principles from Functional Analysis and simple properties of Sobolev spaces of functions in an open subset of $\mathbb{R}^N$. However a long chapter is devoted to the theory of monotonicity in Hilbert space. This theory is illustrated by examples and then used systematically throughout the text.

Emphasis is also made on the method of a priori estimates which governs our strategy to establish the existence of solutions to periodic problems as well as to Cauchy problems.

a) In Chapter A, we give some general ideas concerning existence and uniqueness for the solution of Cauchy problem : since an extensive literature already exists concerning this problem, we do not insist very much on general ideas and our study is centered on semi-linear, quasi autonomous systems. This chapter includes a rather complete study of the dissipative case following closely H.Brezis's book. Examples are given to illustrate every abstract notion, and we close this chapter with the study of special (logarithmic) equations recently introduced by I. Bialynicki-Birula as a model for nonlinear quantum mechanics.

b) In chapter B, concerning the existence of periodic solutions to quasi-autonomous systems, we study with some detail the linear and dissipative cases. The nonlinear, non dissipative case is merely outlined through typical examples.

c) Chapter C is intended to include the main novelties of this book. After recalling some basic features of almost-periodic functions, we mention the results of Dafermos-Slemrod on autonomous dissipative systems. In a second part, we study the asymptotic behavior of quasi-autonomous dissipative periodic systems. Some proofs here are quite technical and make an essential use of the monotonicity theory (Chapter A).

Examples are given in the autonomous and the quasi-autonomous case, especially the hyperbolic case which turns out to be much harder than the parabolic one. Most of the convergence results could be generalized to the case of an almost periodic forcing term, but the <u>existence</u> of almost-periodic solutions still remains unknown in most cases (Cf.Chapter D).

d) Chapter D is devoted to recent, somewhat technical developpements in the field of asymptotic behavior for quasi-autonomous dissipative systems. Several results mentionned here did not appear previously in the literature.

A good knowledge of elementary Banach space theory and some previous acquaintance with Cauchy problem in nonlinear partial differential equations are suitable to read this book. We hope that the reader, far from being afraid by the technical appearance of some chapters, will discover for himself that very few, simple principles lead naturally to the methods developped here. Then this book can play its anticipated role of providing a unified understanding of some classical problems in nonlinear partial differential equations.

TABLE OF CONTENTS

A. THE CAUCHY PROBLEM

I. Generalities and Local Theory

Lecture 1: Generalities, the continuous and linear cases

1. Let X be a real Banach space. Throughout this course we shall be concerned with operators "in X", defined on part of the space X and which may be multivalued at particular points of the domain where they are defined.

The most convenient way of defining such operators is to identify them with a graph

$$\mathscr{G} = G(A) \subset X \times X.$$

For $x \in X$, we then define:

$$Ax = \{y \in X,\ (x,y) \in G(A)\}.$$

And the domain of A is denoted by $D(A)$

$$D(A) = \{x \in X,\ Ax \neq \phi\}$$

The range of A is denoted by $R(A)$

$$R(A) = \bigcup_{x \in X} \{Ax\}.$$

We define A^{-1} by the formula: $(x,y) \in G(A^{-1}) \iff (y,x) \in G[A]$.

It is immediate to check that $R(A) = D(A^{-1})$ and then $R(A^{-1}) = D((A^{-1})^{-1}) = D(A)$.

If A_1 and A_2 are two operators, their sum $A_1 + A_2$ is defined by the formulas

$$D(A_1+A_2) = D(A_1) \cap D(A_2)$$

and

$$(A_1+A_2)(x) = A_1(x) + A_2(x), \quad \forall x \in D(A_1+A_2)$$

$$= \bigcup_{z_1 \in A_1 x, z_2 \in A_2 x} \{z_1+z_2\}.$$

When $\lambda \in \mathbb{R}$, the operator λA is defined in the obvious way. We recall that a function $f: [0,T] \to X$ is said "absolutely continuous" if there exists $g \in L^1([0,T])$ such that

$$\forall (s,t) \in [0,T], \ s \leq t \Rightarrow ||f(t) - f(s)|| \leq \int_s^t g(\sigma)d\sigma.$$

If X is reflexive, every function which is absolutely continuous is differentiable at almost every point of $[0,T]$. For proof, cf. for example [3], Appendix.

Now let X be reflexive, and consider $(A(t))_{t\in[0,T]}$ a family of operators depending on the parameter t. We suppose that $A(t)$ depends "mildly" on t and define a <u>strong</u> solution of the abstract evolution equation:

(1) $$\frac{du}{dt} = A(t)u(t)$$

as a function $u \in C(0,T;X)$ such that

1) u is absolutely continuous on $[\varepsilon,T-\varepsilon]$, $\forall\varepsilon > 0$

2) $u(t) \in \dot{D}(A(t))$ for almost every $t \in [0,T]$

3) $\frac{du}{dt} \in A(t)u(t)$ a.e. on $[0,T]$.

Notice that for $u_0 \in X$, it is generally impossible to find a solution u of (1) such that:

$$u(0) = u_0. \tag{2}$$

For instance, if $A(t) \equiv A_0$ with $G(A_0) = \phi$, for none of $u_0 \in X$ can the equation (1), (2) be solved.

The system (1), (2) is called the Cauchy problem on $[0,T]$ for equation (1) with initial data u_0.

When $A(t) \equiv A$, we say that (1) is autonomous.

When $A(t) \equiv A + f(t)$, we speak of a quasi-autonomous equation.

