Lecture Notes in Mathematics

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Mathematical Foundation of Turbulent Viscous Flows

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Preface

Over two centuries ago, L. Euler (1750) derived an ideal model equation describing the evolution of fluids. Later on, this model was revised under a more realistic basis by H. Navier (1822) and G. Stokes (1845). Finally, with his eponymous equation L. Boltzmann (1872) introduced the foundation of Gas Dynamics. Since then, much progress has been made in the understanding of these physical models. But many fundamental mathematical questions still remain unresolved, such as the existence, uniqueness and stability of solutions to the corresponding equations in three dimensions.

Due to the large number of applications to different fields (such as meteorology, astrophysics, aeronautics, thermodynamics, lasers and plasma physics), the study of these model equations from a purely mathematical point of view plays a crucial role in Applied Mathematics.

The series of lectures contained in this volume reflects five different and complementary approaches to several fundamental questions arising in the study of the Fluid Mechanics and Gas Dynamics equations. These lectures were presented by five well-known mathematicians at the International CIME Summer School held in Martina Franca, Italy, from 1 to 5 September 2003.

P. Constantin presents the Euler equations of ideal incompressible fluids and discusses the blow-up problem for the Navier-Stokes equations of viscous fluids, also describing some of the major mathematical questions of turbulence theory.

These questions are intimately connected to the Caffarelli-Kohn-Nirenberg theory of singularities for the incompressible Navier-Stokes equations, that is explained in detail in *G. Gallavotti*'s lectures.

A. Kazikhov introduces the reader to the theory of strong approximation of weak limits via the method of averaging, applied to the Navier-Stokes equations.

On the other hand, *Y. Meyer*'s lectures focus on several nonlinear evolution equations – in particular the Navier-Stokes ones – and some related unexpected cancellation properties, that are either imposed on the initial condition, or satisfied by the solution itself, whenever it is localized in space or in time variable.

Finally, *S. Ukai* presents the asymptotic analysis theory of fluid equations. More precisely, he discusses the Cauchy-Kovalevskaya technique for the Boltzmann-Grad limit of the Newtonian equation, the multi-scale analysis, giving the compressible and incompressible limits of the Boltzmann equation, and the analysis of their initial layers.

Many Ph. D. students and researchers from all over the world attended the summer school, thereby contributing to its success.

The Apulian landscape with its Romanesque and Baroque cathedrals, castles, rocky settlements, trullis and caves, and the city of Martina Franca, with its Ducal Palace – where the lectures were held – contributed to creating an attractive and pleasant working atmosphere. The summer school would not have taken place without the contagious optimism of Vincenzo Vespri, the efficient coordination of Elvira Mascolo and Pietro Zecca and the precious help of Marco Romito and Veronika Sustik. We would like also to thank here Carla Dionisi, who took care of the final typesetting of the lectures notes.

Finally, let us remember that the CIME summer school benefited from the financial support of Ministero degli Affari Esteri – Direzione Generale per la Promozione e la Cooperazione – Ufficio V, M.U.R.S.T., INdAM and Università Franco-Italiana.

July 2004 *Marco Cannone*
Paris, Kanazawa *Tetsuro Miyakawa*

During the process of proofreading, we learned with sorrow that our colleague and friend Alexandre V. Kazhikhov, one of the authors of this volume, passed away on November 3, while at work in his office in Novosibirsk.

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Euler Equations, Navier-Stokes Equations and Turbulence

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1 Introduction

In 2004 the mathematical world will mark 120 years since the advent of turbulence theory ([80]). In his 1884 paper Reynolds introduced the decomposition of turbulent flow into mean and fluctuation and derived the equations that describe the interaction between them. The Reynolds equations are still a riddle. They are based on the Navier-Stokes equations, which are a still a mystery. The Navier-Stokes equations are a viscous regularization of the Euler equations, which are still an enigma. Turbulence is a riddle wrapped in a mystery inside an enigma ([11]).

