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Peter Imkeller

Two-Parameter Martingales and Their Quadratic Variation

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Author

Peter Imkeller
Mathematisches Institut der Ludwig-Maximilians-Universität München
Theresienstr. 39, 8000 München, Federal Republic of Germany

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Introduction

There are many fields of stochastics where multi-parameter processes can be encountered. For example, to register the positions of the spins of a ferromagnetic substance at a fixed instant of time, one has to attach an appropriate state space to every point of a three-dimensional lattice. Mathematically this leads to a family of random variables indexed by a subset of $\mathbb{R}^3$, a special case of a so-called stochastic field. Correspondingly, formalizing "multivariate observations" may lead to a stochastic process indexed by a set which, according to its order properties, can be interpreted as a multi-time. The infinite dimensional Ornstein - Uhlenbeck process which appears in a variant of Malliavin's variational calculus, may be considered as a stochastic process with a multi-time or a kind of mixed space-time parameter set (see Ikeda, Watanabe [24]). A close relative of it is the "Wiener sheet" which is undoubtedly the most frequently studied among all multi-parameter processes with a continuous parameter set (see Föllmer [21]). Walsh [43] encounters the Wiener sheet in the study of mathematical models which may arise in neurophysiology or also in problems related to heat conduction or electrical cables. We finish our selection with a more recent example. The investigation of the "Poisson chaos" seems to produce a new kind of infinite dimensional Ornstein - Uhlenbeck process which, considered as a stochastic process indexed by a two-dimensional continuous variable, behaves like a Poisson process in one direction and like a Brownian motion or a more general Gaussian process in the other direction (see Ruiz de Chavez [39], Surgailis [41], [42]).

This book is meant as a contribution to the foundations of a general theory of multi-parameter processes and their stochastic calculus. Like most of the authors who have studied this theory since 1975, when a first pioneering paper of Cairoli, Walsh [12] appeared, we restrict our attention to two-parameter processes. We have a good reason to do so, which might be underestimated at present but will become clear along the way. The considerable degree of complication we will have to face may be compensated by the geometrical simplicity of the notions and results in a two-parameter setting. On the other hand, we also have good reasons to hope that our results can be extended to an arbitrary finite number of parameters.

In the one- and two-parameter theory alike, martingales - the processes we will investigate - play an equally important and central role. An integrable process M which, like all two-parameter processes considered here, is a family of random variables on our basic probability space $(\Omega,\mathcal{F},P)$ indexed by $[0,1]^2$, is called martingale with respect to our basic filtration $(\mathcal{F}_t)_{t\in[0,1]^2}$ (i.e. a family of σ-algebras in $\mathcal{F}$, where $\mathcal{F}_t$ represents the information available at $t\in[0,1]^2$, which is increasing with respect to coordinatewise linear order on $[0,1]^2$) if M_t is $\mathcal{F}_t$-measurable (i.e. M is adapted) and for $s \leq t$ conditioning M_t by $\mathcal{F}_s$ gives M_s. One of the primary aims of an advanced martingale theory consists in the development of a "stochastic calculus" which is the basis for the field of stochastic differential or integral equations and the stochastic counterpart of the classical infinitesimal calculus. Its main theorem, known as "Itô's formula", corresponds to the fundamental theorem of calculus relating differentiation and integration. Given a function x of bounded variation on

$[0,1]^2$ which vanishes on the axes, and a C^1-function f, the classical fundamental theorem states that $f(x_{(1,1)})-f(0)$ is given by the integral of f'(x) with respect to the measure dx defined by the variation of x over $[0,1]^2$. Correspondingly, the simplest version of Itô's formula is for processes X on $\Omega\times[0,1]^2$ vanishing on the axes and whose trajectories $X(\omega)$, $\omega\in\Omega$, have bounded variation. It is given by the classical formula, trajectory by trajectory, the random variational measure $dX(\omega)$, $\omega\in\Omega$, replacing dx. As it happens, many interesting stochastic processes, like for example the Brownian motion and its two-parameter analogon, the Wiener sheet, have infinite variation, i.e. their trajectories are non-rectifiable curves. One of the main achievements of early stochastic calculus for one-parameter martingales M was to realize that in its fundamental theorem besides the "variational integral" with respect to dM, which corresponds to the classical one and becomes now a "stochastic integral", a second order term appears. It is an integral of the second derivative of f with respect to the "quadratic variation" of M. Given a sequence of partitions of the interval [0,t] by intervals whose mesh goes to zero, the quadratic variation $[M]_t$ of M at t can be defined as the limit in probability of the sequence of sums of squared increments of M over the intervals of a partition. Taking a key position in Itô's formula, the quadratic variation process is one of the most important basic processes of stochastic analysis. This is equally true for the theory of two-parameter processes, for which quadratic variation is analogously defined with respect to intervals in $[0,1]^2$.

