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G. T. Herman A. K. Louis F. Natterer (Eds.)

Mathematical Methods in Tomography

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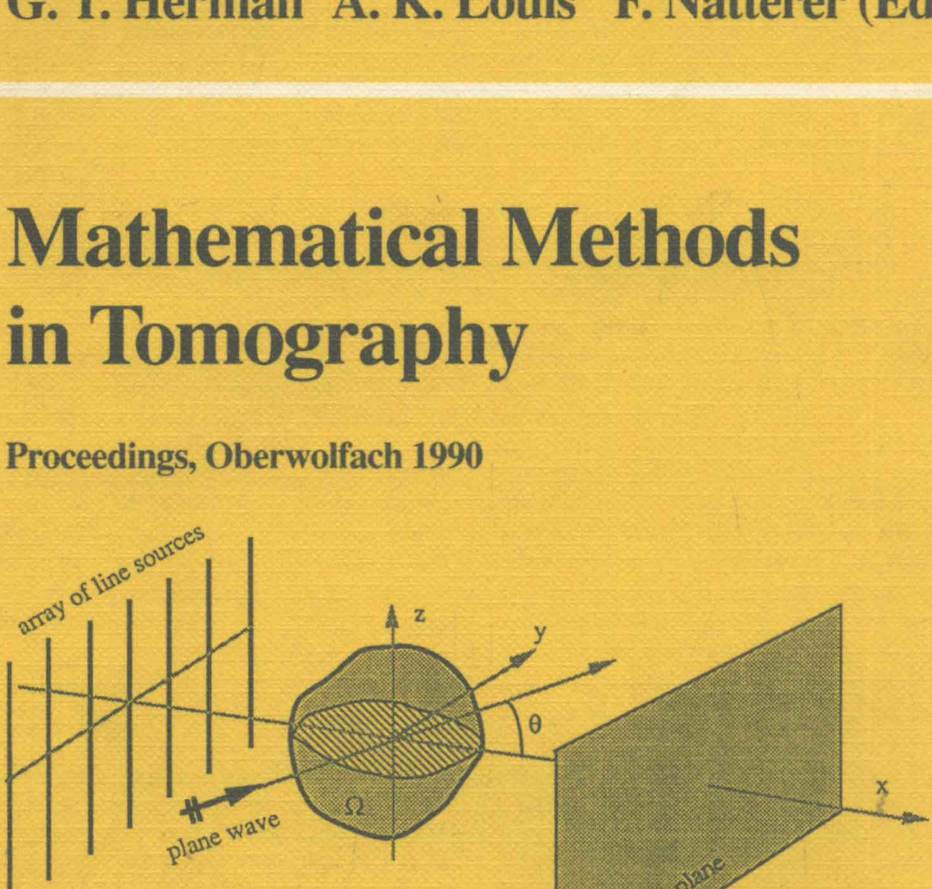

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PREFACE

The word *tomography* means "a technique for making a picture of a plane section of a three-dimensional body." For example, in x-ray computerized tomography (CT), the picture in question is a representation of the distribution in the section of the body of a physical property called x-ray attenuation (which is closely related to density). In CT, data are gathered by using multiple pairs of locations (in the plane of the section) of an x-ray source and an x-ray detector (with the body between them) and measuring for each pair of locations the total attenuation of x-rays between the source and the detector. Mathematically speaking, each measurement provides an approximate value of one sample of the Radon transform of the distribution. (Roughly, the Radon transform of a two-dimensional distribution is a function which associates with any line in the plane the integral of the distribution along that line.) Therefore, the mathematical essence of CT is the *reconstruction* of a distribution from samples of its Radon transform.

Over the years, "tomography" became to be used in a wider sense, namely for any technique of reconstructing internal structures in a body from data collected by detectors (sensitive to some sort of energy) outside the body. Tomography is of interest to many disciplines: physicists, engineers, bioscientists, physicians, and others concern themselves with various aspects of the underlying principles, of equipment design, and of medical or other applications. Mathematics clearly enters in the field where inversion methods are needed to be developed for the various modes of data collection, but we also find mathematicians busily working in many other diverse aspects of tomography, from the theoretical to the applied.