Crucial for the determination of the mean in the Reynolds equation are Reynolds stresses, which are second order moments of fluctuation. The fluctuation requires information about small scales. In order to be able to compute at high Reynolds numbers, in state-of-the-art engineering practice, these small scales are replaced by sub-grid models. "Que de choses il faut ignorer pour 'agir' ! " sighed Paul Valéry ([88]). ("How many things must one ignore in order to 'act' ! ") The effect of small scales on large scales is the riddle in the Reynolds equations. In 1941 Kolmogorov ([65]) ushered in the idea of universality of the statistical properties of small scales. This is a statement about the asymptotics: long time averages, followed by the infinite Reynolds number limit. This brings us to the mystery in the Navier-Stokes equations: the infinite time behavior at finite but larger and larger Reynolds numbers. The small Reynolds number behavior is trivial (or "direct", to use the words of Reynolds himself). Ruelle and Takens suggested in 1971 that deterministic chaos emerges at larger Reynolds numbers ([83]). The route to chaos itself was suggested to be universal by Feigenbaum ([49]). Foias and Prodi discussed finite dimensional determinism in the Navier-Stokes equations already in 1967 ([55]), four years after the seminal work of Lorenz ([68]). The dynamics have indeed finite dimensional character if one confines oneself to flows in bounded regions in two dimensions ([2], [31], [32], [34], [56], [69]). In three dimensions, however, the long time statistics question is muddied by the blow up problem.

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Leray ([67]) showed that there exist global solutions, but such solutions may develop singularities. Do such singularities exist? And if they do, are they relevant to turbulence? The velocities observed in turbulent flows on Earth are bounded. If one accepts this as a physical assumption, then, invoking classical results of Serrin ([84]), one concludes that Navier-Stokes singularities, if they exist at all, are not relevant to turbulence. The experimental evidence, so far, is of a strictly positive energy dissipation rate $0 < \epsilon = \langle \nu |\nabla u|^2 \rangle$, at high Reynolds numbers. This is consistent with large gradients of velocity. The gradients of velocity intensify in vortical activity. This activity consists of three mechanisms: stretching, folding and reconnection of vortices. The stretching and folding are inviscid mechanisms, associated with the underlying incompressible Euler equations. The reconnection is the change of topology of the vortex field, and it is not allowed in smooth solutions of the Euler equations. This brings us to the enigma of the Euler equations, and it is here where it is fit we start.

2 Euler Equations

The Euler equations of incompressible fluid mechanics present some of the most serious challenges for the analyst. The equations are

$$D_t u + \nabla p = 0 \tag{2.1}$$

with $\nabla \cdot u = 0$. The function $u = u(x,t)$ is the velocity of an ideal fluid at the point x in space at the moment t in time. The fluid is assumed to have unit density. The velocity is a three-component vector, and x lies in three dimensional Euclidean space. The requirement that $\nabla \cdot u = 0$ reflects the incompressibility of the fluid. The material derivative (or time derivative along particle trajectories) associated to the velocity u is

$$D_t = D_t(u, \nabla) = \partial_t + u \cdot \nabla. \tag{2.2}$$

The acceleration of a particle passing through x at time t is $D_t u$. The Euler equations are an expression of Newton's second law, $F = ma$, in the form $-\nabla p = D_t u$. Thus, the only forces present in the ideal incompressible Euler equations are the internal forces at work keeping the fluid incompressible. These forces are not local: the pressure obeys

$$-\Delta p = \nabla \cdot (u \cdot \nabla u) = \mathrm{Tr}\left\{ (\nabla u)^2 \right\} = \partial_i \partial_j \left(u_i u_j \right).$$

If one knows the behavior of the pressure at boundaries then the pressure satisfies a nonlocal functional relation of the type $p = F([u \otimes u])$. For instance, in the whole space, and with decaying boundary conditions

$$p = R_i R_j (u_i u_j)$$

where $R_i = \partial_i(-\Delta)^{-\frac{1}{2}}$ are Riesz transforms. (We always sum repeated indices, unless we specify otherwise. The pressure is defined up to a time dependent constant; in the expression above we have made a choice of zero average pressure.)

Differentiating the Euler equations one obtains:

$$D_t U + U^2 + \mathrm{Tr}\left\{(R \otimes R)\, U^2\right\} = 0$$

where $U = (\nabla u)$ is the matrix of derivatives. We used the specific expression for p written above for the whole space with decaying boundary conditions. This equation is quadratic and it suggests the possibly of singularities in finite time, by analogy with the ODE $\frac{d}{dt}U + U^2 = 0$. In fact, the distorted Euler equation

$$\partial_t U + U^2 + \mathrm{Tr}\left\{(R \otimes R)\, U^2\right\} = 0$$

does indeed blow up ([15]). The incompressibility constraint $\mathrm{Tr}U = 0$ is respected by the distorted Euler equation. However, the difference between the Eulerian time derivative ∂_t and the Lagrangian time derivative D_t is significant. One may ask whether true solutions of the Euler equations do blow up. The answer is yes, if one allows the solutions to have infinite kinetic energy. We will give an example in Section Three. The blow up is likely due to the infinite supply of energy, coming from infinity. The physical question of finite time local blow up is different, and perhaps even has a different answer.