However, the problem of the existence of quadratic variation for two-parameter martingales proved to be tough, and the progress in solving it slow. In 1981, Zakai [46] established existence for

L^4-integrable continuous martingales, extending the proof of Cairoli, Walsh [13] for martingales of the Wiener sheet. In 1984, Nualart [36] succeeded in generalizing this result to square integrable continuous martingales. For more general (especially non-continuous) martingales, only few fragmentary results were available (see Frangos, Imkeller [22] and the references there). One

One of the two main subjects of this book is to derive the existence of quadratic variation for square integrable and, more generally, $L \log^+ L$-integrable two-parameter martingales. The second one lies in the method applied to accomplish this aim. It consists in deriving a representation theorem for square integrable martingales by various "pure jump parts" and a "continuous part" and constructing their quadratic variation from those of the parts. This theorem is the end-product of a procedure of "reduction of jumps" which stimulates a deeper study of the structure of martingales and their general theory and is of independent interest, as well. The execution of this entire program, however, attains a degree of complication which makes us prefer to shape its contours at first in the case of a one-parameter martingale M indexed by [0,1] with respect to a filtration $(\mathbb{G}_t)_{t\in[0,1]}$. This will serve as a "red thread" later.

The significance of the concept of "stopping" is one of the big differences between one- and two-parameter theory. Whereas in the former "stopping times" take an eminent place and influence almost all methods of investigation, "stopping points", "stopping lines" and related notions are more peripheric and of limited use in the latter theory. The procedure of separation of jump parts and computation of quadratic variation we are about to sketch, has to take this into account. It will not make any essential use of a stopping notion. Therefore, even in the classical one-parameter martingale theory, although the results are known, the notions and methods we use to derive them seem to bear some novelty.

As one of the most striking elementary phenomena about M, the expected number of up- and downcrossings of a given space interval is bounded. This readily leads to one of the important regularity results of the theory: M possesses a version whose sample paths are right-continuous and have left limits. We denote this regular version again by M and can now talk about "jumps". A point (ω,t) in $\Omega\times[0,1]$ is called jump of M, if $M_t(\omega)-M_{t-}(\omega)\neq 0$, where $M_{t-}(\omega)$ is the left hand limit of $M(\omega)$ at t. Now the observation made above can be put in more stringent terms: for any $n\in\mathbb{N}$, the random set S_n of jumps of M of heights between $\frac{1}{n}$ and $\frac{1}{n-1}$ has $[0,1]$-sections whose cardinalities constitute an integrable random variable. In particular, they are finite a.s..

Assume that M is square integrable in addition. We plan to get a decomposition of M by a jump part and a continuous part (both square integrable martingales) by successively extracting the jumps of M on S_n, $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$ and consider the jump process M_n of M, restricted to S_n, which, at time t, is just the sum of all jumps of M on S_n up to t. The process M_n is of bounded variation, but, of course, need not be a martingale. Yet, the part we want to cut off M in order to obtain a continuous remainder, is to be a martingale. Therefore, our task could be put in the following terms: compensate M_n by a process C_n of integrable variation in such a way that firstly the resulting process M_n-C_n is a martingale and secondly C_n creates no new jumps, i.e. C_n is continuous. The first requirement will ensure that the decomposition we obtain is a martingale decomposition, the second that the remainder after finishing the cut-off procedure is indeed continuous.