This book contains articles based on selected lectures delivered at the August 1990 conference on Mathematical Methods in Tomography held at the Mathematisches Forschungsinstitut (Mathematical Research Center) at Oberwolfach, Germany. The aim of the conference was to bring together researchers whose interests range from the abstract theory of Radon transforms to the diverse applications of tomography. This was the third such conference at Oberwolfach, with the first one held just over ten years earlier [5]. Much has happened in the field of tomography since that first meeting, including the appearance of a number of books (such as [3, 2, 7, 6]) and many special issues (including [4, 1]).

Oberwolfach is a remote place in the Black Forest with excellent conference and housing facilities. Mathematical conferences of one week duration take place there nearly every week.

Participation is by invitation only and there are usually 20–60 participants. The meeting on which this book is based had 41 participants; seventeen came from the USA, seven from Germany, five from France, two from each of Brazil, Sweden, and USSR, and one from each of Belgium, Hungary, Israel, Italy, Japan, and the Netherlands. The lively international scientific atmosphere of the conference resulted in many stimulating discussions; some of which are reflected in the papers that follow.

Thus our book is a collection of research papers reporting on the current work of the participants of the 1990 Oberwolfach conference. Those desiring to obtain an overview of tomography, or even only of its mathematical aspects, would be better served by looking at the already cited literature [3, 2, 4, 7, 6]. However, the collection that follows complements this literature by presenting to the reader the current research of some of the leading workers in the field.

We have organized the articles in the book into a number of sections according to their main topics. The section entitled *Theoretical Aspects* contains papers of essentially mathematical nature. There are articles on Helgason's support theorem and on singular value decomposition for Radon transforms, on tomography in the context of Hilbert space, on a problem of integral geometry, and on inverting three-dimensional ray transforms. The section entitled *Medical Imaging Techniques* is devoted to the mathematical treatment of problems which arise out of trying to do tomography with data collected using various energies and/or geometrical arrangements of sources and detectors. Thus, there are articles on using backscattered photons, on cone-beam 3D reconstruction, and on tomography for diffraction, for diffusion, for scattering, and on biomagnetic imaging. The section on *Inverse Problems and Optimization* discusses mainly the algorithmic aspects of inversion approaches for tomographic data collections. This section contains articles on various formulations of the inverse problems in terms of optimization theory, as well as on iterative approaches for solving the problems. Finally, the section on *Applications* contains articles that have been closely motivated by some practical aspect of tomography; for example, on the determination of density of an aerosol, on nondestructive testing of rockets, and on evaluating the efficacy of reconstruction methods for specific tasks.

Finally, we would like to thank Prof. Martin Barner, the director of the Mathematisches Forschungsinstitut, and his splendid staff for providing us with all the help and just the right ambiance for a successful mathematical conference. We are also grateful to Springer-Verlag for their kind cooperation in publishing this volume.

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Helgason's support theorem for Radon transforms – a new proof and a generalization

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1. Denote by Rf the Radon transform of the function f, i. e. $Rf(H) = \int_H f ds_H$ for, say, continuous functions f on $\mathbb{R}^n$ that decay at least as $|x|^{-n}$ as $|x| \to \infty$, and $H \in G_n$, the set of hyperplanes in $\mathbb{R}^n$; the surface measure on H is denoted ds_H. The well-known support theorem of Helgason [**He1**], [**He2**] states that if $Rf(H) = 0$ for all H not intersecting the compact convex set K and $f(x) = O(|x|^{-m})$ as $|x| \to \infty$ for all m, then f must vanish outside K. In [**He3**] Helgason extended this theorem to the case of Riemannian manifolds with constant negative curvature. Helgason's proofs depend in an essential way on the strong symmetry properties of the Radon transform. Here we will extend the theorem just cited to the case when a real analytic weight function depending on H as well as x is allowed in the definition of the Radon transform (see below), a situation without symmetry. For C^∞ weight functions analogous theorems are not true in general [**B2**]. Our method depends on the microlocal regularity properties of the Radon transform, a method we have already used in [**BQ1**] and [**BQ2**]. The case when f is assumed to have compact support was considered in [**BQ1**]. For rotation invariant (not necessarily real analytic) Radon transforms support theorems were given by E. T. Quinto [**Q3**].