In order to analyze nonlinear PDEs with physical significance one must take advantage of the basic invariances and conservation laws associated to the equation. When properly understood, the reasons behind the conservation laws show the way to useful cancellations.

Smooth solutions of the Euler equations conserve total kinetic energy, helicity and circulation. The total kinetic energy is proportional to the L^2 norm of velocity. This is conserved for smooth flows. The Onsager conjecture ([72], [48]) states that this conservation occurs if and only if the solutions are smoother than the velocities supporting the Kolmogorov theory, (Hölder continuous of exponent 1/3). The "if" part was proved ([28]). The "only if" part is difficult: there is no known notion of weak solutions dissipating energy but with Hölder continuous velocities. The work of Robert ([81]) and weak formulations of Brenier and of Shnirelman are relevant to this question ([85], [6]).

In order to describe the helicity and circulation we need to talk about vorticity and about particle paths. The Euler equations are formally equivalent to the requirement that two first order differential operators commute:

$$[D_t, \Omega] = 0.$$

The first operator $D_t = \partial_t + u \cdot \nabla$ is the material derivative associated to the trajectories of u. The second operator

$$\Omega = \omega(x,t) \cdot \nabla$$

is differentiation along vortex lines, the lines tangent to the vorticity field ω. The commutation means that vortex lines are carried by the flow of u, and is equivalent to the equation

$$D_t\omega = \omega \cdot \nabla u. \tag{2.3}$$

This is a quadratic equation because ω and u are related, $\omega = \nabla \times u$. If boundary conditions for the divergence-free ω are known (periodic or decay at infinity cases) then one can use the Biot-Savart law

$$u = \mathcal{K}_{3DE} * \omega = \nabla \times (-\Delta)^{-1}\omega \tag{2.4}$$

coupled with (2.3) as an equivalent formulation of the Euler equations, the vorticity formulation used in the numerical vortex methods of Chorin ([13], [14]). The helicity is

$$h = u \cdot \omega.$$

The Lagrangian particle maps are

$$a \mapsto X(a,t), \quad X(a,0) = a.$$

For fixed a, the trajectories of u obey

$$\frac{dX}{dt} = u(X,t).$$

The incompressibility condition implies

$$det\,(\nabla_a X) = 1.$$

The Euler equations can be described ([63], [1]) formally as Euler-Lagrange equations resulting from the stationarity of the action

$$\int_a^b \int |u(x,t)|^2\, dxdt$$

with

$$u(x,t) = \frac{\partial X}{\partial t}(A(x,t),t)$$

and with fixed end values at $t = a, b$ and

$$A(x,t) = X^{-1}(x,t).$$

Helicity integrals ([71])

$$\int_T h(x,t)dx = c$$

are constants of motion, for any vortex tube T (a time evolving region whose boundary is at each point parallel to the vorticity, $\omega \cdot N = 0$ where N is the normal to ∂T at $x \in \partial T$.) The constants c have to do with the topological complexity of the flow.

Davydov, and Zakharov and Kuznetsov ([42], [93]) have formulated the incompressible Euler equations as a Hamiltonian system in infinite dimensions in Clebsch variables. These are a pair of active scalars θ, φ which are constant on particle paths,

$$D_t\varphi = D_t\theta = 0$$

and also determine the velocity via

$$u^i(x,t) = \theta(x,t)\frac{\partial\varphi(x,t)}{\partial x_i} - \frac{\partial n(x,t)}{\partial x_i}.$$

The helicity constants vanish identically for flows which admit a Clebsch variables representation. Indeed, for such flows the helicity is the divergence of a field that is parallel to the vorticity $h = -\nabla\cdot(n\omega)$. This implies that not all flows admit a Clebsch variables representation. But if one uses more variables, then one can represent all flows. This is done using the Weber formula ([90]) which we derive briefly below.

In Lagrangian variables the Euler equations are

$$\frac{\partial^2 X^j(a,t)}{\partial^2 t} = -\frac{\partial p(X(a,t),t)}{\partial x_j}. \tag{2.5}$$

Multiplying this by $\frac{\partial X^j(a,t)}{\partial a_i}$ we obtain

$$\frac{\partial^2 X^j(a,t)}{\partial t^2}\frac{\partial X^j(a,t)}{\partial a_i} = -\frac{\partial\tilde{p}(a,t)}{\partial a_i}$$

where $\tilde{p}(a,t) = p(X(a,t),t)$. Forcing out a time derivative in the left-hand side we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial X^j(a,t)}{\partial t}\frac{\partial X^j(a,t)}{\partial a_i}\right] = -\frac{\partial\tilde{q}(a,t)}{\partial a_i}$$

with

$$\tilde{q}(a,t) = \tilde{p}(a,t) - \frac{1}{2}\left|\frac{\partial X(a,t)}{\partial t}\right|^2$$