2. The continuous case

We now recall some classical results when $A(t)x = f(t,x)$, with f continuous on the product space $[0,T] \times X$.

In this case, we can get local existence theorems relying on fixed-point arguments. Thus, we first recall two well-known fixed point results.

Theorem 1 (Picard) V Banach space, $F \subset V$ closed.

$$\mathscr{T}: F \to F$$

such that

$$\exists k < 1, \ ||\mathcal{T}u - \mathcal{T}v|| \leq k||u-v||$$

$$\forall (u,v) \in F \times F.$$

Conclusion: $\exists ! \ u \in F, \quad \mathcal{T}u = u.$

Theorem 2 (Leray-Schauder) We assume that F is bounded, closed and convex, $\mathcal{T}$ is continuous and maps F into a precompact subset of F.

Conclusion: $\exists u \in F, \quad \mathcal{T}u = u.$

From these fixed point results we deduce two existence theorems.

Theorem 3. f continuous on $[0,T] \times X$ and locally lipschitzian with respect to x, uniformly with respect to t. For every $t_0 \in [0,T[$ and each $\phi \in X$, there exists $\delta > 0$ and a unique strong solution on $[t_0, t_0+\delta]$ of the Cauchy problem:

$$\frac{du}{dt} = f(t,u(t)), \quad u(t_0) = \phi.$$

Theorem 4. $X = \mathbb{R}^N$, f continuous on $[0,T] \times X$. Same existence result, but u need not be unique.

Example. $f(t,x) = |x|^{1/2}$ with $X = \mathbb{R}$. Two solutions $u_1(t) = 0$, $u_2(t) = \frac{t^2}{4}$ such that $u(0) = 0$.

Method of Proof: We look for a solution of the integral equation

$$u(t) = \phi + \int_{t_0}^{t} f(s,u(s))ds.$$

Setting $V_\delta = C([t_0,t_0+\delta],X)$, we may consider

$$F = \{u \in V_\delta, \ ||u-\phi||_{V_\delta} \leq \alpha\}.$$

For α small enough, we have

$$\sup_{\substack{s\in[0,T] \\ ||u-\phi||_X\leq\alpha}} ||f(s,u)|| \leq M < +\infty.$$

Introducing $(\mathcal{T} v)(t) = \phi + \int_{t_0}^{t} f(s,v(s))ds$, we therefore see that if $M\cdot\delta \leq \alpha$, $\mathcal{T}$ maps F into F. Furthermore, if α and δ are small enough under the hypotheses of Theorem 3, $\mathcal{T}$ is a strict contraction from F to F. Under the hypothesis of Theorem 4, Ascoli's theorem gives the necessary compactness property.

Remark 5. In the conclusions of Theorems 3-4, one may add that $u \in C^1(t_0,t_0+\delta;X)$.

Remark 6. If f is lipschitzian in X, uniformly with respect to t, we can take $t_0 = 0$ and $\delta = T$. This fact will be proved later (Lecture 4).

It is not true in general for non-lipschitzian f. For example, the solution of the equation in $X = \mathbb{R}$

$$\frac{du}{dt} = u^2 \quad \text{for} \quad u_0 > 0$$

is given by $u(t) = \dfrac{1}{\frac{1}{u_0} - t}$, which "blows up" for $t = \dfrac{1}{u_0}$. For instance if $u_0 = 1$, $T = 2$, it is impossible to solve (1), (2) on $[0,T]$.

3. <u>Linear semi-groups</u>

When $X = \mathbb{R}^N$ and $A(t) = A$ linear, the Remark 6 provides the existence and uniqueness of a solution of (1), (2) with T arbitrarily large. This solution is well-known to be given by the formula:

$$u(t) = \exp(tA)u_0.$$

Setting $T(t) = \exp(tA)$ for $t \geq 0$, we have the following properties

$$(3) \quad \begin{cases} T(0) = I \\ T(t+s) = T(t) \circ T(s), & \forall (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ ||T(t)|| \leq e^{t||A||}, & \forall t \geq 0. \end{cases}$$

When $\dim X = +\infty$, it is very important to solve some equations where A is linear, single valued but <u>unbounded</u>. In many cases of interest, it can be solved by a formula $u(t) = T(t)u_0$ where the family $T(t)$ has properties analogous to (3). More precisely, we shall say that a single valued linear operator $L: X \to X$ is a generator in X if the equation:

$$\frac{du}{dt} = Lu(t), \quad u(0) = u_0 \in D(L)$$

has a unique strong solution for $t \geq 0$ of the form $u(t) = T(t)u_0$, and $T(t)$ extends on $X = \overline{D(L)}$ by a continuous operator such that:

1) $T(0) = I$
2) $T(t+s) = T(t) \circ T(s), \quad \forall (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$
3) $||T(t)|| \leq Me^{\omega t}, \quad \forall t \geq 0.$

There exists a complete characterisation of generators called the Hille-Yosida theorem (cf. [6], p. 624). When L is a generator and X = Hilbert space, L can be constructed from $T(t)$ by the following formulas:

$$\begin{cases} D(L) = \{x \in X, \ \overline{\lim_{t\to 0}} \dfrac{|x-T(t)x|}{t} < +\infty\} \\ \\ Lx = \lim\limits_{\substack{t\to 0 \\ t>0}} \dfrac{T(t)x-x}{t} \quad \text{if } x \in D(L) \end{cases}$$

moreover, if $u_0 \in D(L)$, then $u(t) = T(t)u_0 \in C^1(0,T;X)$.

In the Hilbert framework, it is possible to get convenient sufficient conditions for L to be a generator.