2. Let $\rho = \rho(x, H)$ be a smooth function on the set Z of all pairs (x, H) of $H \in G_n$ and $x \in H$. We define the generalized Radon transform R_ρ by

$$R_\rho f(H) = \int_H f(\cdot)\rho(\cdot, H) ds_H, \quad H \in G_n.$$

If ρ is constant, R_ρ is of course the classical Radon transform. To describe our assumptions on ρ we shall consider $\mathbb{R}^n$ as sitting inside the projective space $\mathbb{P}^n(\mathbb{R})$ in the usual way:

$$\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto (1, x_1, \ldots, x_n) \in \mathbb{P}^n(\mathbb{R}). \tag{1}$$

The manifold Z then becomes imbedded in the manifold

$$\tilde{Z} = \{(x, H); x \in H,\ H \text{ hyperplane in } \mathbb{P}^n(\mathbb{R})\}.$$

Note that $\mathbb{P}^n(\mathbb{R})$ and $\tilde{Z}$ are compact real analytic manifolds. Our assumption will be that ρ can be extended to a positive real analytic function on $\tilde{Z}$. This assumption is of course fulfilled for the constant function, which is the case considered by Helgason in [**He1**].

THEOREM. *Assume ρ is a positive, real analytic function on Z that can be extended to a positive, real analytic function on $\tilde{Z}$. Let K be a compact, convex subset of $\mathbb{R}^n$. Let f be continuous on $\mathbb{R}^n$ and*

$$f(x) = O(|x|^{-m}) \quad \text{as} \quad |x| \to \infty \tag{2}$$

for all m, and assume $R_\rho f(H) = 0$ for all H disjoint from K. Then $f = 0$ outside K.

As pointed out by Helgason, the assumption that f tends to zero rapidly as $|x| \to \infty$ cannot be omitted, even in the case of constant ρ.

The assumption that ρ is analytic at infinity cannot be omitted, even if the decay assumption is considerably strengthened. In fact, using ideas from **[B2]** one can construct examples where ρ is real analytic on Z, $1 \le \rho \le C$, f not identically zero, $R_\rho f = 0$, and, for instance, $f(x) = \mathcal{O}(\exp(-e^{|x|}))$ as $|x| \to \infty$.

The approach adopted here is suggested by the following facts. First, if ρ is analytic and different from zero, it is known that any solution f to $R_\rho f = 0$ must be real analytic, hence vanish if it vanishes in some open set **[B1]**, **[BQ1]**. Second, if one could prove that f, considered as a function on the projective space $\mathbb{P}^n(\mathbb{R})$, must be analytic at infinity, then the assumption that f decays rapidly at infinity would imply that f vanishes identically. Third, the examples showing that the decay assumption cannot be omitted, in two dimensions the functions $\mathrm{Re}(x_1 + ix_2)^{-m}$, are in fact analytic at infinity. An advantage with this approach is, in addition to the fact that the weight function ρ is allowed to be non-constant, that the role of the decay assumption on f is "explained".

We prove the theorem by considering R_ρ as an operator on functions on $\mathbb{P}^n(\mathbb{R})$. The crucial fact is that f, considered as a function on $\mathbb{P}^n(\mathbb{R})$, must have a certain regularity property along the hyperplane at infinity, H_∞; in precise terms, the conormal manifold to H_∞, $N^*(H_\infty)$, must be disjoint from the analytic wavefront set of f (Proposition 1).