Integrating in time, fixing the label a we obtain:

$$\frac{\partial X^j(a,t)}{\partial t}\frac{\partial X^j(a,t)}{\partial a_i} = u^i_{(0)}(a) - \frac{\partial\tilde{n}(a,t)}{\partial a_i}$$

with

$$\tilde{n}(a,t) = \int_0^t \tilde{q}(a,s)ds$$

where

$$u_{(0)}(a) = \frac{\partial X(a,0)}{\partial t}$$

is the initial velocity. We have thus:

$$(\nabla_a X)^* \partial_t X = u_{(0)}(a) - \nabla_a \tilde{n}.$$

where we denote M^* the transpose of the matrix M.
Multiplying by $\left[(\nabla_a X(a,t))^*\right]^{-1}$ and reading at $a = A(x,t)$ with

$$A(x,t) = X^{-1}(x,t)$$

we obtain the Weber formula

$$u^i(x,t) = \left(u^j_{(0)}(A(x,t))\right) \frac{\partial A^j(x,t)}{\partial x_i} - \frac{\partial n(x,t)}{\partial x_i}.$$

This relationship, together with boundary conditions and the divergence-free requirement can be written as

$$u = W[A,v] = \mathbf{P}\left\{(\nabla A)^* v\right\} \tag{2.6}$$

where $\mathbf{P}$ is the corresponding projector on divergence-free functions and v is the virtual velocity

$$v = u_{(0)} \circ A.$$

We will consider the cases of periodic boundary conditions or whole space. Then

$$\mathbf{P} = I + R \otimes R$$

holds, with R the Riesz transforms. This procedure turns A into an *active scalar system*

$$\left\{\begin{array}{r} D_t A = 0, \\ D_t v = 0, \\ u = W[A,v]. \end{array}\right. \tag{2.7}$$

Active scalars ([17]) are solutions of the passive scalar equation $D_t\theta = 0$ which determine the velocity through a time independent, possibly non-local equation of state $u = U[\theta]$.

Conversely, and quite generally: Start with two families of labels and virtual velocities

$$A = A(x,t,\lambda), \quad v(x,t,\lambda)$$

depending on a parameter λ such that

$$D_t A = D_t v = 0$$

with $D_t = \partial_t + u \cdot \nabla_x$. Assume that u can be reconstructed from A, v via a generalized Weber formula

$$u(x,t) = \int \nabla_x A(x,t,\lambda) v(x,t,\lambda) d\mu(\lambda) - \nabla_x n$$

with some function n, and some measure $d\mu$. Then u solves the Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \pi = 0$$

where

$$\pi = D_t n + \frac{1}{2}|u|^2.$$

Indeed, using the kinematic commutation relation

$$D_t \nabla_x f = \nabla_x D_t f - (\nabla_x u)^* \nabla_x f$$

and differentiating the generalized Weber formula we obtain:

$$D_t u = D_t (\int \nabla_x A v d\mu - \nabla_x n) =$$

$$-\int ((\nabla_x u)^* \nabla_x A) v d\mu - \nabla_x (D_t n) + (\nabla_x u)^* \nabla n =$$

$$-\nabla_x (D_t n) - (\nabla_x u)^* \left[\int (\nabla_x A) v d\mu - \nabla_x n \right] =$$

$$-\nabla_x (D_t n) - (\nabla_x u)^* u = -\nabla_x (\pi).$$

The circulation is the loop integral

$$C_\gamma = \oint_\gamma u \cdot dx$$

and the conservation of circulation is the statement that

$$\frac{d}{dt} C_{\gamma(t)} = 0$$

for all loops carried by the flow. This follows from the Weber formula because

$$u^j (X(a,t)) \frac{\partial X^j}{\partial a_i} = u^i_{(0)}(a) - \frac{\partial \tilde{n}(a,t)}{\partial a_i}.$$

The important thing here is that the right hand side is the sum of a time independent function of labels and a label gradient. Viceversa, the above formula follows from the conservation of circulation. The Weber formula is equivalent thus to the conservation of circulation.