The solution of this compensation problem requires a closer study of measurability concepts in the product $\Omega\times[0,1]$. To see which σ-algebra might play a role, let us try to find conditions under which we face the simplest case - the case in which M_n is already a martingale and compensation unnecessary. First assume that $S_n = \Omega\times\{t_o\}$ is a deterministic set, for some $t_o\in[0,1]$. Let $(t_m)_{m\in\mathbb{N}}$ be a strictly increasing sequence in parameter space which converges to t_o, and let $X^m = (M_{t_o}-M_{t_m})\,1_{[t_o,1]}$. Now the martingale property of M states that for any pair (u,v) of times, $u\leqq v$, conditioning M_v by $\mathfrak{G}_u$ gives M_u. This implies that X^m_v, conditioned by $\mathfrak{G}_u$, may only differ from X^m_u if $t_m<u<t_o$. But, as becomes evident from Doob's maximal inequality, the convergence $X^m\to M_n$ as $m\to\infty$ is dominated and thus M_n turns out to be a martingale. Next assume that S_n is a random one point set. It can easily be seen to be just the graph of a stopping time T_o with respect to $(\mathfrak{G}_t)_{t\in[0,1]}$. Since the conditioning property characterizing a martingale extends from a pair of deterministic times (u,v) to a pair of stopping times (U,V), $U\leqq V$, we see that M_n is a martingale, if, as above, T_o can be "predicted" by a strictly increasing sequence $(T_m)_{m\in\mathbb{N}}$ of stopping times. Of course, this need not necessarily be the case. It is true if and only if T_o is measurable with respect to the σ-algebra of "previsible sets" which is generated by all continuous adapted processes. "Continuity" ensures that stopping times in this σ-algebra can be predicted. There is another important σ-field in $\Omega\times[0,1]$, in which this is typically not the case. It is generated by the regular adapted processes, hence contains the previsible sets and is called σ-algebra of "optional sets". Of course, M and therefore S_n is optional. And, as is suggested by what has been said above, M_n

is a martingale if S_n is previsible.

Let us return to the problem of compensation of M_n. A version of the decomposition theorem of Doob-Meyer points in the direction of a possible solution. This theorem which we will refer to as "projection theorem" is of central importance for our analysis and states that for any integrable increasing process A there exists a unique previsible increasing process A^p, called its "dual previsible projection" such that $A-A^p$ is a martingale. We may apply it to the increasing processes $\overline{M}_n$, $\underline{M}_n$ of positive resp. negative jumps of M on S_n. Then $C_n = \overline{M}_n^p - \underline{M}_n^p$ makes $M_n - C_n$ a martingale. But to be a good candidate for a compensator, C_n has to be continuous, too. This need not be true in general. Yet, the above discussion already indicates what might be crucial for the problem. According to it, the previsible case is already solved. Assume now, on the contrary, that S_n intersects any previsible random set of the same geometric type (i.e. its [0,1]-sections are a.s. finite) only on a negligible set. In this case, S_n is said to be "totally inaccessible", and C_n has to be continuous for the following reasons. In consequence of its previsibility and regularity, it is unable to realize the totally inaccessible set S_n and therefore cannot jump on it. On the other hand, it cannot jump outside S_n, since it can be seen without effort that a previsible martingale of bounded variation, like a hypothetical jump of $M_n - C_n$ on a previsible set outside S_n, vanishes. This important observation indicates how the problem can be attacked in general: partition the optional set of jumps of M by random sets which are as simple as S_n, but which are either previsible or totally inaccessible. Of course, this partition need not coincide with $(S_n)_{n \in \mathbb{N}}$.

We first state more precisely what we mean by "simplicity". Call an optional random set S in $\Omega \times [0,1]$ simple, if the cardinalities of its [0,1]-sections define an integrable random variable. The crux of the construction of the desired partition is then to see how a given simple set (like S_n) can be covered by a sequence of pairwise disjoint either previsible or totally inaccessible sets. To this end, associate with a given simple set S the integrable increasing process $\Gamma(S)$ which counts the points of S, i.e. $\Gamma(S)(\omega)$ at time t is just the cardinality of the intersection of S_ω and $[0,t]$, $\omega\in\Omega$. Consider its dual previsible projection $\Gamma(S)^p$. Since this process is previsible, its jumps can be arranged on a countable union of pairwise disjoint simple random sets $(T_n)_{n\in\mathbb{N}}$ which are previsible. Since $\Gamma(S)-\Gamma(S)^p$ is a martingale, the union of the T_n is already the essential supremum of previsible simple sets in S and its complement in S totally inaccessible. Hence, an analysis of the jumps of the dual previsible projection of the increasing process associated with a simple set gives the desired covering sequence.

Assume now a partition $(U_n)_{n\in\mathbb{N}}$ of the set of jumps of M by either previsible or totally inaccessible simple sets is given. Let M_n denote again the jump process of M on U_n, C_n its compensator. We know C_n is a continuous process of bounded variation. In the Hilbert space of square integrable martingales whose norm is defined by the usual L^2-norm, the martingales M_n-C_n prove to be pairwise orthogonal due to the pairwise disjointness of the U_n. The orthogonal complement of the sum M^o of these compensated jumps defines a square integrable continuous martingale M^c, due to the continuity of the compensators. This finishes the procedure of decomposing M by a pure jump martingale M^o and a continuous martingale M^c.