3. We are now going to express R_ρ in terms of a Radon transform on $\mathbb{P}^n(\mathbb{R})$. For this purpose we need to introduce some more notation. Set $X = \mathbb{R}^n$, $Y = G_n$, denote $\mathbb{P}^n(\mathbb{R})$ by $\tilde{X}$ and the set of hyperplanes in $\mathbb{P}^n(\mathbb{R})$ by $\tilde{Y}$. Denote the map (1) from X to $\tilde{X}$ by α. This map induces maps $Y \to \tilde{Y}$ and $X \times Y \to \tilde{X} \times \tilde{Y}$, which we will also denote by α. As models for $\tilde{X}$ and $\tilde{Y}$ we shall use the unit sphere S^n with opposite points identified. Thus a model for $\tilde{Z} \subset \tilde{X} \times \tilde{Y}$ will consist of all pairs $(u, \omega) \in S^n \times S^n$ such that $u \cdot \omega = 0$, all four pairs $(\pm u, \pm\omega)$ identified. On the plane $L(\omega) = \{u;\ u \cdot \omega = 0\}$ we choose the measure ds_L equal to the (push-forward of) $n - 1$-dimensional surface measure on S^n. We use the notation $\omega = (\omega_0, \omega')$, and we note that the plane at infinity, L_∞, is represented by $\omega = (\pm 1, 0, \dots, 0)$.

Examples of functions ρ satisfying the hypothesis of the theorem can easily be constructed as follows. Let $a(z, \omega)$ be real analytic and positive on $\{(z, \omega); z \cdot \omega = 0\} \subset \mathbb{R}^{2n+2}$, even and homogeneous of degree zero in both variables (separately), let $H_{\theta,p}$ be the plane $x \cdot \theta = p$, $\theta \in S^{n-1}$, and set

$$\rho(x, H_{\theta,p}) = a(1, x, -p, \theta), \tag{3}$$

for $x \in H_{\theta,p}$. Then a (restricted to $S^n \times S^n$) represents the extension, $\tilde{\rho}$, of ρ. Conversely, every ρ satisfying our assumptions can obviously be represented in the form (3).

Let τ be a positive real analytic function on $\tilde{Z}$. For continuous functions g on $\tilde{X}$ we define the generalized Radon transform $\tilde{R}_\tau$ by

$$\tilde{R}_\tau(g)(L) = \int_L g(\cdot)\tau(\cdot, L)\, ds_L, \quad L \in \tilde{Y}.$$

If f is a function on X, sufficiently small at infinity, $\tilde{f} = f \circ \alpha^{-1}$, $\tilde{\rho} = \rho \circ \alpha^{-1}$, and $L = \alpha(H)$, then

$$R_\rho f(H) = \int_H f(\cdot)\rho(\cdot, H)\, ds_H = \int_L \tilde{f}(\cdot)\tilde{\rho}(\cdot, L)\, \alpha_*(ds_H)$$

$$= \int_L \tilde{f}(\cdot)\tilde{\rho}(\cdot, L) b(\cdot, L)\, ds_L; \tag{4}$$

here $\alpha_*(ds_H) = b(u, L) ds_L$ is the push-forward of the measure ds_H under α. It is very important for us that the density $b(u, L)$ can be factored, $b(u, L) = b_0(u) b_1(L)$, into a function depending only on u and one depending only on the plane L. This fact is well known; see e. g. [**GGG**], pp. 64-66.

LEMMA 1. *The measure $\alpha_*(ds_H)$ is equal to $b(u, L)\, ds_L$, where*

$$b(u, \omega) = b(u, L(\omega)) = c|u_0|^{-n}\sqrt{1 - \omega_0^2} = c|u_0|^{-n}|\omega'|, \quad u_0 \neq 0, \quad |\omega'| \neq 0. \tag{5}$$

for some positive constant c.