Differentiating the Weber formula, one obtains

$$\frac{\partial u^i}{\partial x_j} = \mathbf{P}_{ik} \left(Det \left[\frac{\partial A}{\partial x_j}; \frac{\partial A}{\partial x_k}; \omega_{(0)}(A) \right] \right).$$

Here we used the notation

$$\omega_{(0)} = \nabla \times u_{(0)}.$$

Taking the antisymmetric part one obtains the Cauchy formula:

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\left(Det\left[\frac{\partial A}{\partial x_j}; \frac{\partial A}{\partial x_k}; \omega_{(0)}(A)\right]\right)$$

which we write as

$$\omega = \mathcal{C}[\nabla A, \zeta] \tag{2.8}$$

with ζ the Cauchy invariant

$$\zeta(x,t) = \omega_{(0)} \circ A.$$

Therefore the active scalar system

$$\begin{cases} D_t A = 0, \\ D_t \zeta = 0, \\ u = \nabla \times (-\Delta)^{-1}(\mathcal{C}[\nabla A, \zeta]) \end{cases} \tag{2.9}$$

is an equivalent formulation of the Euler equations, in terms of the Cauchy invariant ζ. The purely Lagrangian formulation (2.5) of the Euler equations is in phrased in terms of independent label variables (or ideal markers) a, t, except that the pressure is obtained by solving a Poisson equation in Eulerian independent variables x, t. The rest of PDE formulations of the Euler equations described above were: the Eulerian velocity formulation (2.1), the Eulerian vorticity formulation (2.3), the Eulerian-Lagrangian virtual velocity formulation (2.7) and the Eulerian-Lagrangian Cauchy invariant formulation (2.9). The Eulerian-Lagrangian equations are written in Eulerian coordinates x, t, in what physicists call "laboratory frame". The physical meaning of the dependent variables is Lagrangian.

The classical local existence results for Euler equations can be proved in either purely Lagrangian formulation ([45]), in Eulerian formulation ([70]) or in Eulerian-Lagrangian formulation ([19]). For instance one has

Theorem 2.1. *([19]) Let $\alpha > 0$, and let u_0 be a divergence free $C^{1,\alpha}$ periodic function of three variables. There exists a time interval $[0,T]$ and a unique $C([0,T]; C^{1,\alpha})$ spatially periodic function $\ell(x,t)$ such that*

$$A(x,t) = x + \ell(x,t)$$

solves the active scalar system formulation of the Euler equations,

$$\frac{\partial A}{\partial t} + u \cdot \nabla A = 0,$$

$$u = \mathbf{P}\left\{(\nabla A(x,t))^* u_0(A(x,t))\right\}$$

with initial datum $A(x,0) = x$.

A similar result holds in the whole space, with decay requirements for the vorticity. As an application, let us consider rotating three dimensional incompressible Euler equations

$$\partial_t u + u \cdot \nabla u + \nabla \pi + 2\Omega e_3 \times u = 0.$$

The Weber formula for relative velocity is:

$$u(x,t) = \mathbf{P}(\partial_i A^m(x,t) u_0^m(A(x,t),t))$$

$$+\Omega \mathbf{P}\left\{(\widehat{z}; A(x,t), \partial_i A(x,t)) - (\widehat{z}; x; e_i)\right\}.$$

Here Ω is the constant angular velocity (not $\omega \cdot \nabla$), and e_i form the canonical basis of $\mathbf{R}^3$. We consider the Lagrangian paths $X(a,t)$ associated to the relative velocity u, and their inverses $A(x,t) = X^{-1}(x,t)$, obeying

$$(\partial_t + u \cdot \nabla)\, A = 0.$$

As a consequence of the Cauchy formula for the total vorticity $\omega + 2\Omega e_3$ one can prove that the direct Lagrangian displacement

$$\lambda(a,t) = X(a,t) - a$$

obeys a time independent differential equation. The Cauchy formula for the total vorticity (the vorticity in a non-rotating frame) follows from differentiation of the Weber formula above and is the same as in the non-rotating case

$$\omega + 2\Omega e_3 = \mathcal{C}[\nabla A, \zeta + 2\Omega e_3]$$

Composing with X the right hand side is

$$\mathcal{C}[\nabla A, \zeta + 2\Omega e_3] \circ X = \left(\omega_{(0)} + 2\Omega e_3\right) \cdot \nabla_a X.$$

Rearranging the Cauchy formula we obtain

$$\partial_{a_3} \lambda(a,t) + \frac{1}{2}\rho_0(a)\xi_0(a) \cdot \nabla_a \lambda(a,t) =$$

$$= \frac{1}{2}\left(\rho_t(a)\xi(a,t) - \rho_0(a)\xi(a,0)\right)$$

where

$$\rho_t(a) = \frac{|\omega(X(a,t),t)|}{\Omega}$$

is the local Rossby number and $\xi = \frac{\omega}{|\omega|}$ is the unit vector of relative vorticity direction. This fact explains directly ($\partial_{a_3}\lambda = O(\rho)$) the fact that strong rotation inhibits vertical transport ([24]). In particular, one can prove rather easily