Given this decomposition, it is not hard to compute the quadratic variation of M. One simply has to compute the quadratic variations of M^o and M^c and add them: the fact that all compensators are continuous and of bounded variation, hence cannot contribute, and that M^c is continuous, makes the quadratic variations of the orthogonal parts also "orthogonal". Now the quadratic variation of M^o is just the sum of the squares of the jumps of M. To obtain the quadratic variation of M^c, one may take resort to a more general version of the theorem of Doob-Meyer than the above projection theorem. It states that for any nonnegative submartingale X there exists a unique previsible increasing process A, also called "dual previsible projection" of X, such that X-A is a martingale. It applies to $(M^c)^2$ and allows to identify the dual previsible projection of this process as the quadratic variation of M^c.

To summarize, the procedure of investigating the structure of square integrable martingales M and their quadratic variations which has been outlined consists of the following main steps:

1) by application of an appropriate version of the decomposition theorem of Doob-Meyer (called projection theorem in the case of increasing processes) find the dual previsible projections of increasing processes (like $\Gamma(S)$ for simple S) and of submartingales (like M^2),
2) partition the random set of jumps of M by a sequence $(U_n)_{n \in \mathbb{N}}$ of either previsible or totally inaccessible simple sets,
3) find the compensators C_n of the jump process M_n of M on U_n and show they are continuous,
4) show that the compensated jump processes $M_n - C_n$ are pairwise orthogonal, subtract their sum M^o from M to obtain the continuous part,
5) compute the quadratic variations of M^o and M^c and sum up to obtain the quadratic variation of M.

Guided by the above 5 step program we will now give an outline of an analogous procedure of gradual elimination of jumps and computation of quadratic variation for two-parameter martingales, as presented in this book. Of course, the notions and statements figuring in 1)-5) have to be reinterpreted in a two-parameter setting. As a usual phenomenon which might be realized rather soon, the theory hereby becomes essentially more complicated. For this reason, we emphasize that our "sketch" can be considered as being of a more than purely introductory character. On one hand, we felt that at places it might become too tough to read without offering the reader the opportunity to jump to the details in the text. We therefore have provided it with hints indicating the numbers of corresponding main theorems (T) or propositions (P) of the text. On the other hand, given that the procedure proposed is not of a straightforward kind, this outline might be helpful for a better understanding. It strictly follows the strain of ideas involved in the step-by-step order of 1)-5), not necessarily the order in which the results are presented in the text and which is imposed by logical or formal aspects. Whenever the reader looks for a "red thread" he may jump back to this sketch to regain orientation.

Our analysis starts with a crucial assumption stated first by Cairoli, Walsh [12] in their pioneering paper and adopted ever since by most of the authors concerned with this theory. It says that the information gained in the future of a given time point t in one direction is independent of the information gained in the other direction, given what happened up to t. This "conditional independence" assumption is vital for what follows. It finds its most useful interpretation for our purposes in terms of "optional and previsible (dual) projections". To explain these notions, let us briefly return to the one-parameter case, where we already encountered the dual previsible projection of an integrable increasing process. The theorem stating its existence can be viewed in an alternative way. For a given integrable increasing process A, let m_A denote the measure on the product $\Omega\times[0,1]$ defined by $m_A(S) = E(\int 1_S \, dA)$, S a product measurable set. Then it states that there is exactly one previsible increasing process A^p such that the integrals of any bounded previsible process with respect to m_A and m_{A^p} are the same. An optional version of this theorem says that there is exactly one optional increasing process A^o (the "dual optional projection" of A) such that the integrals of any bounded optional process with respect to m_A and m_{A^o} coincide. Finally, the statements of the projection theorems can be dualized. Given a bounded product measurable process X, there is exactly one bounded optional (previsible) process ${}^oX$ (${}^pX$) such that X and ${}^oX$ (X and ${}^pX$) cannot be distinguished by measures associated with optional (previsible) integrable increasing processes. ${}^oX$ resp. ${}^pX$ is called "optional" resp. "previsible" projection of X. Projection theorems and the martingale notion are linked by the following elementary observation. If X is a bounded random variable, considered as a process with a trivial time dependence, and M is a regular version of the martingale generated by taking conditional expectations of X, then ${}^oX=M$, ${}^pX=M_-$, where M_- is the process of left limits of M. This identification is the starting point for our study of projections for two-parameter processes.