Formula (5) shows that the measure $\alpha_*(ds_H)$ has a strong singularity along the plane at infinity. However, the fact that the density function $b(u, L)$ factors as expressed by (5), $b(u, L) = b_0(u) b_1(L)$, implies that this singularity is harmless in our context. In fact, using (4) and (5) we can write

$$R_\rho f(H) = b_1(L) \int_L \tilde{f}(u) b_0(u) \tilde{\rho}(u, L)\, ds_L = b_1(L)(\tilde{R}_\tau g)(L),$$

where $g(u) = \tilde{f}(u) b_0(u) = \tilde{f}(u)|u_0|^{-n}$, and $\tau(u, L) = \tilde{\rho}(u, L)$. Note that $g(u)$ tends to zero as $u_0 \to 0$, since f is rapidly decreasing; hence g is extendible to a continuous function on all of $\tilde{X}$, which vanishes on L_∞. Thus, in particular, if $R_\rho f(H) = 0$ for all H not intersecting K, then $\tilde{R}_\tau g$ must vanish in some neighbourhood of L_∞.

4. We will now turn our attention to the microlocal regularity properties of solutions to the equation $\tilde{R}_\tau g = 0$. The result that we shall need, Proposition 1 below, is well known, but it is not easy to find it in the literature. The analogous statement for the C^∞ category is clearly contained in the very general theory in [**H1**] as well as in [**GS**]. The additional arguments needed for the real analytic case can be found for instance in [**Bj**], ch. 4. Generalized Radon transforms as Fourier integral operators are discussed in [**GS**]; see also [**GU**] and [**Q2**]. In particular, Radon transforms on projective spaces are considered in [**Q1**]. For definition and basic properties of the analytic wavefront set, see [**H2**], ch. 8. If E is a smooth submanifold of the manifold M, one denotes by $N^*(E)$ the conormal manifold to E, that is, the set of all $(x, \xi) \in T^*(M) \setminus 0$ such that $x \in E$ and ξ is conormal to the tangent space to E at x.

PROPOSITION 1. *Assume $\tau(u, L)$ is real analytic and positive on $\tilde{Z}$ and that*

$$\tilde{R}_\tau g(L) = 0$$

for all L in some neighbourhood of $L_0 \in \tilde{Y}$. Then

$$N^*(L_0) \cap \mathrm{WF_A}(g) = \emptyset.$$

We finally need the following lemma, related to an important theorem of Hörmander, Kawai, and Kashiwara ([**H2**], Theorem 8.5.6.).

LEMMA 2. *Let S be the spherical surface $\{x;\ |x| = 1\}$ in $\mathbb{R}^m$, and let f be continuous in some neighbourhood of S. Assume*

$$f(x) = O((|x| - 1)^N) \quad \text{as} \quad |x| \to 1 + 0 \quad \textit{for all} \quad N, \tag{6}$$

(or as $|x| \to 1 - 0$), and

$$N^*(S) \cap \mathrm{WF_A}(f) = \emptyset. \tag{7}$$

Then $f = 0$ in some neighbourhood of S.

PROOF: Let S_t be the sphere with radius t and define the function h by

$$h(t) = \int_{S_t} f\, ds,$$

where ds is surface measure on S_t. By basic facts about the wavefront set it follows from (7) that none of the elements above $t = 1$ can be contained in $\mathrm{WF_A}(h)$, i. e. h is analytic at $t = 1$. But (6) implies that h is rapidly decreasing as $t \to 1 + 0$. Hence h must vanish in some neighbourhood of $t = 1$.

Let $\varphi(x)$ be any real analytic function defined on a neighbourhood of S. Multiplying f by φ clearly preserves the properties (6) and (7). Applying our reasoning to φf we conclude that

$$\int_{S_t} \varphi f\, ds = 0$$

for t near 1 and for all bounded and analytic functions φ. This implies that $f = 0$ in some neighbourhood of S. The lemma is proved.

PROOF OF THE THEOREM: Let f be a function satisfying the hypotheses of the theorem, and consider again the function g on $\tilde{X}$ defined by

$$g(u) = |u_0|^{-n} \tilde{f}(u) = |u_0|^{-n} f(\alpha^{-1}(u)).$$

We have seen that $\tilde{R}_\tau g(L)$ must vanish for L near L_∞. But then Proposition 1 implies that

$$N^*(L_\infty) \cap \mathrm{WF_A}(g) = \emptyset. \tag{8}$$

Now we want to use Lemma 2 to infer that g vanishes near L_∞. The fact that L_∞, considered as a hypersurface in $\tilde{X}$, is non-orientable is a slight inconvenience for us; we therefore move up to S^n, the double cover of $\tilde{X}$. We will use the same notation on S^n as on $\tilde{X}$, so that points will be denoted by u and the function g pulled back to S^n will again be denoted g; this will cause no confusion. Thus g is an even function defined in a neighbourhood of the equator $\Sigma = \{u;\ u_0 = 0\} \subset S^n$. It is clear that (8) holds with Σ in place of L_∞. Now, the stereographic map takes S^n with the north pole removed into $\mathbb{R}^n$ and Σ into an $n-1$-sphere in $\mathbb{R}^n$. An application of Lemma 2 now proves that $g = 0$ in a neighbourhood of Σ, hence $f = 0$ outside some compact set in $\mathbb{R}^n$. An application of the theorem in [**BQ1**] therefore finishes the proof. We prefer, however, to complete the proof with another application of the arguments already used here. Since a compact, convex set is equal to the intersection of all closed balls that contain it, we may assume that K is a ball. We may also assume that its center is the origin; let R be its radius. Let S_r be the sphere with radius r, centered at the origin, and let r_0 be the smallest r for which $f = 0$ outside S_r. Assume $r_0 > R$. Applying Proposition 1 (or the analogous statement for the Radon transform R_ρ on $\mathbb{R}^n$) to all tangentplanes to S_{r_0} we find that $N^*(S_{r_0}) \cap \mathrm{WF_A}(f)$ must be empty. Lemma 2 now shows that f must vanish in some neighbourhood of S_{r_0}. This contradicts the definition of r_0, hence the proof is complete.

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Singular Value Decompositions for Radon Transforms

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1 Introduction

In this paper simple techniques are developed for the construction of singular value decompositions (SVD) for rotationally invariant Radon transforms in euclidean spaces. First we introduce the definition of a SVD.

Definition 1. Let A be a linear operator between (separable) Hilbert spaces X, Y

$$A : X \rightarrow Y \quad .$$

The triple $\{u_n, v_n, \sigma_n\}_{n\geq 0}$ is called a **Singular Value Decomposition (SVD)** of the operator A if
$\{u_n\}_{n\geq 1}$ is a complete orthonormal system in X,
$\{v_n\}_{n\geq 1}$ is an orthonormal system in Y,
$\{\sigma_n\}$ is a set of non-negative real numbers,

$$Au_n = \sigma_n v_n \quad and \quad A^* v_n = \sigma_n u_n \quad .$$

The singular values σ_n are usually ordered such that $\sigma_1 \geq \sigma_2 \geq .. \geq 0$.

Sometimes one refers to the singular functions $\{u_n\}$ as generalized Eigenfunctions, since

$$(A^* A)u_n = \sigma_n^2 u_n \quad .$$

The importance of a SVD comes form its ability to express the action of A by orthogonal series expansions, $f = \sum f_n u_n$ with $f_n =< f, u_n >_X$:

$$Af = A(\sum f_n u_n) = \sum f_n \sigma_n v_n \quad .$$

From this one immediately obtains an inversion formula, range characterizations and results on the ill posedness of the inverse problem.

SVD's for Radon transforms have a long history in computerized tomography. The early results of Marr [6] and Cormack [2] show that the 2-dimensional Radon transform maps a set of certain orthogonal functions to products $U_m(s)e^{il\omega}$ of Tchebycheff polynomials with trigonometric functions. Subsequently these results have been extended to arbitrary dimensions, weighted L_2-spaces, functions of unbounded support and Radon transforms modelling limited data problems in tomography, see Perry [8], Quinto [9], Louis [3,4], Maass [5]. Quite different techniques have been used in these papers. Our aim is to demonstrate two elementary and general ways of obtaining SVD's for Radon transform with rotational symmetry, e.g. for full, exterior and interior Radon transforms.