We first prove that "one-directional" (dual) projections exist

(T(4.1), T(4.2)). More precisely, for a given product measurable bounded process X on $\Omega\times[0,1]^2$ the family of optional (previsible) projections of the one-parameter processes $X_{(\cdot,r)}$ in i-direction, r being the (fixed) parameter of the complementary direction, can be chosen measurably with respect to the product $\mathfrak{G}^i$ $(\mathfrak{P}^i)$ of the optional (previsible) sets in i-direction with the Borel sets in the remaining one, called σ-algebra of "i-optional" ("i-previsible") sets, i=1,2. A dual statement can be made for integrable increasing processes A, where "increasing" means that they define random measures on the Borel sets of $[0,1]^2$ in the usual way. The resulting processes ${}^{\gamma_i}X$ $({}^{\pi_i}X)$ resp. A^{γ_i} (A^{π_i}) are called (dual) i-optional (i-previsible) projections. A series of regularity results (T(6.1), T(6.2)) shows that X inherits regularity properties to its projections ${}^{\gamma_i}X$ $({}^{\pi_i}X)$. They in turn imply that any $L\log^+L$-integrable martingale M possesses a version whose trajectories are continuous for approach in the right upper quadrant and possess limits for approach in the left upper and right lower quadrants (P(8.2)). Strangely enough, only an additional investigation of a stochastic integral for square integrable martingales which shows that they can be "stopped" on more complicated random sets (P(9.3), P(9.4)) reveals that M also possesses limits for the left lower quadrant (P(9.5)). To sum up, $L\log^+L$-integrable martingales have "regular" versions (T(9.1)). Once this is established, it is not hard to link two-parameter projections and martingales in the same way as above. If X is a bounded random variable, M a regular version of the martingale generated by taking conditional expectations of X, then ${}^{\gamma_1\gamma_2}X = M = {}^{\gamma_2\gamma_1}X$, ${}^{\pi_1\gamma_2}X = M^{-\bullet} = {}^{\gamma_2\pi_1}X$, ${}^{\gamma_1\pi_2}X = M^{\bullet-} = {}^{\pi_2\gamma_1}X$, ${}^{\pi_1\pi_2}X = M^{--} = {}^{\pi_2\pi_1}X$ (T(10.1)). Here "-" always means that left limits have to be taken in the respective direction, for example: $M^{-\bullet}$ is the process defined by the left limits of M in the first direction. This result has very important consequences, once it is extended to general bounded product measurable processes and dualized. It says that it is immaterial in which order iterated (dual) projections are taken and allows us to define (dual) previsible (i.e. 1- and 2-previsible, denoted by $^{\pi}X$ resp. A^{π}) projections, 1-previsible, 2-optional projections, 2-previsible, 1-optional projections, and optional (i.e. 1- and 2-optional, denoted by $^{\gamma}X$, A^{γ}) projections by just the product of two one-directional projections in an arbitrary order (P(10.1)). Here is what we mean by the most useful interpretation of the conditional independence assumption: projections and dual projections in different directions are "independent" of each other, commute and their compositions define "two-directional" (dual) projections in an unambiguous way (T(10.2), T(10.3)). Implicitly we also have accomplished the first half of step 1 of our program. Given an integrable increasing process A, we know that $A - A^{\pi_1}$ is a martingale in direction 1, whose dual previsible projection is $A^{\pi_2} - A^{\pi_1 \pi_2} = A^{\pi_2} - A^{\pi}$. Hence $A - (A^{\pi_1} + A^{\pi_2} - A^{\pi})$ is a martingale in both directions, i.e. a martingale (cor. 1 of T (11.2)). At this stage, the second half of step 1 is not very difficult any more. Call an integrable process X submartingale, if it is a submartingale in the order sense, i.e. if it is a one-parameter submartingale on every increasing path in $[0,1]^2$, and a weak submartingale, if the increments $\Delta_J X$ of X over rectangles J in parameter space, conditioned by the information available at their lower left boundary points, are nonnegative. We concentrate on the construction of the counterpart of A^{π} for X. The process A^{π} is the uniquely determined previsible increasing process such that $m_A|\mathcal{P} = m_{A^{\pi}}|\mathcal{P}$, where $\mathcal{P} = \mathcal{P}^1 \cap \mathcal{P}^2$ is the σ-algebra of previsible sets and m_B defined for integrable increasing B like in the one-para-