It will be more convenient to work with the adjoint of the Radon transform, the backprojection operator, and to translate the results to the original Radon transform at the end. The next section recalls the invariance of the spherical harmonics under the action of the backprojection operator for rotationally invariant Radon transforms. This reduces the original problem posed for the n-dimensional Radon transform to finding SVD's for 1-dimensional integral operators.

Even in a simple construction of a SVD we must be able to define the sets of orthogonal functions $\{u_n\}$ and $\{v_n\}$. Since they are built from special functions, e.g. orthogonal polynomials, they are usually defined by either 3-term recurrence relations or differential equations.

Chapter 3 starts with the definition of the functions $\{u_n\}$ as solutions of differential equations $D_1 u_n = 0$. From there a second differential operator D_2, intertwining with the action of the adjoint Radon transform, is constructed. This determines the images of $\{u_n\}$ as solutions of differential equations and immediately gives the desired orthogonality properties. The procedure is examplified by carrying out the computations for the full Radon transform.

At the beginning of Chapter 4 the functions $\{u_n\}$ are introduced by 3-term recurrence relations. The corresponding recurrence relations for the functions $\{v_n\}$ can be obtained in a surprisingly simple way. Examples include a SVD for the interior Radon transform.

2 The adjoint operator $\mathcal{R}^*$

We use the standard notation for the Radon transform as a map between weighted L_2-spaces, i.e. a real valued function f is mapped to its integrals over hyperplanes, which are parametrized by a normal unit vector ω and the signed distance s from the origin. The function f may either be defined on $\mathbb{R}^n$ or on a subset, e.g. the unit ball. For the full Radon transform, the integrals over all hyperplanes are measured. For limited data problems, the integrals over some hyperplanes are missing.

$$\begin{aligned} \mathcal{R} \;&: L_2(\Omega, W^{-1}) \rightarrow L_2(Z, w^{-1}) \\ f \;&\mapsto \int_{\omega^\perp} f(s\omega + y)\, dy \quad . \end{aligned}$$

There are many meaningful choices for Ω and Z. We are interested in Radon transforms where the set of accessible hyperplanes is rotationally invariant, i.e.

$$\Omega = \mathbb{R}^n \quad or \quad \Omega = B_a := \{\, x \in \mathbb{R}^n \mid \| x \| \leq a \,\}$$

$$and \;\; \mathbf{Z} = S^{n-1} \times I \quad , I \subset \mathbb{R} \quad .$$

The following cases are of particular importance.

$\Omega = I\!R^n \quad , \quad I = I\!R$:	the full Radon transform for functions with unbounded support,
$\Omega = B_1 \quad , \quad I = [-1,1]$:	the full Radon transform for functions with compact support,
$\Omega = I\!R^n \quad , \quad I = \{s \in I\!R \mid \mid s \mid \geq 1\}$:	the exterior Radon transform,
$\Omega = I\!R^n \quad , \quad I = [-1,1]$:	the interior Radon transform .

Since Ω, Z are both invariant under rotations it is natural to assume that the weight functions W, resp. w, only depend on $|x|$, resp. s. It will be more convenient to work with the adjoint operator. Obviously a SVD for the adjoint operator also gives a SVD for the operator itself.
The adjoint Radon transform is a continuous linear operator hence we can compute the transform of g for each term in the sum separately. The following Lemma states the well known property that spherical harmonics are invariant under the action of $\mathcal{R}^*$.

Lemma 2. *Let* $g_l \in L_2(I, w)$ *then*

$$\mathcal{R}^*(wg_lY_l)(x) = |S^{n-2}| \ W(|x|)f_l(| \ x \ |) \ Y_l(\frac{x}{| \ x \ |})$$

where

$$f_l(r) = \int_{-1}^{1} g_l(rs)(w_\nu C_l^\nu)(s)ds \quad ,$$

the integral extending over those values of s *where* $g_l(| \ x \ | \ s)$ *is defined. Here* $(w_\nu C_l^\nu)$ *is the weighted Gegenbauer polynomial with* $\nu = (n-2)/2$.

This Lemma is a direct consequence of the Funk-Hecke Theorem, e.g. see [7]. We will call T_l the radial operators of the Radon transform. The orthogonality relations for spherical harmonics of different degree reduce the problem of finding a SVD for $\mathcal{R}^*$ to the construction of SVDs for the integral opertors transforming the radial parts in the expansion $g = \sum wg_lY_l$. The weight function W will be choosen appropriately later and we ignore the constant factor $| \ S^{n-2} \ |$. The radial tranforms are then given by

$$(T_lg)(r) := \int_{-1}^{1} g(rt)(w_\nu C_l^\nu)(t)dt \quad .$$

Again the integration extends only over those values of s where the integrand is defined. For the interior Radon transform this yields

$$(T_lg)(r) := \int_{max(-1/r,-1)}^{min(1/r,1)} g(rt)(w_\nu C_l^\nu)(t)dt \quad ,$$

and for the exterior Radon transform

$$(T_lg)(r) := \int_{1/r \leq |t| \leq 1} g(rt)(w_\nu C_l^\nu)(t)dt \quad .$$

In any of these cases the Radon transform decomposes into a series of 1-dimensional integral operators.

Corollary 3. *Let $g \in L_2(\mathbf{Z}, w^{-1})$, i.e. $g(\omega, s) = \sum w(s) g_l(s) Y(\omega)$, then*

$$(\mathcal{R}^* g)(x) = W(x) \mid S^{n-2} \mid \sum (T_l g_l)(\mid x \mid) Y_l\left(\frac{x}{\mid x \mid}\right) \quad ,$$

this implies that given SVD's $\{u_{ml}, v_{ml}, \sigma_{ml}\}$ for the radial transforms T_l we have a SVD for the adjoint Radon transform by, $\{Y_{lk}\}$ enumerates an orthonormal basis for the spherical harmonics,

$$\{u_{ml}(s) Y_{lk}(\omega), v_{ml}(\mid x \mid) Y_{lk}\left(\frac{x}{\mid x \mid}\right), \mid S^{n-2} \mid \sigma_{ml}\} \quad .$$

3 Intertwining differential operators

In the previous chapter the original problem, i.e. how to find a SVD for Radon transforms, was reduced to the construction of SVD's for the integral transforms T_l, see Corollary 4. The idea is to construct intertwining pairs of differential operators (D_1, D_2) such that

$$T_l D_1 = D_2 T_l \quad .$$

Then given a function g, which satisfies $D_1 g = 0$, its transform $T_l g$ is the solution of $D_2 f = 0$ with appropriate boundary conditions. Thus if we can find a complete set of intertwining differential operators, i.e. a pair (D_1, D_2) for any element of an orthogonal basis $\{u_m\}$; the images $\{T_l u_m\}$ are characterized as solutions of ordinary differential equations. The structure of these differential equations will not only allow to compute $T_l u_m$ explicitly it will also determine the weight function $W(\mid x \mid)$ for which the transformed functions are orthogonal. This procedure was used to obtain a SVD for the interior Radon transform; see [5]. We will demonstrate this technique with the full Radon transform. The operator T_l is defined by

$$(T_l g)(r) = \int_{-1}^{1} g(rt)\ C_l^{(n-2)/2}(t)(1-t^2)^{(n-3)/2} dt \quad .$$

Functions g which have a parity opposite to the parity of l are mapped to zero. Let us start with a general class of differential operators D_1 :

$$(D_1 g)(t) = (1-t^2) g''(t) + \alpha t g'(t) + \beta g(t) \quad . \tag{1}$$

For example all classical polynomials obey differential equations built with operators of this kind. We aim to combine derivatives of $T_l g$ in such a way that $D_1 g$ occurs under the integral. From the definition of $T_l g$ it follows

$$\frac{d}{dr}[(T_l g)](r) = (1/r)[T_l(t g')](r) \quad , \tag{2}$$

and

$$\frac{d^2}{dr^2}[(T_l g)](r) = (1/r^2)[T_l(t^2 g'')](r) \quad . \tag{3}$$